数理解析研究所講究録 1464 巻 2006 年 66-81



# Stability of steady-state solutions with transition layers for a bistable reaction-diffusion equation [1]

Michio URANO (浦野 道雄) [2]

Department of Mathematical Science,
School of Science and Engineering, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, JAPAN
(早稲田大学 大学院 理工学研究科 数理科学専攻)

## 1 Introduction

In this paper we will consider the following reaction-diffusion problem:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(x,u), & 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

Here $\varepsilon$ is a positive parameter and

$$f(x,u) = u(1-u)(u-a(x)),$$

where $a$ is a $C^2[0,1]$-function with the following properties:
(A1) $0 < a(x) < 1$ in $[0,1]$,
(A2) if $\Sigma$ is defined by

$$\Sigma := \{x \in (0,1)\,;\, a(x) = 1/2\}, \tag{1.2}$$

then $\Sigma$ is a finite set and $a'(x) \neq 0$ at any $x \in \Sigma$,
(A3) $a'(0) = a'(1) = 0$.

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[1] This is a joint work with Professors Kimie NAKASHIMA (Tokyo University of Marine Science and Technology) and Yoshio YAMADA (Waseda University).

[2] E-mail: `michio_u@akane.waseda.jp`

It is well known that (1.1) describes phase transition phenomena in various fields such as physics, chemistry and mathematical biology. This problem is a gradient system with the following energy functional:

$$E(u) := \int_0^1 \left\{ \frac{1}{2}\varepsilon^2 |u_x|^2 + W(x,u) \right\} dx,$$

where

$$W(x,u) := -\int_0^u f(x,s)ds.$$

For every solution of (1.1), $E(u(\cdot,t))$ is decreasing with respect to $t$ and it is well known that $u(x,t)$ is convergent to a solution of the corresponding steady-state problem as $t \to \infty$. The graph of $W$ has two local minimums at $u = 0$ and $u = 1$; so that we can regard both $u = 0$ and $u = 1$ as stable states when $\varepsilon$ is sufficiently small. Furthermore, the minimal energy state changes according as $a(x)$ is greater than $1/2$ or not; if $a(x) < 1/2$, then $W$ attains its minimum at $u = 1$, while if $a(x) > 1/2$, then the minimum of $W$ is attained at $u = 0$. The interaction of the bistability and the spatial inhomogeneity yields a complicated structure of solutions to (1.1).

In this point of view, one of the most important problems for (1.1) is to know the structure of steady state solutions. So we will mainly consider the following steady state problem associated with (1.1):

$$\begin{cases} \varepsilon^2 u'' + f(x,u) = 0 & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where ' $'$ ' denotes the derivative with respect to $x$.

Among all solutions of (1.3), we are interested in a solution with transition layers. We have complete information about the locations of transition layers. Here **transition layer** is a part of a solution $u$ where $u(x)$ drastically changes from 0 to 1 or 1 to 0 when $x$ varies in a very small interval. For (1.3), we can observe a cluster of transition layers. This is called a **multi-layer**, while a single transition layer is called a **single-layer**. It is known that any single- or multi-layer appears only in a vicinity of a point in $\Sigma$. These results are proved by Ai, Chen and Hastings [1] (see also Urano, Nakashima and Yamada [7], whose method of proof is different from the method in [1]), and they are given in Theorems 2.6 and 2.7. It should be noted that the existence of such solutions is also discussed in [1] by shooting method. Furthermore, they have also discussed the stability problem of such solutions with use of Sturm's comparison theorem (Proposition 3.1). The study of stability properties of such solutions is also a great important problem.

For (1.3), Angenent, Mallet-Paret and Peletier [3] proved that there exist solutions with single-layers in the form of transitions from minimal energy state to minimal energy state when $\varepsilon$ is sufficiently small. They also showed that all solutions with such transition layers are stable. See also Hale and Sakamoto [4], who discussed solutions with single-layers connecting from non-minimal energy state to nonminimal energy state; all of their solutions are unstable. In a special case that $\int_0^1 f(x,u)du = 0$, which is called a balanced case, Nakashima [5, 6] has shown the existence of solutions with transition layers. Especially, in [6] she has proved the existence of a solution with multi-layers and obtained its stability property.

