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On the Palais-Smale condition and the L^∞-global bounds for global solutions of some semilinear parabolic problems with critical Sobolev exponent

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1 Introduction

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\lambda \in \mathbb{R}$, $q \in (2, 2^*)$ ($2^* := 2N/(N-2)$ denotes the critical exponent of the Sobolev embedding $H_0^1 \hookrightarrow L^r$) and $u_0 \in L^\infty$. In this paper, we are concerned with the existence of an L^∞-global bounds of global-in-time solutions of the following parabolic problems:

$$(\mathrm{P}) \quad \begin{cases} \partial u/\partial t = \Delta u + \lambda u + u|u|^{q-2} & \text{in} \quad \Omega \times (0, T_m), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T_m), \\ u = u_0 & \text{in} \quad \Omega \times \{0\} \end{cases}$$

where T_m denotes the maximal existence time of the classical solution of (P).

It is well-known that (P) appears as a model which describes various kinds of nonlinear phenomena. Therefore it is important to analyze the asymptotic behavior of solutions of (P). As for global solutions, to establish the existence of an L^∞-global bounds is a first step. Concerning this problem, there still seem to exist some mysteries in the critical case while the subcritical problem is well-understood so far.

We here briefly review some known results. For the sake of simplicity, we assume that $\lambda = 0$ in the rest of this section. Here we recall that a global-in-time solution u of (P) is said to have an L^∞-global bounds if there exists $C > 0$ such that $\sup_{t \geq 0} \|u(t)\|_\infty < C$.

Proposition 1.1 (Subcritical case) [2]
Suppose that $q \in (2, 2^*)$. Then any global-in-time solution u of (P) *has an L^∞-global bounds.*

On the other hand, the existence of a priori bounds as in the subcritical case does not hold in the critical case:

Proposition 1.2 (Critical case) [3]
Suppose that $q = 2^*$. Let Ω be a ball. Then there exists a radially symmetric function $u_0 \in L^\infty$ which gives a global-in-time solution u of (P) *with*

$$\|u(t)\|_\infty \to \infty \text{ as } t \to \infty.$$

Observe that, by Proposition 1.2, we cannot expect the existence of a priori L^∞-global bounds for global-in-time solutions in the critical case. Therefore it is important to seek the condition which assures the existence of such a global bounds.

The main purpose of this note is to shed some new light on and to give an answer for this problem from the variational analytical point of view.

2 Main Result

Hereafter we always assume that u denotes a global-in-time solution of (P).

Multiplying (P) by $\partial u(t)/\partial t$ and integrating it over Ω, we have

$$\left\| \frac{\partial u(t)}{\partial t} \right\|_2^2 = -\frac{d}{dt} J_\lambda(u(t)), \tag{2.1}$$

where J_λ denotes the energy (or Lyapunov) functional associated to (P) defined by

$$J_\lambda(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{q}\|u\|_q^q.$$

Hence $J_\lambda(u(t))$ is nonincreasing in t. Moreover, it is well known that

$$\text{if } T_m < \infty, \text{ then } J_\lambda(u(t)) \geq 0 \text{ for } t \in [0, \infty), \tag{2.2}$$

see e.g. [11].

By (2.1) and by (2.2), for any (global) solution u of (P), there exists $d \geq 0$ such that

$$\lim_{t\to\infty} J_\lambda(u(t)) = d. \tag{2.3}$$

In order to state our main result, we have to recall some notion from variational analysis.

Definition 2.1 ((PS)-condition)

Let $S \subset H_0^1$.

(a) *(u_n) is said to be a Palais-Smale sequence of J_λ at level d in S ($(\mathrm{PS})_d$-sequence in S) if*

$$(u_n) \subset S, \quad J_\lambda(u_n) \to d, \quad (dJ_\lambda)_{u_n} \to 0 \text{ in } (H_0^1)^*$$

where $(dJ_\lambda)_{u_n}$ denotes the Fréchet derivative of J_λ at u_n in H_0^1.

(b) *J_λ is said to satisfy the Palais-Smale condition at level d in S ($(\mathrm{PS})_d$-condition in S) if any $(\mathrm{PS})_d$-sequence in S contains a strongly convergent subsequence in H_0^1.*

Let u be a (global) solution of (P). We introduce the $(\mathrm{PS})_d$-condition along the orbit u.

Definition 2.2 ((PS)-condition along the orbit)

J_λ is said to satisfy the Palais-Smale condition along u ((PS)-condition along u) when J_λ satisfies the $(\mathrm{PS})_d$-condition in $S = \{u(t); t \in (0,\infty)\}$ where d is given by (2.3).

