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Radially symmetric solutions of a chemotaxis model in the critical case

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1 The formulation of the problem

This is a report on a joint work with Grzegorz Karch (Wrocław), Philippe Laurençot (Toulouse) and Tadeusz Nadzieja (Zielona Góra), cf. a part of published results in [5].

We investigate properties and large time asymptotics of radially symmetric solutions of a parabolic-elliptic model of chemotaxis (the simplified Keller–Segel system) either in a disc of $\mathbb{R}^2$ or in the whole plane $\mathbb{R}^2$, in the subcritical and critical cases.

Denoting by $u = u(x,t) \geq 0$ the density of microorganisms (e.g. amoebae), and by $\varphi = \varphi(x,t)$ the concentration of a chemoattractant secreted by themselves, the simplified Keller-Segel system we study herein reads

$$u_t = \nabla \cdot (\nabla u + u \nabla \varphi), \tag{1.1}$$

$$\varphi = E_2 * u, \tag{1.2}$$

with the space variable x ranging either in $B(0,R) \equiv \{x \in \mathbb{R}^2,\ |x| < R\}$, $R > 0$, or $\mathbb{R}^2$, and the time variable $t \in (0,\infty)$. Here $E_2(z) = \frac{1}{2\pi} \log |z|$

denotes the fundamental solution of the Laplacian in $\mathbb{R}^2$, so that (1.2) leads to the Poisson equation $\Delta\varphi = u$. The system is supplemented with either the no flux boundary condition

$$\frac{\partial u}{\partial \overline{\nu}} + u\frac{\partial \varphi}{\partial \overline{\nu}} = 0, \tag{1.3}$$

where $\overline{\nu}$ denotes the unit normal vector field to the boundary of $B(0,R)$, or a suitable decay condition $u(x,t) \to 0$ as $|x| \to \infty$ implying the integrability condition $\int_{\mathbb{R}^2} u(x,t)\,dx < \infty$. Moreover, an initial condition

$$u(x,0) = u_0(x) \geq 0 \tag{1.4}$$

is added. After a suitable reduction, see [5, (1.5)–(1.7)] (or [4]), the problem may be posed as a nonlinear nonuniformly parabolic equation for the cumulated mass variable $M(s,t) = \int_{B(0,\sqrt{s})} u(x,t)\,dx$

$$M_t = 4\,s\,M_{ss} + \frac{1}{\pi}\,M\,M_s \tag{1.5}$$

with a nondecreasing continuous initial condition

$$M(s,0) = M_0(s) \tag{1.6}$$

on either the interval $(0,1)$ or the half-line $(0,\infty)$, together with the boundary conditions:

$$M(0,t) = 0, \quad M(1,t) = \widehat{M}, \tag{1.7}$$

or

$$M(0,t) = 0, \quad M(\infty,t) = \widehat{M}, \tag{1.8}$$

respectively. We study the problem (1.5)–(1.6) and either (1.7) or (1.8) when the total mass parameter $\widehat{M}$ belongs to the interval $[0, 8\pi]$.

As it is well known, in the supercritical case $\widehat{M} > 8\pi$ there occurs a lost of the boundary condition at $s = 0$: $\lim_{s\to 0} M(s,t) > 0$ for $t \geq T$ with some

$T > 0$, cf. e.g. [2], [11]. This is interpreted as a blow up of solutions of the original chemotaxis system (at $x = 0$ for radially symmetric solutions)

$$\lim_{t\nearrow T} \|u(t)\|_{H^1} = \lim_{t\nearrow T} |u(t)|_{L^p} = \lim_{t\nearrow T} \int_\Omega u(x,t)\log u(x,t)\,dx = \infty$$

for each $p > 1$, cf. [4, 3, 6]. A fine description of blowing up solutions is fairly complicated, see [12], but for radially symmetric solutions the situation is much simpler. The degeneracy of the elliptic operator $4\,s\,M_{ss}$ at $s = 0$ does not allow the diffusion to compensate the growth induced by the convection term $\frac{1}{\pi}\,M\,M_s$ and $M(0,t) \neq 0$ for $t > T$ holds. On the one hand, we will show that, in the critical case $\widehat{M} = 8\pi$, the blow up in the disc does not take place in a finite time but occurs in infinite time, i.e. the whole mass concentrates at $s = 0$ as $t \to \infty$. We also obtain some temporal decay estimates on $|M(t) - 8\pi|_{L^1}$ for large times. On the other hand, if $\widehat{M} \in [0, 8\pi)$, we show the exponential convergence of $M(t)$ towards the unique stationary solution to (1.5)–(1.7) in the disc. The situation is completely different in the case of the whole plane.

