数理解析研究所講究録 1464 巻 2006 年 164-172

Large time behavior of unbounded global solutions to some nonlinear diffusion equations

中村 健一 (電気通信大学)
KEN-ICHI NAKAMURA
University of Electro-Communications
Chofu, Tokyo 182-8585, Japan
nakamura@im.uec.ac.jp

荻原 俊子 (城西大学)
TOSHIKO OGIWARA
Josai University
Sakado, Saitama 350-0295, Japan
toshiko@math.josai.ac.jp

1 Introduction

We consider a one-dimensional quasilinear parabolic equation of the form

$$\begin{cases} u_t = a(x,u,u_x)u_{xx} + f(x,u,u_x), & x \in I := (0,1),\ t>0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0. \end{cases} \tag{1.1}$$

In this paper only real-valued classical solutions are considered. Under some assumptions on a and f, the solution of problem (1.1) with initial data $u(x,0) = u_0(x)$ in I exists globally in time, provided that u_0 belongs to a suitable function space X.

Once the global existence of a solution is known, then its long-time behavior becomes an important subject for mathematical studies of (1.1). There are two possibilities which can occur:

(I) u is bounded in X;

(II) u is unbounded in X.

Concerning the asymptotic behavior of solutions as $t \to +\infty$ in the former case, extensive studies have been made in earlier works including [12] and [6]. Applying the result of Zelenyak ([12]) to (1.1), we see that there exists a nontrivial Lyapunov functional of the form

$$E[u(\cdot,t)] := \int_0^1 \Phi(x, u(x,t), u_x(x,t))dx$$

satisfying $\frac{d}{dt}E[u(\cdot,t)] \leq 0$ and that any solution $u(x,t)$ with bounded $C^{2+\alpha}$-norm converges to an equilibrium solution of (1.1) as $t \to +\infty$ in $C^2(\bar{I})$. See also [7] for a different method of constructing a Lyapunov functional for one-dimensional quasilinear parabolic equations.

The aim of this paper is to investigate the large time behavior of solutions of (1.1) in the latter case (II) under the supposition that a and f are periodic in u. For example, if the nonlinearity f is strictly positive, then the comparison theorem immediately implies that every solution $u(x,t)$ tends to $+\infty$ as $t \to +\infty$ and that no equilibrium solution exists. In such a case, what is the typical behavior of unbounded global solutions of (1.1)? More precisely, is there any special solution $U(x,t)$ of (1.1) such that $u(x,t)$ tends to $U(x,t)$ in some function space as $t \to +\infty$? We will see below that the periodicity of a and f in u helps us to find such a special solution with time-periodic growth speed and profile.

2 Main Results

Our hypotheses are as follows:

(A1) $a(x,u,p) > 0$ and $f(x,u,p)$ are smooth functions and are periodic in u with least period $L > 0$;

(A2) there exists a positive constant M such that if $u(x,t)$ is a solution of (1.1) on $[0,T]$ with initial data u_0 satisfying $\|u_{0x}\|_\infty \leq M$ then the gradient estimate $\|u_x(\cdot,t)\|_\infty \leq M$ holds for all $t \in [0,T]$.

By the theory of quasilinear parabolic equations ([3, 4]), (1.1) has a local solution $u(x,t)$ on $[0,T]$ for every sufficiently smooth initial data u_0 satisfying the compatibility condition $u_0'(0) = u_0'(1) = 0$. Condition (A1) and the comparison theorem imply the following growth bound:

$$\|u(\cdot,t)\|_\infty \leq Kt + \|u_0\|_\infty, \tag{2.1}$$

where $K := \sup\{f(x,u,0) \mid x \in \bar{I},\ u \in [0,L]\} < +\infty$.

Since u_x solves a linear parabolic equation with divergence form, applying the interior and boundary Hölder estimates for linear parabolic equations ([3, 4]), we obtain the Hölder gradient estimates

$$\|u_x\|_{C^{\alpha,\alpha/2}(\bar{I}\times[0,T])} \leq C.$$

Here, condition (A2) guarantees that the constants $\alpha \in (0,1)$ and $C > 0$ do not depend on T. Consequently, under the conditions (A1) and (A2), the solution $u(x,t)$ of (1.1) with initial data u_0 exists globally in $t \geq 0$ if $u_0 \in X$, where

$$X := \{u_0 \in C^{2+\alpha}(\bar{I}) \mid u_0'(0) = u_0'(1) = 0, \ \|u_0'\|_\infty \leq M\}.$$

Furthermore we assume the existence of an unbounded solution:

(A3) there exists a $\overline{u}_0 \in X$ such that the solution $\overline{u}(x,t)$ of (1.1) with initial data $\overline{u}_0$ satisfies

$$\limsup_{t\to+\infty} \max_{x\in\bar{I}} \overline{u}(x,t) = +\infty.$$

Condition (A3) and the periodicity of a and f ensure that for any $u_0 \in X$ the solution $u(x,t)$ diverges to $+\infty$ everywhere as $t \to +\infty$ and that no equilibrium solution exists.

