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On the existence of periodic traveling wave solutions for the Ostrovsky equation

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概 要

We are concerned with the Ostrovsky equation, which is derived from the theory of weakly nonlinear long surface and internal waves in shallow water under the presence of rotation. Based on a variational method, we show the existence of periodic traveling wave solutions.

1 Introduction

Waves in shallow water have been the subject for intensive studies. Well known examples include the famous Korteweg-de Vries (KdV) equation, which is derived as a model for weakly nonlinear long waves. If the fluid is rotating and the wave frequency is greater than the Coriolis frequency, then the evolution is described by the so called Ostrovsky equation [9]

$$(u_t - \beta u_{xxx} + (u^2)_x)_x = \varepsilon^2 u, \quad u = u(x,t), \quad x \in \boldsymbol{R},\ t > 0, \tag{1}$$

where $\varepsilon > 0$, $\beta \in \boldsymbol{R}$ are constant coefficients. The equation (1) is also referred to as the rotation-modified KdV equation [1].

In a recent nice paper [8], Y. Liu and V. Varlamov investigated the existence and stability of solitary waves for (1). Here a solitary wave solution of (1) means a traveling wave solution; namely, by abuse of notation, the solution has the form $u(x,t) = u(x-ct)$ with a parameter $c \in \boldsymbol{R}$ which represents the velocity and $u = u(x)$ verifies

$$(-cu - \beta u_{xx} + u^2)_{xx} = \varepsilon^2 u, \quad x \in \boldsymbol{R}. \tag{2}$$

Part of main accomplishments in [8] states that if $\beta < 0$ and $c < \sqrt{140|\beta|}|\varepsilon|$ then (2) has no nontrivial solitary wave, while $\beta > 0$ and $c < 2\sqrt{\beta}|\varepsilon|$ then (2) admits a nontrivial one.

This note, on the other hand, is focused on the existence of periodic traveling waves for the Ostrovsky equation. To be specific, we deal with the existence of periodic solutions to (2). Although a family of periodic traveling waves for the Ostrovsky equation is numerically indicated to exist [3][7], there seems little analytical attempt so far; we make up for such lack of issues. For related nonlinear, dispersive wave equations, we refer to [2] for instance.

Before formulating our main achievements, we transform (2) in order to clarify the point of the problem. In (2) we make a change $u \to -u$, $c \to -c$ so that (2) becomes

$$(-cu + \beta u_{xx} + u^2)_{xx} = -\varepsilon^2 u.$$

Therefore the sign of β corresponds to the sign of the coefficient ε^2, and it is legitimate to assume $\beta > 0$ without loss of generality. Finally the change of variables $x \to \sqrt{\beta}x$ and $\varepsilon^2 \to \varepsilon^2/\beta$ brings us to the equation

$$(-cu + u^2 + u_{xx})_{xx} = \pm\varepsilon^2 u, \quad u = u(x), \quad x \in \boldsymbol{R}. \tag{3}$$

We intend to prove the existence of periodic solutions, whose period will be denoted by L. We recall once again that ε^2 is a fixed constant and c is a parameter representing the velocity.

Now our main results of this article reads as follows.

Theorem 1 *In the + sign case, there exists a periodic solution to* (3) *for every period $L > 0$. Furthermore if $L > 2\sqrt{6(1+\varepsilon^{-2})}$, the velocity c does not vanish. While in the − sign case, there exists a periodic traveling wave solution for every period $L > 0$. Furthermore if $L < \min\{1, |\varepsilon|/2\}$, then there holds $c < -(L^{-2} - L)/2 < 0$.*

The principal tool of our proof is a variational technique; we present two methods. One is to increase the number of unknown variable and to seek for a critical point of a certain functional of two unknown variables. The strategy is then akin to the one developed in other higher-order equation [4][5][6]. The other is to utilize an integral term, which is somewhat familiar in this field of researches. In our case, however, each step is elementary and much transparent.