2 (Sub)critical case in the disc

The problem (1.5)–(1.7) on $(0,1)$ is well posed whenever $\widehat{M} \in [0, 8\pi]$.

Theorem 2.1 *Consider $\widehat{M} \in [0, 8\pi]$ and a continuous nondecreasing function M_0 satisfying*

$$M_0(0) = 0 \quad \textit{and} \quad M_0(1) = \widehat{M}. \tag{2.1}$$

There exists a unique function $M \in \mathcal{C}([0,\infty); L^2(0,1)) \cap \mathcal{C}^{2,1}_{s,t}((0,1)\times(0,\infty))$ such that

$$0 \le M(s,t) \le \widehat{M},\quad M_s(s,t) \ge 0 \ \text{ for } \ (s,t) \in (0,1)\times(0,\infty)\,, \tag{2.2}$$

$$M^*(t) \equiv \inf_{s\in(0,1)} M(s,t) = 0 \ \text{ a.e. in } \ (0,\infty)\,, \tag{2.3}$$

and

$$
\begin{aligned}
M_t &= 4\,s\,M_{ss} + \frac{1}{\pi}\,M\,M_s\,, \quad (s,t)\in(0,1)\times(0,\infty)\,, &(2.4)\\
M(1,t) &= \widehat{M}\,, \quad t\in(0,\infty)\,, &(2.5)\\
M(s,0) &= M_0(s)\,, \quad s\in(0,1)\,. &(2.6)
\end{aligned}
$$

Moreover, if there is $\delta\in(0,1)$ such that $M_0(s)\le(8\pi s)/\delta$ for $s\in(0,1)$, then $M^*(t)=0$ for each $t\ge0$. Observe that if the derivative of M_0 is finite: $M_{0,s}(0)<\infty$, then the above condition on M_0 is satisfied with a suitable $\delta>0$.

The idea of the proof of Theorem 2.1 is to consider a uniformly parabolic regularized problem

$$
\begin{aligned}
M_{\varepsilon,t} &= 4\,(s+\varepsilon)\,M_{\varepsilon,ss} + \frac{1}{\pi}\,M_\varepsilon\,M_{\varepsilon,s}\,,\ (s,t)\in(0,1)\times(0,\infty), &(2.7)\\
M_\varepsilon(0,t) &= \widehat{M} - M_\varepsilon(1,t) = 0\,, \quad t\in(0,\infty)\,, &(2.8)\\
M_\varepsilon(s,0) &= M_{0\varepsilon}(s)\,, \quad s\in(0,1)\,. &(2.9)
\end{aligned}
$$

This problem has a unique solution

$$M_\varepsilon\in\mathcal{C}([0,1]\times[0,\infty))\cap\mathcal{C}^{2,1}_{s,t}((0,1)\times(0,\infty)),$$

and we infer from (2.1), (2.7)–(2.8), and the comparison principle that

$$0\le M_\varepsilon(s,t)\le\widehat{M} \quad\text{and}\quad M_{\varepsilon,s}(s,t)\ge0 \quad\text{for}\quad (s,t)\in[0,1]\times(0,\infty)\,. \tag{2.10}$$

Moreover, classical parabolic regularity results imply that

$$\|M_\varepsilon\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}_{s,t}([\delta,1]\times[\tau,T])}\le C(\alpha,\delta,\tau,T) \tag{2.11}$$

for each $T>0$, $\tau\in(0,T)$ and $\alpha\in(0,1)$, where $0<C(\alpha,\delta,\tau,T)<\infty$ is a constant depending on α, δ, τ and T but independent of $\varepsilon\in(0,1)$.

The key estimate which allows us to control the behavior of solutions for small $s>0$ is

$$0\le\int_0^T\int_0^1\frac{M_\varepsilon(s,t)\,\left(8\pi-M_\varepsilon(s,t)\right)}{s+\varepsilon}\,ds\,dt\le C_1(T) \tag{2.12}$$

for every $\varepsilon \in (0,1)$ and a constant $0 < C_1(T) < \infty$ independent of ε. This is obtained by multiplying (2.7) by $-\log(s+\varepsilon)$ and integrating over $(0,1)$. Here we use crucially the relation $0 \le M_\varepsilon \le \widehat{M} \le 8\pi$.