A sufficient condition for (A2) and (A3) is that $f = f(u_x)$ with $f(0) > 0$. Other examples will be given later.

The following is the main theorem of the present paper:

Theorem 1

(i) *There exists an entire solution* $U(x,t)$ *of* (1.1) *such that*

$$U(x, t+T_0) = U(x,t) + L, \quad x \in \bar{I}, \ t \in \mathbb{R} \tag{2.2}$$

for some positive constant T_0.

(ii) $U_t(x,t) > 0$ *for all* $x \in \bar{I}$ *and* $t > 0$.

(iii) *The solution* U *is asymptotically stable up to time shift, that is, for any* $u_0 \in X$ *there exists a constant* τ_0 *such that the solution* $u(x,t)$ *of* (1.1) *with initial data* u_0 *satisfies*

$$\lim_{t\to+\infty} \|u(\cdot,t) - U(\cdot, t+\tau_0)\|_{C^2(\bar{I})} = 0. \tag{2.3}$$

Remark 2 By (2.2), we see that the solution $U(x,t)$ is written in the form

$$U(x,t) = \phi(x,t) + \frac{L}{T_0}t, \tag{2.4}$$

where ϕ is T_0-periodic in t. We call the quantity L/T_0 the *average growth speed* of U. Namah and Roquejoffre studied the existence and stability of solutions of similar form to (2.4) (they call such solutions *periodic fronts*) for a semilinear parabolic equations in $\mathbb{R}^n$ ([8]). They showed the existence of such solutions by using the Leray-Schauder degree theory.

Remark 3 In an earlier paper [11] the authors have studied the long-time behavior of solutions for the semilinear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (A(x)\nabla u) + f(x,u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν is the outer normal unit vector of $\partial\Omega$, $A(x)$ is a smooth positive function on $\overline{\Omega}$ and $f(x,u)$ is a smooth function which is L-periodic in u. The existence and the monotonicity of a solution U satisfying (2.2) are also valid for the semilinear case. As for the asymptotic stability of U, the following was proved instead of (2.3): there exists a constant $\mu > 0$ such that for any $u_0 \in C(\overline{\Omega})$ the solution $u(x,t)$ of (2.5) with initial data u_0 satisfies

$$\|u(\cdot,t) - U(\cdot,t+\tau_0)\|_\infty \le M_0 e^{-\mu t}, \quad t \ge 0, \tag{2.6}$$

where $\tau_0 \in \mathbb{R}$ and $M_0 > 0$ are constants depending on u_0. See also [9, 10] for related results.

Remark 4 In [1], Giga, Ishimura and Kohsaka studied a weakly anisotropic curvature flow in an annulus $\{x \in \mathbb{R}^2 \mid \rho < |x| < R\}$ and considered spiral-shaped solutions of the form $\Gamma(t) = \{(r\cos\theta(r,t), r\sin\theta(r,t)) \mid \rho \le r \le R\}$. Then the function $\theta(r,t)$ satisfies the following equation:

$$\begin{cases} \theta_t = M(\boldsymbol{n}) \left(\dfrac{a(\boldsymbol{n})(r\theta_{rr} + r^2\theta_r^3 + 2\theta_r)}{r(1+r^2\theta_r^2)} + \dfrac{V_0(1+r^2\theta_r^2)^{1/2}}{r} \right), \\ \theta_r(\rho,t) = \theta_r(R,t) = 0. \end{cases} \tag{2.7}$$

Here, $M(\boldsymbol{n})$ is called the mobility which depends on the unit normal vector $\boldsymbol{n}$ represented as

$$\boldsymbol{n} = \boldsymbol{n}(r, \theta, \theta_r) = \frac{1}{(1 + r^2\theta_r^2)^{1/2}} \begin{pmatrix} -\sin\theta - r\theta_r \cos\theta \\ \cos\theta - r\theta_r \sin\theta \end{pmatrix},$$

$a(\boldsymbol{n})$ is a positive coefficient which comes from the anisotropic curvature of $\Gamma(t)$ and V_0 is a positive constant corresponding to a driving force term.

They have shown that conditions (A2) and (A3) are satisfied for (2.7). In view of these conditions they proved, among other things, the same statements as those of Theorem 1 (i) and (ii) (which imply the existence of a spiral solution for (2.7) and the uniqueness up to translation of time). As for the stability, they only proved the stability of the spiral solution in the sense of Lyapunov. The method used in [1] is basically the same as that of [9]. Applying Theorem 1 (iii) to (2.7), one can obtain the asymptotic stability of the spiral solution.