Remark 2.1

It is easy to see that J_λ satisfies the (PS)-condition along u if there exists U such that J_λ satisfies the $(\mathrm{PS})_d$-condition in U and $\{u(t);\ t \in (0,\infty)\} \subset U$.

Our main theorem gives a sufficient and necessary condition (on u_0 or u and J_λ) for the existence of an L^∞-global bounds of u in terms of the (PS)-condition. Observe that our main theorem does not require the subcriticality of q.

Theorem 2.1 (Main Theorem)

Let $q \in (2, 2^*]$ *and* $d = \lim_{t\to\infty} J_\lambda(u(t))$*. Then the following assertion* (a) *and* (b) *are equivalent.*
(a) J_λ *satisfies the* $(\mathrm{PS})_d$*-condition along* u*.*
(b) u *has an* L^∞*-global bounds.*

Now we shall see some corollaries which follow easily from the main theorem.

For $q < 2^*$, it is well known that J_λ satisfies the $(\mathrm{PS})_d$-condition for any $d \in \mathbb{R}$ and for any $\lambda \in \mathbb{R}$, see e.g. [12, Chapter II, Proposition 2.2]. Hence by Remark 2.1 and by Theorem 2.1, we again obtain Proposition 1.1.

Corollary 2.1 (Subcritical case, Proposition 1.1)

Let $q \in (2, 2^*)$ *and let* $\lambda \in \mathbb{R}$*. Then* u *has an* L^∞*-global bounds.*

Let $q = 2^*$ and $d < S^{N/2}/N$. Then it is well known that J_λ satisfies the $(\mathrm{PS})_d$-condition, see [8]. Hence Remark 2.1 and Theorem 2.1 yield the following.

Corollary 2.2 (Critical case, Brezis-Nirenberg type)

Let $q = 2^*$, $\lambda \in \mathbb{R}$ *and* $d < S^{N/2}/N$*. Then* u *has an* L^∞*-global bounds.*

Let $\Omega_a := \{x \in \mathbb{R}^m;\ 1 < |x|_{\mathbb{R}^m} < 2\}$ be an annulus, $k \in \mathbb{N}$ and $\Omega := \Omega_a \times \cdots \times \Omega_a$ (k times). Also let $G := SO(m) \oplus \cdots \oplus SO(m)$ (k fold). Here we recall that J_λ satisfies the $(\mathrm{PS})_d$-condition in the G-invariant subspace of H_0^1. It is also obvious that if u_0 is G-invariant, then the corresponding solution of (P) is also G-invariant. Hence by Remark 2.1 and Theorem 2.1, we have:

Corollary 2.3 (Critical, G-invariant case)

Let $q = 2^*$ *and* $\lambda \in \mathbb{R}$*. Let* Ω *and* G *be as above and* u_0 *be a* G*-invariant function. Then* u *has an* L^∞*-global bounds.*

As for the solution which blows up in infintie time (see e.g. Proposition 1.2), Theorem 2.1 yields:

Corollary 2.4 (Infinite time blow up solution)

Let $q \in (2, 2^*]$*. Assume that* u *blows up in infinite time in the* L^∞*-sense. Then* J_λ *does not satisfy the* $(\mathrm{PS})_d$*-condition along* u*.*

3 Proof of Theorem 2.1

Now let us give the sketch of the proof of Theorem 2.1. In the following, $q_0 := N(q-2)/2$ (which is the critical exponent of (P) as a parabolic problem, see [6]).

The proof of (a)$\Rightarrow$(b) consists of two steps. The first step, Proposition 3.1, involves the compactness propertiy of the orbit in L^{q_0}. In the latter step, we establish the relation between the existence of an L^∞-global bounds and the compactness of the orbit in L^{q_0} (Proposition 3.2). The proof of Theorem 2.1 is in the last of this section. In this section, u always denotes a global-in-time solution of (P).

Proposition 3.1

Let $q \in (2, 2^*]$. Assume that J_λ satisfies $(\mathrm{PS})_d$-condition along u. Then for any $t_n \to \infty$, there exists a subsequence of (t_n) (still denoted by the same symbol) and $u \in L^{q_0}$ such that $u(t_n) \to u$ strongly in L^{q_0}.

Proof.