The behaviour of M_ε for small times can be inferred from the estimate

$$\int_0^T \int_0^1 (s+\varepsilon)\, |M_{\varepsilon,s}(s,t)|^2 \, ds\, dt + \int_0^T \|M_{\varepsilon,t}(t)\|_{H^{-1}}^2 \, dt \le C_2(T) \qquad (2.13)$$

for every $\varepsilon \in (0,1)$ and a constant $0 < C_2(T) < \infty$ independent of ε.

The above estimates permit us to pass to the limit $\varepsilon \longrightarrow 0$ with the approximate solutions M_ε and obtain a solution M. $\square$

In fact, for each continuous increasing initial data $M^*(t) = 0$ holds for every $t \in (0,\infty)$, not merely for a.e. t. Moreover there is a regularizing parabolic effect for (1.5) on the derivatives of solutions. Namely, the estimate $M_s(s,t) \le C/t$ holds for each $s > 0$ and $t > 0$. These properties are shown by a local comparison with self-similar solutions discussed in Section 3.

Remark. Using the methods above, similar existence and regularity results can be obtained for the "star problem" considered in [6, Theorem 1(i)] and describing a cloud of self-attracting particles in the gravitational field of a fixed point mass ("star"). Namely, the equation (1.5) with the boundary conditions $M(0,t) = m^* \in (0,4\pi)$, $M(1,t) = \widehat{M} \le 8\pi - m^*$, and suitable initial conditions, has global solutions satisfying properties similar to those in Theorem 2.1.

Since (1.5) is a convection-diffusion equation, we anticipate that it may enjoy some contraction property with respect to some L^1-norm. We actually show the following L^1-stability property for solutions.

Theorem 2.2 *If M, $\bar{M}$ are two solutions to (1.5)–(1.7) (as in Theorem 2.1) with initial data M_0 and $\bar{M}_0$ satisfying (2.1) with the same $\widehat{M}$, $\widehat{M} \in [0,8\pi]$, then $t \longmapsto |\varrho(M(t) - \bar{M}(t))|_{L^1}$ is a nonincreasing function of time for each* *nonnegative, nonincreasing and concave weight* $\varrho \in W^{2,\infty}(0,1)$. *Furthermore, if* $\widehat{M} \in [0, 8\pi)$,

$$|M(t) - \bar{M}(t)|_{L^1} \leq 2\,|M_0 - \bar{M}_0|_{L^1}\, e^{-(4-(\widehat{M}/2\pi))t}\,. \tag{2.14}$$

To prove Theorem 2.2 we consider the difference $N = M - \bar{M}$ which satisfies the equation

$$N_t = \frac{\partial}{\partial s}\left(4sN_s + \frac{1}{2\pi}N(M + \bar{M} - 8\pi)\right) \tag{2.15}$$

with $N(0,t) = N(1,t) = 0$ for a.e. $t \in (0,\infty)$. We prove the $L^1((0,1);\varrho(s)\,ds)$ contraction property of solutions. For $\delta \in (0,1)$ and $r \in \mathbb{R}$, we use a convex approximation of $r \longmapsto |r|$, e.g.,

$$\Phi_\delta(r) \equiv \begin{cases} \dfrac{1}{\delta}\left(|r| - \dfrac{\delta}{2}\right)_+^2 & \text{if} \quad |r| \in [0,\delta]\,, \\[2ex] |r| - \dfrac{3}{4}\delta & \text{if} \quad |r| \in (\delta,\infty)\,, \end{cases}$$

We multiply (2.15) by $\varrho\,\Phi'_\delta(N)$ and integrate over $(0,1)$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \varrho(s)\Phi_\delta(N)\,ds \\
= \;& 4s\varrho(s)N_s\Phi'_\delta(N)\,|_0^1 + \frac{1}{2\pi}\varrho(s)\Phi'_\delta(N)N(M+\bar{M}-8\pi)\,|_0^1 \\
& - \int_0^1 4s\varrho(s)\Phi''_\delta(N)N_s^2\,ds - \int_0^1 4s\varrho'(s)\Phi'_\delta(N)N_s\,ds \\
& - \frac{1}{2\pi}\int_0^1 \varrho(s)\Phi''_\delta(N)N_sN(M+\bar{M}-8\pi)\,ds \\
& - \frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi'_\delta(N)N(M+\bar{M}-8\pi)\,ds \\
\leq \;& -\frac{1}{2\pi}\int_0^1 \varrho(s)\Phi''_\delta(N)NN_s(M+\bar{M}-8\pi)\,ds \\
& - \frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi'_\delta(N)N(M+\bar{M}-16\pi)\,ds \\
& + 4\int_0^1 s\varrho''(s)\Phi_\delta(N)\,ds + 4\int_0^1 \varrho'(s)(\Phi_\delta(N) - N\Phi'_\delta(N))\,ds\,.
\end{aligned}$$