2 Proof of the Theorem

Our aim is to show the existence of periodic solutions for the fourth-order equation (3). First we deal with the + sign case and introduce an auxiliary variable

$$v = -cu + u^2 + u_{xx}, \tag{4}$$

which makes it possible that transforms the single equation (3) into the system of second-order equation

$$\begin{cases} u_{xx} + u^2 - cu = v \\ v_{xx} = \varepsilon^2 u. \end{cases} \tag{5}$$

We remark that to recover (3) from (5), auxiliary variable v in (4) is allowed to be up to additive constants; namely, with regard to u variable, in place of (4), $v = -cu + u^2 + u_{xx} - \lambda$ ($\lambda \in \boldsymbol{R}$) works as well.

To proceed further, we fix an interval $(0, l)$ ($l > 0$) for simplicity, which turns out to be without loss of generality. Define functionals

$$J(u, v) := \int_0^l \left(\frac{1}{2}u_x^2 + \frac{1}{2\varepsilon^2}v_x^2 + uv\right) dx - \frac{1}{3}\int_0^l u^3\, dx. \tag{6}$$

The functionals J is handled on a function space

$$\mathcal{A} := \left\{(u, v) \in (H^1(0, l))^2 \mid \int_0^l u\, dx = 0,\ \int_0^l u^2\, dx = 1\right\}. \tag{7}$$

It can be seen that the critical point (u, v) of J among $\mathcal{A}$ verifies

$$\begin{cases} u_{xx} = v - u^2 + cu + \lambda & \text{in } 0 < x < l \\ v_{xx} = \varepsilon^2 u & \text{in } 0 < x < l \\ u_x = v_x = 0 & \text{at } x = 0, l, \end{cases} \tag{8}$$

where constants c and λ originate in the Lagrange multiplier of the constraints $\int_0^l u^2\,dx = 1$ and $\int_0^l u\,dx = 0$, respectively. In particular, u realizes a solution to (3) with $u_x = u_{xxx} = 0$ at $x = 0, l$.

If we extend u over the interval $(0, 2l)$ by the reflection

$$u(x) := \begin{cases} u(x) & \text{for } 0 \le x \le l \\ u(2l - x) & \text{for } l \le x \le 2l, \end{cases}$$

then we obtain a desired periodic solution with period $L := 2l$. Here, with abuse of notation, the extended u has been denoted by the same.

Now the following proposition will be settled.

Proposition 2 *There exists a global minimizer $(\bar{u}, \bar{v})$ of J on $\mathcal{A}$. Moreover, if $l > \sqrt{6(1+\varepsilon^{-2})}$, then the Lagrange multiplier c in* (8) *does not vanish.*

Proof. First we ascertain that J is bounded below on $\mathcal{A}$. To do so, we compute

$$\begin{aligned} \left| \int_0^l u^3\,dx \right| &\le |u|_{L^\infty(0,l)} \int_0^l u^2\,dx \le \sqrt{l}|u_x|_{L^2(0,l)} \\ \left| \int_0^l uv\,dx \right| &= \left| \int_0^l u(x)\Big(v(0) + \int_0^x v_x(y)\,dy\Big)\,dx \right| \\ &\le \int_0^l |u(s)|\sqrt{l}|v_x|_{L^2(0,l)} dx \le l|v_x|_{L^2(0,l)}, \end{aligned}$$

by virtue that $\int_0^l u\,dx = 0$, $\int_0^l u^2\,dx = 1$ for $(u, v) \in \mathcal{A}$. We thus infer that

$$J(u,v) \ge \frac{1}{2}|u_x|^2_{L^2(0,l)} - \sqrt{l}|u_x|_{L^2(0,l)} + \frac{1}{2\varepsilon^2}|v_x|^2_{L^2(0,l)} - l|v_x|_{L^2(0,l)}. \qquad (9)$$

This proves that J is bounded below on $\mathcal{A}$.

Next we take a minimizing sequence $\{(u_n, v_n)\}_{n \in \mathbf{N}} \subset \mathcal{A}$ for J. We may assume that $\int_0^l v_n\,dx = 0$ with replacing v_n by $v_n - l^{-1}\int_0^l v_n\,dx$ if necessary since $J(u_n, v_n) = J(u_n, v_n - l^{-1}\int_0^l v_n\,dx)$. Invoking (9) and $\int_0^l u_n\,dx = 0$, we conclude that there exists a subsequence (u_{n_m}, v_{n_m}) such that

$$u_{n_m} \to \bar{u}, \quad v_{n_m} \to \bar{v} \qquad \text{as } n_m \to \infty$$

weakly in $H^1(0,l)$ as well as strongly in $L^2(0,l)$. The lower semicontinuity of J yields

$$\liminf_{n_m\to\infty} J(u_{n_m}, v_{n_m}) \geq J(\bar{u}, \bar{v}),$$

which implies that $(\bar{u}, \bar{v})$ gives a global minimizer.