Take any $t_n \to \infty$ and let $u_n(s) := u(t_n - 1/2 + s)$ for $s \in [0,1]$. Then it is easy to see that there exists a subsequence of (t_n) (still denoted by the same symbol) and $L \subset [0,1]$ with measure zero such that, for all $s \in [0,1] \setminus L$,

$$(u_n(s)) \text{ is a } (\mathrm{PS})_d\text{-sequence}, \tag{3.1}$$

$$\|u_n(s)\|_q^q \to d/(1/2 - 1/q) \text{ as } n \to \infty. \tag{3.2}$$

By the assumption of the Proposition and (3.1), $(u_n(s))$ is relatively compact in H_0^1 for $s \in [0,1] \setminus L$. Hence by the continuity of $H_0^1 \hookrightarrow L^q$, $K(s) := \overline{\{u_n(s)\}}^{L^q}$ is a compact set in L^q. Then by Theorem 1 of [6], for any $\varepsilon > 0$ and for any $s \in [0,1] \setminus L$, there exists $\delta(\varepsilon, s) := \delta(\varepsilon, K(s)) > 0$ such that

$$\|u_n(s+\sigma)\|_q^q \le \|u_n(s)\|_q^q + \varepsilon/2, \quad \forall n, \ \forall \sigma \in [0, \delta(\varepsilon, s)]. \tag{3.3}$$

Consequently, we find that

$$\|u_n(s)\|_q^q \le d/(1/2 - 1/q) + \varepsilon, \quad \forall s \in [1/4, 3/4], \ \forall n > N \tag{3.4}$$

for some N. Hence, the decreasing property of $J_\lambda(u(t))$ in t together with (3.4) yields

$$\|\nabla u_n(s)\|_2 \le C, \quad \forall s \in [1/4, 3/4], \ \forall n > N \tag{3.5}$$

for some $C > 0$.

Case 1. Assume that $q < 2^*$. Then $q_0(:= N(q-2)/2) < 2^*$. Hence by the compactness of $H_0^1 \hookrightarrow L^{q_0}$ and by (3.5) with $s = 1/2$, we have the conclusion (recall that $u_n(1/2) = u(t_n)$).

Case 2. Assume that $q = 2^*$. Note that, in this case, $q_0 = q = 2^*$. Hereafter we denote both of q and q_0 by 2^*. By (3.5), by the continuity of $H_0^1 \hookrightarrow L^{2^*}$ and by the compactness of $H_0^1 \hookrightarrow L^2$, we can find $u(1/2)$ such that

$$u_n(1/2) \rightharpoonup u(1/2) \text{ weakly in } L^{2^*}, \tag{3.6}$$

$$u_n(1/2) \to u(1/2) \text{ strongly in } L^2 \tag{3.7}$$

as $n \to \infty$, taking subsequence if necessary (recall that $u_n(1/2) = u(t_n)$). Especially by (3.6) and (3.4),

$$\|u(1/2)\|_{2^*}^{2^*} \le \|u_n(1/2)\|_{2^*}^{2^*} + o(1) \le d/(1/2 - 1/q) + o(1) \tag{3.8}$$

as $n \to \infty$.

Take any $\sigma \in [1/4, 3/4] \setminus L$. Then by (3.1) and by the assumption of the Proposition, $(u_n(\sigma))$ has a strongly convergent subsequence in H_0^1. Hence, there exists $u(\sigma) \in H_0^1$ such that

$$u_n(\sigma) \to u(\sigma) \text{ strongly in } L^{2^*} \text{ and in } L^2 \tag{3.9}$$

taking further subsequence if necessary. Especially by (3.2) and by (3.9), we have

$$d/(1/2 - 1/q) = \|u_n(\sigma)\|_{2^*}^{2^*} + o(1) = \|u(\sigma)\|_{2^*}^{2^*} \tag{3.10}$$

as $n \to \infty$.

Moreover, by (3.7) and (3.9),

$$\begin{aligned} \|u(\sigma) - u(1/2)\|_2 &\le \|u(\sigma) - u_n(\sigma)\|_2 + \left\| \frac{\partial u_n}{\partial s} \right\|_{L^2(0,1;L^2)} \\ &\quad + \|u(1/2) - u_n(1/2)\|_2 \\ &= o(1), \end{aligned} \tag{3.11}$$

thus we have

$$u(1/2) = u(\sigma). \tag{3.12}$$

Hence by (3.10), (3.12) and (3.8),

$$\begin{aligned} d/(1/2-1/q) &= \|u(\sigma)\|_{2^*}^{2^*} = \|u(1/2)\|_{2^*}^{2^*} \le \|u_n(1/2)\|_{2^*}^{2^*} + o(1) \\ &\le d/(1/2-1/q) \end{aligned}$$

as $n \to \infty$. Therefore combining this relation with (3.6), we have $u(t_n) = u_n(1/2) \to u(1/2)$ strongly in $L^{2^*} = L^{q_0}$, thus the conclusion. ∎

Proposition 3.2

Assume that for any $t_n \to \infty$, there exists a subsequence of (t_n) (still denoted by the same symbol) and u such that $u(t_n) \to u$ in L^{q_0}. Then u has an L^∞-global bounds.