Observe that N_s belongs to $L^\infty((0,\infty);L^1(0,1))$, M, $\bar{M}$ and N are bounded, and $r \longmapsto r\ \Phi_\delta''(r)$ is bounded and converges a.e. towards zero as $\delta \to 0$. Thus, the Lebesgue dominated convergence theorem ensures that the first term of the right-hand side of the above inequality converges to zero as $\delta \to 0$. On the other hand, both $r \longmapsto \Phi_\delta(r)$ and $r \longmapsto r\,\Phi_\delta'(r)$ converge uniformly towards $r \longmapsto |r|$ on $\mathbb{R}$. Thanks to the boundedness of M, $\bar{M}$ and N, we can pass to the limit as $\delta \to 0$ in the other terms of the above inequality, and end up with

$$\begin{aligned}\frac{d}{dt}\int_0^1 \varrho(s)|N|\,ds \quad &\le \quad -\frac{1}{2\pi}\int_0^1 \varrho'(s)|N|(M+\bar{M}-16\pi)\,ds \\ &\quad + 4\int_0^1 s\varrho''(s)|N|\,ds\,. \end{aligned} \tag{2.16}$$

Since $M+\bar{M} \le 2\widehat{M} \le 16\pi$ and ϱ' and ϱ'' are both nonpositive, the right-hand side of (2.16) is nonpositive, from which the first assertion of Theorem 2.2 follows.

We now turn to the decay rate (2.14) and assume that $\widehat{M} \in [0,8\pi)$. We take $\varrho(s) = 2-s$ in (2.16). Since $M+\bar{M} \le 2\widehat{M} < 16\pi$, we infer from (2.16) that

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \le \frac{1}{2\pi}\int_0^1 |N|(2\widehat{M}-16\pi)\,ds \le \frac{\widehat{M}-8\pi}{2\pi}\int_0^1 (2-s)|N|\,ds\,,$$

whence

$$\int_0^1 (2-s)|N(t)|\,ds \le \int_0^1 (2-s)|N(0)|\,ds\ \ e^{-(4-(\widehat{M}/2\pi))t}\,,$$

from which (2.14) readily follows.

An immediate consequence of (2.14) with $\bar{M} = M_b$ — the (unique) steady state such that $M_b(1) = \widehat{M}$, i.e.

$$M_b(s) = 8\pi\frac{s}{s+b}\,, \quad s \in (0,1)\,, \quad \text{with} \quad b = \frac{8\pi}{\widehat{M}} - 1 > 0\,, \tag{2.17}$$

is the exponential decay

$$|M(t) - M_b|_{L^1} \leq 2\ |M_0 - M_b|_{L^1}\ e^{-(4-(\widehat{M}/2\pi))t}\,.$$

The exponential decay rate does not hold true for the critical case $\widehat{M} = 8\pi$ but the following weaker assertion is available

$$|M(t) - 8\pi|_{L^1} \leq \frac{8\pi}{t}\,. \tag{2.18}$$

For the proof, we put $N(s,t) = M - 8\pi$, $\varrho(s) = 2 - s$. We notice that N solves

$$N_t = \frac{\partial}{\partial s}\left(4sN_s + \frac{1}{2\pi}NM\right) \tag{2.19}$$

with $N(0,t) = -8\pi$ and $N(1,t) = 0$ for a.e. $t \in (0,\infty)$. Keeping the notations from the proof of Theorem 2.2, we multiply (2.19) by $\varrho\ \Phi'_\delta(N)$ and integrate over $(0,1)$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \varrho(s)\Phi_\delta(N)\,ds \\
\leq\ & -\frac{1}{2\pi}\int_0^1 \varrho(s)\Phi''_\delta(N)NN_sM\,ds - \frac{1}{2\pi}\int_0^1 \varrho'(s)\Phi'_\delta(N)NM\,ds \\
& +4\int_0^1 s\varrho''(s)\Phi_\delta(N)\,ds + 4\int_0^1 \varrho'(s)\Phi_\delta(N)\,ds\,,
\end{aligned}$$