Remark 5 Conditions (A2) and (A3) are also fulfilled for a quasilinear parabolic equation related to a curvature-dependent motion of curves in a 2-dimensional cylinder with saw-toothed boundary ([5]).

3 Proof of Theorem 1

Proof of Theorem 1 (i). For each $t \geq 0$, we define

$$\tau(t) := \inf\{s \geq 0 \mid \overline{u}(x,t) + L \leq \overline{u}(x, t+s) \text{ for all } x \in \bar{I}\}.$$

Condition (A3) and the comparison theorem imply that the function $\tau(t)$ is well-defined and is monotone decreasing in $t \geq 0$. Thus the limit $T_0 = \lim_{t\to+\infty} \tau(t) \geq 0$ exists.

In view of (A3), there exists a sequence $0 < t_1 < t_2 < \cdots \to +\infty$ such that

$$\max_{x\in\bar{I}} \overline{u}(x, t_n) = \max_{x\in\bar{I}} \overline{u}_0(x) + nL.$$

For $n \in \mathbb{N}$ we define $u_n(x,t) := \overline{u}(x, t+t_n) - nL$. Then u_n solves (1.1) and satisfies $\|u_{nx}(\cdot,t)\|_\infty \leq M$ for $t \geq -t_n$.

We fix $T > 0$. By the growth bound (2.1), there exists a positive constant K_1 independent of n such that $\|u_n(\cdot,t)\|_\infty \leq KT + K_1$ for any $n \in \mathbb{N}$ with

$t_n > T$ and $t \in [-T, T]$. Therefore the Hölder gradient estimates and the Schauder estimates imply

$$\|u_n\|_{C^{2+\alpha,1+\alpha/2}(\bar{I}\times[-T,T])} \leq C_T$$

for some constant $C_T > 0$. Hence one can find a subsequence $\{t_{n_j}\}_{j\in\mathbb{N}}$ and a function $U(x,t)$ defined in $\bar{I} \times \mathbb{R}$ such that u_{n_j} converges to U as $j \to \infty$ in $C^{2,1}(\bar{I} \times [-T, T])$ for any $T > 0$. This means that $U(x,t)$ is an entire solution of (1.1).

By the definition of $\tau(t)$, $\overline{u}(x, t_n) + L \leq \overline{u}(x, t_n + \tau(t_n))$ for all $n \in \mathbb{N}$ and $x \in \bar{I}$. Subtracting nL from the above inequality and letting $n \to \infty$, we obtain $U(x,0) + L \leq U(x, T_0)$ for $x \in \bar{I}$. This implies $T_0 > 0$.

Suppose that $U(x, 0) + L \not\equiv U(x, T_0)$. Then by the comparison theorem, for any fixed $\delta > 0$, we have $U(x,\delta) + L < U(x, T_0 + \delta)$ for all $x \in \bar{I}$. Therefore, for sufficiently large $j \in \mathbb{N}$,

$$\overline{u}(x, t_{n_j} + \delta) + L < \overline{u}(x, t_{n_j} + \delta + T_0), \quad x \in \bar{I}.$$

This implies that $\tau(t_{n_j} + \delta) < T_0$, which contradicts the definition of T_0. Thus we obtain $U(x, 0) + L \equiv U(x, T_0)$, and hence (2.2) holds. □

Proof of Theorem 1 (ii). Fix $t \in \mathbb{R}$ arbitrarily and set

$$t_0 := \inf\{s > 0 \mid U(x,t) \leq U(x, t+s) \text{ for all } x \in \bar{I}\}.$$

Clearly $0 \leq t_0 < T_0$. Suppose $t_0 > 0$. Then, since $U(x,t) \leq U(x, t + t_0)$ for $x \in \bar{I}$ and since $U(x,t) \not\equiv U(x, t+t_0)$, it follows from the comparison theorem that

$$U(x, t + T_0) < U(x, t + t_0 + T_0), \quad x \in \bar{I}.$$

By (2.2), this implies $U(x,t) < U(x, t+t_0)$ for $x \in \bar{I}$, which contradicts the definition of t_0. Therefore $t_0 = 0$ and hence $U_t(x,t) \geq 0$ for $x \in \bar{I}$ and $t \in \mathbb{R}$. Moreover, by the strong maximum principle, we obtain $U_t(x,t) > 0$ for all $x \in \bar{I}$ and $t \in \mathbb{R}$. □

Proof of Theorem 1 (iii). For $n \in \mathbb{N}$ we define $w_n(x,t) := u(x, t+nT_0) - nL$. Arguing as in the proof of (i), we see that there exists a constant τ_0 such that

$$\lim_{n\to\infty} \|w_n - U(\cdot, \cdot + \tau_0)\|_{C^{2,1}(\bar{I}\times[-T,T])} = 0. \tag{3.1}$$