The main purpose of this paper is to study stability properties of a solution $u_\varepsilon$ of (1.3) which possesses transition layers by using different approach from Ai, Chen and Hastings [1]. Consider the following linearized problem:

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x,u_\varepsilon)\phi = \lambda\phi & \text{in } (0,1), \\ \phi'(0) = \phi'(1) = 0. \end{cases} \tag{1.4}$$

We will show that all solutions with transition layers are non-degenerate. We also study the stability property of $u_\varepsilon$ in terms of Morse index. The notion of non-degeneracy and Morse index is defined as follows:

**Definition 1.1 (Non-degeneracy).** Let $u_\varepsilon$ be a solution of (1.3). If (1.4) does not admit zero eigenvalue, then $u_\varepsilon$ is said to be **non-degenerate**.

**Definition 1.2 (Morse index).** Let $u_\varepsilon$ be a solution of (1.3). The Morse index of $u_\varepsilon$ is defined by the number of negative eigenvalues of (1.4).

In general, the stability property of $u_\varepsilon$ has a close relationship to its profile. In particular, the results of Angenent, Mallet-Paret and Peletier [3], and Hale and Sakamoto [4] (Proposition 4.1) tell us that the stability of solutions with single-layers is determined by the direction of each transition layer. Therefore we can expect that such facts remain valid for solutions with multi-layers. Indeed, we can show that the Morse index of a solution with multi-layers is equal to the number of transition layers from nonminimal energy state to nonminimal energy state (Theorem 4.2). Our method of proof is based on the Courant min-max principle and is different from that of Ai, Chen and Hastings [1].

The content of this paper is as follows: In Section 2, we will collect some information on profiles of solutions with transition layers . In Section 3 we will recall the theory of Sturm-Liouville for the eigenvalue problem. Finally, Section 4 is devoted to the stability analysis for solutions with transition layers.

## 2 Profiles of steady-state solutions with transition layers

In this section, we will give some important properties concerning to the profiles of solutions with transition layers. Such oscillating solutions have at most a finite number of intersecting points with $a$ in $(0,1)$. So, we take account of the number of these points. Let $u_\varepsilon$ be a solution of (1.3) and set

$$\Xi := \{x \in (0,1)\,;\, u_\varepsilon(x) = a(x)\}. \tag{2.1}$$

We now introduce the notion of $n$-mode solutions.

**Definition 2.1.** Let $u_\varepsilon$ be a solution of (1.3) and set $\Xi$ by (2.1). If $\#\Xi = n$, then $u_\varepsilon$ is called an ***n*-mode solution.**

In what follows, we denote the set of all of $n$-mode solutions by $S_{n,\varepsilon}$. We collect some properties of solutions in $S_{n,\varepsilon}$. By the maximum principle, one can easily see that any $u_\varepsilon \in S_{n,\varepsilon}$ satisfies $0 < u_\varepsilon(x) < 1$ in $(0,1)$.

**Lemma 2.2.** *For $u_\varepsilon \in S_{n,\varepsilon}$, assume $\Xi = \{\xi_k\}_{k=1}^n$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. Then there exist exactly $n-1$ critical points $\{\zeta_k\}_{k=1}^{n-1}$ of $u_\varepsilon$ satisfying*

$$0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \zeta_{n-1} < \xi_n < 1,$$

*provided that $\varepsilon$ is sufficiently small.*

**Lemma 2.3.** *For $u_\varepsilon \in S_{n,\varepsilon}$, let $\xi^\varepsilon$ be any point in $\Xi$ and define $U_\varepsilon$ by $U_\varepsilon(t) = u_\varepsilon(\xi^\varepsilon + \varepsilon t)$. Then there exists a subsequence $\{\varepsilon_k\} \downarrow 0$ such that $\xi_k = \xi^{\varepsilon_k}$ and $U_k = U_{\varepsilon_k}$ satisfy*

$$\lim_{k\to\infty} \xi_k = \xi^* \quad \text{and} \quad \lim_{k\to\infty} U_k = U \quad \text{in} \quad C^2_{loc}(\mathbb{R}),$$

*with some $\xi^* \in [0,1]$ and $U \in C^2(\mathbb{R})$. Furthermore, if $\xi^* \in \Sigma$ and $\dot{U}(\xi^*) > 0$ (resp. $\dot{U}(\xi^*) < 0$), then $U$ is a unique solution of the following problem:*

$$\begin{cases} \ddot{U} + U(1-U)(U-1/2) = 0 & \text{in } \mathbb{R}, \\ \dot{U} > 0 \quad (\text{resp. } \dot{U} < 0) & \text{in } \mathbb{R}, \\ U(-\infty) = 0,\ U(\infty) = 1 \quad (\text{resp. } U(-\infty) = 1,\ U(\infty) = 0), & \\ U(0) = 1/2, & \end{cases}$$