since Φ'_δ vanishes on a neighbourhood of 0 and $M^*(t) = 0$, so the boundary terms vanish. We then proceed as in the proof of (2.16) to pass to the limit as $\delta \to 0$ and end up with

$$\frac{d}{dt}\int_0^1 \varrho(s)|N|\,ds \ \leq\ \frac{1}{2\pi}\int_0^1 \varrho'(s)(8\pi - M)|N|\,ds\,,$$

i.e.

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \ \leq\ -\frac{1}{2\pi}\int_0^1 |N|^2\,ds\,.$$

We infer from the Cauchy–Schwarz inequality that

$$\frac{d}{dt}\int_0^1 (2-s)|N|\,ds \leq -\frac{1}{2\pi}\left(\int_0^1 |N|\,ds\right)^2 \leq -\frac{1}{8\pi}\left(\int_0^1 (2-s)|N|\,ds\right)^2,$$

whence

$$|M(t) - 8\pi|_{L^1} \le \int_0^1 (2-s)|N(t)|\, ds \le \frac{8\pi}{t + 4\pi|8\pi - M_0|_{L^1}^{-1}}\,.$$

$\square$

3 The problem in the whole plane

The equation (1.5) for $s \in (0, \infty)$ is invariant under the space-time scaling

$$s \longmapsto Rs, \quad t \longmapsto Rt, \quad R > 0. \tag{3.1}$$

This property has important consequences for the analysis of the problem (1.5)–(1.6) on $(0, \infty) \times (0, \infty)$.

The global in time existence of solutions of that problem can be proved using the ideas of regularizations of the nonlinear term in [11]. An alternative way is to use our previous construction in Theorem 2.1 and the scaling property (3.1) of (1.5). More precisely, if $0 \le M_0 \nearrow \widehat{M} \le 8\pi$ is a subcritical initial data, then we consider its restriction to the interval $(0, R)$. Rescaling M_0 to M_{0R} defined on $(0,1)$, $M_{0R}(s/R) = M_0(s) \le \widehat{M}$ for $s \in (0, R)$, we construct the solution M_R of (1.5)–(1.7) with the initial condition $M_R(s, 0) = M_{0R}(s)$. For each $s \in (0,1)$ the functions $M_{0R}(s) \le \widehat{M}$ increase with $R \nearrow \infty$ so that, by the comparison principle, $M_R(s,t) \le \widehat{M}$ are also increasing with respect to R. The functions $\widetilde{M_R}(s,t) = M_R(s/R, t/R)$ defined for $(s,t) \in (0,R) \times (0, \infty)$ solve the equation (1.5) with $\widetilde{M_R}(s, 0) = M_0(s)$, $s \in (0, R)$. To obtain a global in time solution with analogous regularity properties as in Theorem 2.1, we perform the passage with $\widetilde{M_R}$ to the limit $R \to \infty$.

Since (1.5) is invariant under the scaling (3.1) it is natural to consider self-similar solutions of (1.5), i.e. those satisfying $M(Rs, Rt) \equiv M(s,t)$ for each $R > 0$. They have the form $M(s,t) = m(s/t)$ for a function m. The existence of self-similar solutions in the range $\widehat{M} \in [0, 8\pi)$ has been established in, e.g., [2] and [10] (not necessarily radially symmetric case of the chemotaxis system).

Let us briefly recall the reasoning from [2, Prop. 3, i)]. For $M(s,t) = 2\pi\zeta(s/t)$ (1.5) reads

$$\zeta'' + \frac{1}{4}\zeta' + \frac{1}{2y}\zeta\zeta' = 0 \text{ with } y = \frac{s}{t}. \tag{3.2}$$

The change of variables $\tau = \frac{1}{2}\log y$, $v(\tau) = 2y\frac{d\zeta}{dy}(y)$, $w(\tau) = \zeta(y)$ transforms (3.2) into the nonautonomous problem for (u, v) in the plane

$$\begin{aligned} v' &= (2-w)v - \frac{e^{2\tau}}{2}v, \quad w' = v, \quad ' = \frac{d}{d\tau}, \\ v(-\infty) &= 0, \quad w(-\infty) = 0. \end{aligned} \tag{3.3}$$

Evidently, $\lim_{\tau\to\infty} w(\tau) < 4$ because the function $(w-2)^2 + 2v$ is strictly decreasing along the phase trajectories of the above system.