Proof.

Assume that the conclusion is false. Then there exist $(x_n) \subset \Omega$ and $t_n \to \infty$ such that

$$\|u(t_n)\|_\infty \to \infty,\ \sup_{t\in(0,t_n]} \|u_n(t)\|_\infty = \|u(t_n)\|_\infty,\ \|u(t_n)\|_\infty/2 \le |u(x_n,t_n)| \quad (3.13)$$

Let y, s, v_n be

$$y = \lambda_n(x-x_n),\ s = \lambda_n^2(t-t_n),\ \lambda_n^{2/(q-2)} v_n(y,s) = u(x,t)$$

for λ_n with $\lambda_n^{2/(q-2)} = \|u(t_n)\|_\infty$. Note that by virtue of the choice of λ_n and (3.13), we have $\lambda_n \to \infty$ and

$$\sup_{s\in[-1,0]} \|v_n(s)\|_\infty \le \|v_n(0_s)\|_\infty = 1, \quad (3.14)$$

$$|v_n(0_y, 0_s)| \ge 1/2. \quad (3.15)$$

By the boundedness of Ω and the homogeneous Dirichlet condition, we can assume that $x_n \to x \in \operatorname{int}\Omega$ taking subsequence if necessary, see e.g. [5] or [9]. By (3.14), $\|v_n\|_{L^\infty(-1,\delta;L^\infty)} < 2$ holds for some $\delta > 0$ which is independent of n. Then, by the standard parabolic estimate, we see that

$$v_n \to v \text{ in } C_{\text{loc}}(\mathbb{R}^N \times (-1,\delta)) \quad (3.16)$$

holds for $v \in C_{\text{loc}}(\mathbb{R}^N \times (-1,\delta))$.

Also by the straightforward calculation using (2.3),

$$\begin{aligned}\left\|\frac{\partial v_n}{\partial s}\right\|^2_{L^2(-1,\delta;L^2)} &= \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(t_n-1/\lambda_n^2,t_n+\delta/\lambda_n^2;L^2)} \\ &= J_\lambda(u(t_n - 1/\lambda_n^2)) - J_\lambda(u(t_n + \delta/\lambda_n^2)) \\ &\to d - d = 0\end{aligned}$$

follows. Hence the same argument as in (3.11) implies that v is independent of s. Moreover by (3.15) and by (3.16), $|v(0_y)| \geq 1/2$. Therefore there exists $R > 0$ sufficientlly small such that

$$\|v\|_{q_0,B(0;R)} =: \eta > 0. \tag{3.17}$$

Since $x \in \operatorname{int}\Omega$, $B(x;\varepsilon) \subset \Omega$ holds for small ε. Observe that for large n, $B(x_n; R/\lambda_n) \subset B(x,\varepsilon)$. Then by (3.17) and (3.16),

$$\begin{aligned}0 &< \eta = \|v\|_{q_0,B(0;R)} = \|v_n(0_s)\|_{q_0,B(0;R)} + o(1) \\ &= \|u(t_n)\|_{q_0,B(x_n;R/\lambda_n)} + o(1) \leq \|u(t_n)\|_{q_0,B(x;\varepsilon)} + o(1)\end{aligned} \tag{3.18}$$

for small $\varepsilon > 0$.

On the other hand, the assumption of the Proposition yields

$$\|u(t_n)\|_{q_0,B(x;\varepsilon)} \overset{n\to\infty}{\to} \|u\|_{q_0,B(x;\varepsilon)} \overset{\varepsilon\to 0}{\to} 0$$

along an appropriate subsequence, which is absurd in view of (3.18). ∎

Proof of Theorem 2.1 The assertion (a) ⇒ (b) immediately follows from Proposition 3.1 and 3.2.

The assertion (b) ⇒ (a) follows from a typical argument for the verification of (PS)-condition in the variational analysis. ∎

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