*where '$\dot{}$' denotes the derivative with respect to $t$.*

**Theorem 2.4.** *For $u_\varepsilon \in S_{n,\varepsilon}$, let $\xi_1, \xi_2$ be successive points in $\Xi$ satisfying $\xi_1 < \xi_2$ and $(\xi_2 - \xi_1)/\varepsilon \to \infty$ as $\varepsilon \to 0$ and let $\zeta \in (\xi_1, \xi_2)$ be a critical point of $u_\varepsilon$. Furthermore, set*

$$d(x) = \begin{cases} x - \xi_1 & \text{if } \xi_1 \le x \le \zeta, \\ \xi_2 - x & \text{if } \zeta \le x \le \xi_2. \end{cases}$$

*Then one of the following assertions holds true:*
*(i) If $u_\varepsilon$ attains its local maximum at $\zeta$, then there exist positive constants $C_1, C_2, r, R$ with $C_1 < C_2$ and $r < R$ such that*

$$C_1 \exp\left(-\frac{Rd(\zeta)}{\varepsilon}\right) < 1 - u_\varepsilon(x) < C_2 \exp\left(-\frac{rd(x)}{\varepsilon}\right) \quad \text{in } [\xi_1, \xi_2]. \tag{2.2}$$

*(ii) If $u_\varepsilon$ attains its local minimum at $\zeta$, then there exist positive constants $C_1', C_2', r', R'$ with $C_1' < C_2'$ and $r' < R'$ such that*

$$C_1' \exp\left(-\frac{R'd(\zeta)}{\varepsilon}\right) < u_\varepsilon(x) < C_2' \exp\left(-\frac{r'd(x)}{\varepsilon}\right) \quad \text{in } [\xi_1, \xi_2]. \tag{2.3}$$

**Remark 2.5.** Theorem 2.4 tells us that $u_\varepsilon(x)$ and $1 - u_\varepsilon(x)$ are very small when $x$ does not lie in an $O(\varepsilon)$-neighborhood of a point in $\Xi$. On the contrary, one can see that $u_\varepsilon$ has a sharp transition in a small neighborhood of a point in $\Xi$.

**Theorem 2.6.** *For $u_\varepsilon \in S_{n,\varepsilon}$, define $\Xi$ by* (2.1) *and assume that $u_\varepsilon$ forms a transition layer near $\xi \in \Xi$. Then there exists a positive number $\varepsilon_0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $\xi - z = O(\varepsilon|\log\varepsilon|)$ with some $z \in \Sigma$.*

We also give a result on multi-layers. For this purpose, we decompose $\Sigma$ into the following subsets:

$$\Sigma^+ = \{x \in \Sigma\,;\, a'(x) > 0\}, \quad \Sigma^- = \{x \in \Sigma\,;\, a'(x) < 0\}.$$

**Theorem 2.7.** *For $u_\varepsilon \in S_{n,\varepsilon}$, assume that $u_\varepsilon$ has a multi-layer near $z \in \Sigma$ when $\varepsilon$ is sufficiently small. Then there exists a positive number $K$ such that $\#(\Xi \cap (z - K\varepsilon|\log\varepsilon|, z + K\varepsilon|\log\varepsilon|)) = 2m - 1$ with some $m \in \mathbb{N}$. Furthermore, if the multi-layer is a multi-layer from* 0 *to* 1 *(resp. from* 1 *to* 0*), then $z \in \Sigma^+$ (resp. $z \in \Sigma^-$).*

**Remark 2.8.** Theorem 2.7 gives us more precise information on the profile of $u_\varepsilon$. Set $\Xi \cap (z - K\varepsilon|\log\varepsilon|, z + K\varepsilon|\log\varepsilon|) = \{\xi_k\}_{k=1}^{2m-1}$ with $\xi_1 < \xi_2 < \cdots < \xi_{2m-1}$ and let $\{\zeta_k\}_{k=0}^{2m-1}$ be a set of critical points of $u_\varepsilon$ satisfying $\zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1}$. Then, by Theorem 2.7, there exists a positive constant $M$ such that $\zeta_{k+1} - \zeta_k < M\varepsilon|\log\varepsilon|$ for each $k = 1, 2, \ldots, 2m-3$.