We consider also an autonomous system

$$\underline{v}' = (2-\underline{w})\underline{v} - \varepsilon\underline{v}, \quad \underline{w}' = \underline{v},$$

where $\varepsilon > 0$, $\underline{v} = \underline{v}_\varepsilon$, $\underline{w} = \underline{w}_\varepsilon$, with the same condition at $\tau = -\infty$. A comparison of these vector fields gives the relation $\underline{w}(\tau) \le w(\tau)$ for all $\tau \le \tau_\varepsilon$ with $e^{2\tau_\varepsilon} = 2\varepsilon$. Since $\underline{w}(\tau) = 2(2-\varepsilon)Ae^{(2-\varepsilon)\tau}\left(1 + Ae^{(2-\varepsilon)\tau}\right)^{-1}$ with an arbitrary $A > 0$ is a solution of the auxiliary system, so $\underline{w}(\tau_\varepsilon) = 2(2-\varepsilon)A(2\varepsilon)^{1-\varepsilon/2}\left(1 + A(2\varepsilon)^{1-\varepsilon/2}\right)^{-1}$ and $\sup Z = \lim_{y\to\infty}\zeta(y) = \sup w(\tau) \ge \limsup_{\varepsilon\to 0,\, \tau\le\tau_\varepsilon,\, A>0} \underline{w}(\tau) = 4$. $\square$

We prove that the asymptotics of general solutions of (1.5)–(1.6), (1.8) for $0 < \widehat{M} < 8\pi$ is described by that of self-similar solutions, i.e.

$$0 \le \frac{m(s/t) - M(s,t)}{m(s/t)} \to 0 \text{ as } t \to \infty.$$

Here m denotes the self-similar solution with $m(\infty) = \widehat{M}$. The proof involves analysis of the family of suitable scalings of the solution M, and the uniqueness property of self-similar solutions with a given mass $\widehat{M} \in [0, 8\pi)$. A related result for the original chemotaxis system has been recently announced in [8].

Looking at the problem on a finite interval $(0,1)$, one might suspect that $M(s,t) \to 8\pi$ as $t \to \infty$ but for $s \in (0,\infty)$ the picture is much more complicated. First of all, nontrivial solutions of the steady state problem (1.5)–(1.6), (1.8) on $(0,\infty)$ exist for $\widehat{M} = 8\pi$ (only!) and are parametrized by $b > 0$:

$$M_b(s) = 8\pi \frac{s}{s+b}, \quad b > 0. \tag{3.4}$$

Second, if M_0 satisfies the condition $\int_0^\infty (8\pi - M(s,t))\, ds < \infty$, the solution $M(.,t)$ converge pointwise to 8π as $t \to \infty$, but does not converge to 8π in the L^1 sense. Indeed, for those solutions (they correspond to solutions u of the original chemotaxis system (1.1)–(1.2) possessing the second moment, i.e. $\int_{\mathbb{R}^2} |x|^2 u(x,t)\, dx < \infty$) we have

$$\frac{d}{dt}\int_0^\infty (8\pi - M(s,t))\, ds = 32\pi - \frac{(8\pi)^2}{2\pi} = 0.$$

since $4sM_s(s,t) \to 0$ as $s \to 0$ and as $s \to \infty$. To prove the above, we begin with M_0 such that $(8\pi - M_0)$ has compact support in $[0,\infty)$. From the construction of M as the limit of $\widetilde{M_R}$'s, it is easy to conclude using comparison principle that $M(s,t) \to 8\pi$ for each $s > 0$ when $t \to \infty$. The remaining part follows from the L^1 contraction property $|M(t) - \bar{M}(t)|_{L^1} \le |M_0 - \bar{M}_0|_{L^1}$ proved as in Theorem 2.2 with $\varrho(s) \equiv 1$. Indeed, M_0 such that $(8\pi - M_0) \in L^1(0,\infty)$ can be approximated by initial data with $(8\pi - M_0)$ of compact support. Combining monotonicity properties of M's and the L^1 contraction property, the desired pointwise convergence follows.