For $t \geq 0$, we define $n(t) \in \mathbb{N} \cup \{0\}$ and $r(t) \in [0, T_0)$ by $t = n(t)T_0 + r(t)$. Then, since

$$\begin{aligned} u(x,t) - U(x,t+\tau_0) &= u(x, n(t)T_0 + r(t)) - U(x, r(t) + \tau_0) - n(t)L \\ &= u_{n(t)}(x, r(t)) - U(x, r(t) + \tau_0), \end{aligned}$$

we have

$$\|u(\cdot,t) - U(\cdot, t+\tau_0)\|_{C^2(\bar{I})} \leq \|u_{n(t)} - U(\cdot, \cdot + \tau_0)\|_{C^{2,1}(\bar{I}\times[0,T_0])} \to 0$$

as $t \to +\infty$. □

Remark 6 Since the method used in the proof of Theorem 1 is based on the comparison theorem for classical solutions, the statements of the theorem remain true even for a class of quasilinear parabolic equations in higher space dimension including

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(x,u,\nabla u)\Delta u + b(x,u,\nabla u), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.2}$$

where $a(x,u,p)$ and $b(x,u,p)$ are L-periodic in u with some additional conditions for the global existence of classical solutions.

4 Variational Principles for Growth Speed

In this section we derive a characterization for the average growth speed of U in Theorem 1. To our problem we apply the idea of [2] for a min-max characterization for the traveling wave velocity in inhomogeneous media. Roughly speaking, the average growth speed is characterized as the growth speed of the fastest subsolution and as the growth speed of the slowest supersolution.

Theorem 7 *Let U be the solution of* (1.1) *in Theorem* 1 *and let* $c = L/T_0$ *be the average growth speed of* U. *We define the set*

$$K = \left\{ v \in C^{2,1}(I \times \mathbb{R}) \;\middle|\; \begin{array}{l} v_t(x,t) > 0, \ v(x,t+1) = v(x,t) + 1, \\ v_x(0,t) = v_x(1,t) = 0 \end{array} \right\}$$

and the function

$$\Psi[v](x,t) = \frac{a(x, v(x,t), v_x(x,t))v_{xx}(x,t) + f(x, v(x,t), v_x(x,t))}{v_t(x,t)}$$

for $v \in K$. *Then we have*

$$\sup_{v\in K} \inf_{(x,t)\in\Omega\times\mathbb{R}} \Psi[v](x,t) = c = \inf_{v\in K} \sup_{(x,t)\in\Omega\times\mathbb{R}} \Psi[v](x,t).$$

The proof is almost identical to that of [2, Theorem 2] and is therefore omitted.

References

[1] Y. Giga, N. Ishimura and Y. Kohsaka, *Spiral solutions for a weakly anisotropic curvature flow equation*, Adv. Math. Sci. Appl., **12** (2002), 393–408.

[2] S. Heinze, G. Papanicolaou and A. Stevens, *Variational principles for propagation speeds in inhomogeneous media*, SIAM Appl. Math., **62** (2001), 129–148.

[3] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, Rhode Island, 1968.

[4] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[5] B. Lou, H. Matano and K. -I. Nakamura, *Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit*, in preparation.

[6] H. Matano: *Convergence of soltustions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ., **18** (1978), 221–227.

[7] H. Matano, *Asymptotic behavior of solutions of semilinear heat equations on* S^1, "Nonlinear Diffusion Equations and Their Equilibrium States", II (W.-M. Ni, L. A. Peletier and J. Serrin eds.), Math. Sci. Res. Inst. Publ., **13** (1988), 139–162.

[8] G. Namah and J.-M. Roquejoffre, *Convergence to periodic fronts in a class of semilinear parabolic equations*, Nonlinear differ. equ. appl., **4** (1997), 521–536.

[9] T. Ogiwara and K.-I. Nakamura, *Spiral traveling wave solutions of some parabolic equations on annuli*, Josai Mathematical Monographs, **2** (2000), 15–34.

[10] T. Ogiwara and K.-I. Nakamura, *Spiral traveling wave solutions of nonlinear diffusion equation related to a model of spiral crystal growth*, Publ. Res. Inst. Math. Sci., **39** (2003), pp.767–783.

[11] T. Ogiwara and K.-I. Nakamura, *Asymptotic behavior of solutions for nonlinear diffusion equations*, Proceedings of the second international conference on nonlinear analysis and convex analysis, (2003), pp.363–372.

[12] T. I. Zelenyak, *Stabilisation of solution of boundary value problems for a second-order equation with one space variable*, Differ. Equations, **4** (1968), 17-22.