The proofs of Lemmas and Theorems in this section can be found in [7].

# 3 Basic theory for Sturm-Liouville eigenvalue problem

In this section, we recall the Sturm-Liouville theory for (1.4).

**Proposition 3.1.** *There exist infinitely number of eigenvalues of* (1.4) *and all of them are real and simple. Furthermore, if $\lambda_j$ denotes the $j$-th eigenvalue of* (1.4)*, then it holds that*

$$-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \to \infty \quad \text{as } j \to \infty$$

*and the eigenfunction corresponding to $\lambda_j$ has exactly $j-1$ zeros in $(0,1)$.*

The following results is well known as the Courant min-max principle:

**Proposition 3.2.** *Let $\lambda_j$ be the $j$-th eigenvalue of* (1.4)*. Then $\lambda_j$ is characterized by*

$$\lambda_1 = \inf_{\phi \in H^1(0,1)\setminus\{0\}} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}},$$

$$\lambda_j = \sup_{\psi_1,\dots,\psi_{j-1}\in L^2(0,1)} \inf_{\phi \in X[\psi_1,\dots,\psi_{j-1}]} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}} \quad \text{for } j = 2, 3, \dots, \tag{3.1}$$

*where*

$$\mathscr{H}(\phi) := \int_0^1 \left\{\varepsilon^2|\phi'(x)|^2 - f_u(x, u_\varepsilon(x))|\phi(x)|^2\right\}dx$$

*and*

$$X[\psi_1,\dots,\psi_{j-1}] := \{\phi \in H^1(0,1)\setminus\{0\}\,;\ (\phi,\psi_i)_{L^2(0,1)} = 0\,(i = 1, 2, \cdots, j-1)\}.$$

**Remark 3.3.** If $\psi_i$ is the eigenfunction corresponding to the $i$-th eigenvalue $\lambda_i$ of (1.4) for every $i = 1, 2, \dots, j-1$ in (3.1), then $\lambda_j$ is characterized by

$$\lambda_j = \inf_{\phi \in X[\psi_1,\dots,\psi_{j-1}]} \frac{\mathscr{H}(\phi)}{\|\phi\|^2_{L^2(0,1)}}.$$

It is possible to prove the following result from Proposition 3.2:

**Proposition 3.4.** *Let $\lambda_j$ be the $j$-th eigenvalue of* (1.4) *and let $\widetilde{\lambda}_j$ be the $j$-th eigenvalue of the following eigenvalue problem:*

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi + p(x)\phi = \lambda\phi & \text{in } (0,1), \\ \phi'(0) = \phi'(1) = 0, \end{cases}$$

*where $p \in C([0,1])$. If $p(x) \geq 0$ (resp. $p(x) \leq 0$) and $p(x) \not\equiv 0$ in $(0,1)$, then $\widetilde{\lambda}_j > \lambda_j$ (resp. $\widetilde{\lambda}_j < \lambda_j$).*

# 4 Stability of solutions with transition layers

We will study stability properties of solutions with transition layers. In order to study a solution with transition layers, assume that a solution $u_\varepsilon$ of (1.3) does not have any oscilation in $(0,1)$. For such $u_\varepsilon$, we can choose a positive constant $M$ and a subset $\{z_i\}_{i=1}^l$ of $\Sigma$ satisfying

$$\Xi \cap (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|) \neq \emptyset \tag{4.1}$$

and

$$\#(\Xi \cap (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|)) = 2m_i - 1 \tag{4.2}$$

with some $m_i \in \mathbb{N}$ for each $i = 1, 2, \ldots, l$, and

$$\Xi = \Xi \cap \bigcup_{i=1}^{l} (z_i - M\varepsilon|\log\varepsilon|, z_i + M\varepsilon|\log\varepsilon|), \tag{4.3}$$

provided that $\varepsilon$ is sufficiently small. We should note that, if $m_i = 1$, then $u_\varepsilon$ forms a single-layer near $z_i$, while, if $m_i \geq 2$, then $u_\varepsilon$ forms a multi-layer near $z_i$.

In the case that $m_i = 1$ for each $i = 1, 2, \ldots, l$, the stability or instability of $u_\varepsilon$ has been established by Angenent, Mallet-Paret and Peletier [3] and Hale and Sakamoto [4].