To prove the stability of steady states (3.4), we will interprete (1.5) as a nonlinear Fokker–Planck type equation considered in [1], and we will employ a family of Lyapunov functionals for the dynamical system associated with (1.5)–(1.6), (1.8) in the $L^1(0,\infty)$-metric.

Theorem 3.1 *The function* $\mathcal{W}_b(M) = \int_0^\infty w_b(M(s,t))\,ds$, *where the entropy density* w_b *is defined as*

$$w_b(M) = M\log\frac{M}{M_b} + (8\pi - M)\log\frac{8\pi - M}{8\pi - M_b}, \tag{3.5}$$

is finite for each M *such that* $(M - M_b) \in L^1(0,\infty)$, $M_{b_1} \le M \le M_{b_2}$ *for some* $b_1 > b > b_2 > 0$. *Moreover, this is nonincreasing along the trajectories* $M(t) = M(.,t)$ *of the dynamical system (1.5)–(1.6), (1.8)*

$$\frac{d\mathcal{W}_b}{dt} \le -\frac{1}{2\pi}\int_0^\infty s\,M(8\pi - M)\left|\frac{\partial}{\partial s}\left(\log\frac{M}{8\pi - M}\frac{8\pi - M_b}{M_b}\right)\right|^2 ds \le 0. \tag{3.6}$$

This implies that if M_0 *is such that* $W_b(M_0) < \infty$ *and* $(M_0 - M_b) \in L^1(0,\infty)$ *for some* $b > 0$, *then* $\lim_{t\to\infty}\mathcal{W}_b(t) = 0$, *and therefore (by a Csiszár–Kullback type lemma)*

$$\lim_{t\to\infty} |M(t) - M_b|_{L^1} = 0.$$

Local attracting property of the stationary solutions M_b is a rather weak property. In particular, this does not give any information on the asymptotic behavior of solutions starting from data like, e.g., $M_0(s) = 8\pi\frac{s}{s+2+\cos s}$ which satisfy the relation $M_3 \le M_0 \le M_1$, but $M_0 - M_b \notin L^1(0,\infty)$ for any $b > 0$. All this shows that the long time behavior of solutions in the critical case may be extremely complicated and even chaotic. □

Remark. The problem of the chemotaxis (1.1)–(1.4) in the whole plane in the subcritical case $\widehat{M} < 8\pi$, without radial symmetry assumptions, has been recently studied in [9]. In particular, the authors proved the global in time existence of solutions using logarithmic Sobolev inequalities.

Using the approach via radially symmetric decreasing rearrangements in [7] we might use the results here to give an alternative construction of global in time solutions for $\widehat{M} \le 8\pi$, and to give a flavor of the diversity of locally attracting solutions for the problem without radial symmetry. Indeed, results from [7] imply that, roughly speaking, the existence of solutions of (1.1)–(1.4) is controlled by the existence of solutions to the radially symmetric problem given by (1.5)–(1.6), (1.8) with the initial condition M_0 obtained from the radially symmetric decreasing rearrangement of u_0.

4 Supercritical case in $\mathbb{R}^2$

Let us recall some results from the preprint [11] (Theorems 2.7, 3.5, 4.4) related to the supercritical case of equation (1.5) on $(0,\infty)$, i.e. for $\widehat{M} > 8\pi$.

First, the classical solution of (1.5) (that possesses the second moment — which was not explicitly stated in [11], cf. [3], [4]) blows up in a finite time: there is $0 < T < \infty$ such that $\lim_{t\nearrow T} M(s,t) \geq 8\pi$ for each $s > 0$. This means that the boundary condition at $s = 0$ is lost, $M^*(t)$ jumps to 8π instantaneously at $t = T$.

Moreover, there exists a continuation of M, $M \in \mathcal{C}^\infty(0,\infty) \times (0,\infty))$, past the blow up time T, satisfying (1.5), (1.6) for all $t > 0$, and the quantity $M^*(t)$ strictly increases for $t > T$. Such a global in time smooth solution — a continuation of the classical solution for $t < T$ — is unique in $\mathcal{C}^\infty((0,\infty)\times(0,\infty))$, and satisfies $\lim_{t\to\infty} M(s,t) = \widehat{M}$ for each $s \geq 0$. Moreover, $\lim_{t\to\infty} M^*(t) = \widehat{M}$: the whole mass concentrates at the origin in the infinite time, unlike the critical $\widehat{M} = 8\pi$ (nontrivial steady states exist) and subcritical cases $M^* < 8\pi$ (mass spreads to infinity).

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