Finally we establish that $c \neq 0$ if $l > \sqrt{6(1+\varepsilon^{-2})}$. For this purpose, multiplying $\bar{u}$ and $\bar{v}$ equation of (8) by $\bar{u}$ and $\bar{v}$, respectively, we deduce, after integration,

$$\begin{aligned}\int_0^l \bar{u}\bar{v}\,dx &= \int_0^l (-\bar{u}_x^2 - c\bar{u}^2 + \bar{u}^3)\,dx = \int_0^l (-\bar{u}_x^2 + \bar{u}^3)\,dx - c \\ &= -\frac{1}{\varepsilon^2}\int_0^l \bar{v}_x^2\,dx,\end{aligned}$$

from which we find that

$$J(\bar{u},\bar{v}) = \int_0^l \frac{1}{2}(\bar{u}_x^2 + \bar{u}\bar{v}) + \frac{1}{2}(\frac{1}{\varepsilon^2}\bar{v}_x^2 + \bar{u}\bar{v})\,dx - \frac{1}{3}\int_0^l \bar{u}^3\,dx = \frac{1}{6}\int_0^l \bar{u}^3\,dx - \frac{c}{2}.$$

On the other hand, $J(-\bar{u}, -\bar{v}) \geq J(\bar{u}, \bar{v})$ leads to $\int_0^l \bar{u}^3\,dx \geq 0$. Therefore it follows that $J(\bar{u},\bar{v}) \geq 0$ so long as $c = 0$.

A simple test function, however, reveals the absurdity. If we put $u^0(x) := \alpha(x - 2^{-1}l)$ and $v^0(x) := -u^0(x)$ with $\alpha^2 = 12l^{-3}$, then we discover

$$J(u^0, v^0) = -1 + 6(1+\varepsilon^{-2})l^{-2} < 0$$

if $l^2 > 6(1+\varepsilon^{-2})$. This contradicts with the fact that $(\bar{u},\bar{v})$ is a global minimizer. Consequently $c > 0$ and the proof is completed. □

Remark. The reason why we introduce the two-component functional J is that it facilitates for us to choose a test function.

Next we turn our attention to the $-$ sign case. This time we minimize

$$\mathcal{J}(u) := \int_0^l \Big(\frac{1}{2}u_x^2 - \frac{\varepsilon^2}{2}(\partial_x^{-1}u)^2\Big)\,dx - \frac{1}{3}\int_0^l u^3\,dx, \tag{10}$$

over $\mathcal{A}_u$, where $\partial_x^{-1}u := \int_0^x u(y)dy$ and

$$\mathcal{A}_u := \left\{ u \in H^1(0,l) \mid \int_0^l u\,dx = 0,\ \int_0^l u^2\,dx = 1 \right\}.$$

Proposition 3 *There exists a critical point $\bar{u}$ of $\mathcal{J}$ on $\mathcal{A}_u$ for every $l > 0$. Moreover if $l < \min\{1/2, |\varepsilon|/4\}$ then it follows that $c < -8^{-1}l^{-2} + l < 0$.*

Proof. First we treat the existence of a critical point. Consider the minimization problem $\min_{u\in\mathcal{A}_u} \mathcal{J}(u)$. Since $(\partial_x^{-1}u)(0) = (\partial_x^{-1}u)(l) = 0$, we have

$$|\partial_x^{-1}u|_{L^2(0,l)} \le \frac{l}{\pi}|u|_{L^2(0,l)} \le \frac{l}{\pi}.$$

So a functional $\mathcal{J}(u)$ is coercive and there exists a $u_0 \in \mathcal{A}_u$ satisfying $\mathcal{J}(u_0) = \min_{u\in\mathcal{A}_u} \mathcal{J}(u)$. The critical point $\bar{u}$ satisfies

$$\int_0^l \left(\bar{u}_x\eta_x + \varepsilon^2\partial_x^{-1}\bar{u}\partial_x^{-1}\eta - \bar{u}^2\eta\right) dx = c\int_0^l \bar{u}\eta dx \tag{11}$$

for every $\eta \in H^1(0,l)$ with $\int_0^l \eta dx = 0$, where c is a Lagrange multiplier. Integrating by part, we have

$$\begin{cases} \partial_x^2\bar{u} - \varepsilon^2 \displaystyle\int_l^x \int_0^y \bar{u}(s)dsdy - c\bar{u} + \bar{u}^2 = 0, \\ \partial_x\bar{u}(0) = \partial_x\bar{u}(l) = 0. \end{cases} \tag{12}$$

Differentiating (12), we obtain $\bar{u}_{xxx}(0) = \bar{u}_{xxx}(l) = 0$ and (3).

Next we show an estimate for the Lagrange multiplier c. We recall that the period of $\bar{u}$ is $L := 2l$ and hence it is better to consider the equation satisfied by $\bar{u}$ on the interval $[0, L]$.

$$(-c\bar{u} + \bar{u}^2 + \bar{u}_{xx})_{xx} = -\varepsilon^2\bar{u} \qquad \text{on } 0 < x < L = 2l. \tag{13}$$

Multiplying (13) by $\bar{u}$ and noting that $\int_0^L \bar{u}^2\, dx = 2$, we have

$$|\bar{u}_{xx}|^2_{L^2(0,L)} - 2\int_0^L \bar{u}\bar{u}_x^2\, dx + c|\bar{u}_x|^2_{L^2(0,L)} + 2\varepsilon^2 = 0. \tag{14}$$

Here the sign of c takes effects and we divide our reasoning according to it.

If $c > 0$, then we derive

$$2\varepsilon^2 + c|\bar{u}_x|^2_{L^2(0,L)} + |\bar{u}_{xx}|^2_{L^2(0,L)} \le \sqrt{L}|\bar{u}_x|^3_{L^2(0,L)}$$

in light of $|\bar{u}|_{L^\infty(0,L)} = |\bar{u}|_{L^\infty(0,l)} \le \sqrt{l}|\bar{u}_x|_{L^2(0,l)} \le \sqrt{L}|\bar{u}_x|_{L^2(0,L)}/2$. Taking account that

$$|\bar{u}_x|^2_{L^2(0,L)} \le |\bar{u}|_{L^2(0,L)}|\bar{u}_{xx}|_{L^2(0,L)} = \sqrt{2}|\bar{u}_{xx}|_{L^2(0,L)},$$

we infer that $|\bar{u}_x|_{L^2(0,L)} \leq 2\sqrt{L}$ and as a by-product $c < 2L$ and $\varepsilon^2 \leq 4L^2$ must be fulfilled.

If $c < 0$ in (14), we find that

$$2\varepsilon^2 + |\bar{u}_{xx}|^2_{L^2(0,L)} \leq \sqrt{L}|\bar{u}_x|^3_{L^2(0,L)} + |c||\bar{u}_x|^2_{L^2(0,L)}.$$

A similar procedure as above leads to

$$|\bar{u}_x|^4_{L^2(0,L)} \leq 2\sqrt{L}|\bar{u}_x|^3_{L^2(0,L)} + 2|c||\bar{u}_x|^2_{L^2(0,L)}$$

and therefore

$$|\bar{u}_x|_{L^2(0,L)} \leq \sqrt{L} + \sqrt{L + 2|c|} \leq 2\sqrt{L + 2|c|}$$
$$2 = |\bar{u}|^2_{L^2(0,L)} \leq L^2|\bar{u}_x|^2_{L^2(0,L)} \leq 2L^2(L + 2|c|).$$

To summarize, if there holds $l = L/2 < \min\{1/2, |\varepsilon|/4\}$,then we conclude that $c < -(L^{-2} - L)/2 < 0$. This completes the proof. □

Remark A straight modification of the functional $\mathcal{J}$ can be applied to prove the existence of solutions as well in the + sign case.

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