**Proposition 4.1** ([3], [4]). *Let $u_\varepsilon$ be a solution of* (1.3) *satisfying* (4.1), (4.2) *and* (4.3) *with $m_i = 1$ for every $i = 1, 2, \ldots l$. Then the following statements hold true:*
*(i) If $u'_\varepsilon(z_i)a'(z_i) < 0$ for all $i$, then $u_\varepsilon$ is stable.*
*(ii) If $u'_\varepsilon(z_i)a'(z_i) > 0$ for all $i$, then $u_\varepsilon$ is unstable. Furthermore,*

$$\textit{the Morse index of } u_\varepsilon = l.$$

We will discuss stability properties of a solution $u_\varepsilon$ in the case where $m_i \geq 1$. The stability property of such $u_\varepsilon$ is described as follows:

**Theorem 4.2.** *Let $u_\varepsilon$ be a solution of* (1.3). *Assume that there exist a positive constant $M$ and a subset $\{z_i\}_{i=1}^l$ of $\Sigma$, which satisfy* (4.1), (4.2) *and* (4.3). *Then the following assertions hold true:*
*(i) If $m_i = 1$ and $u'_\varepsilon(z_i)a'(z_i) < 0$ for all $i = 1, 2, \ldots, l$, then $u_\varepsilon$ is stable.*
*(ii) If there exists an $i \in \{1, 2, \ldots, l\}$ which satisfies either $m_i \geq 2$ or $m_i = 1$*

*with $u'_\varepsilon(z_i)a'(z_i) > 0$, then $u_\varepsilon$ is unstable. Furthermore, $u_\varepsilon$ is non-degenerated and*

$$\text{the Morse index of } u_\varepsilon = \sum_{i\in\{1,2,\ldots,l\}\setminus\mathscr{I}} m_i,$$

*where*

$$\mathscr{I} := \{i \in \{1,2,\ldots,l\}\,;\, m_i = 1 \text{ and } u'_\varepsilon(z_i)a'(z_i) < 0\}\,.$$

**Remark 4.3.** Proposition 4.1 is a special case of Theorem 4.2; so Theorem 4.2 is generalization of Proposition 4.1.

**Remark 4.4.** The same result as Theorem 4.2 has been obtained by Ai, Chen and Hastings [1] with use of Sturm's comparison theorem (Proposition 3.1). In this paper, we will show a different approach based on the Courant min-max principle (Proposition 3.2).

We will discuss the simplest case, $l = 1$, in Theorem 4.2. We should note that $m_1 = 1$ implies that $u_\varepsilon$ has only one single-layer, while $m_1 \geq 2$ implies that $u_\varepsilon$ has only one multi-layer in $(0,1)$. We will prove the following theorem in place of Theorem 4.2:

**Theorem 4.5.** *Under the same assumptions as in Theorem 4.2 with $l = 1$ and $m_1 = m \geq 2$, $u_\varepsilon$ is non-degenerate and unstable. Furthermore, the Morse index of $u_\varepsilon$ is exactly $m$.*

In what follows, we denote the $j$-th eigenvalue of (1.4) by $\lambda_j$. By virtue of Proposition 3.1, it is sufficient to show the following two lemmas to prove Theorem 4.5:

**Lemma 4.6.** *Under the same assumptions as in Theorem 4.5, it holds that*

$$\lambda_m < 0.$$

**Lemma 4.7.** *Under the same assumptions as in Theorem 4.5, it holds that*

$$\lambda_{m+1} > 0.$$

We will give the essential idea of proofs of Lemmas 4.6 and 4.7. For details, see [9].

*Proof of Lemma 4.6.* We will consider the case that $a'(z_1) > 0$. It follows from Theorem 2.7 that $u_\varepsilon$ forms a multi-layer from 0 to 1 near $z_1$. Since $u_\varepsilon$ and $a$ have $2m-1$ intersecting points in $(z_1 - M\varepsilon|\log\varepsilon|, z_1 + M\varepsilon|\log\varepsilon|)$, we can denote these points by $\{\xi_k\}_{k=1}^{2m-1}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{2m-1} < 1$. In this case, there exist critical points $\{\zeta_k\}_{k=0}^{2m-1}$ of $u_\varepsilon$ satisfying

$$0 = \zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_{2m-1} < \zeta_{2m-1} = 1.$$

Define $\{w_k\}_{k=1}^{m}$ by

$$w_k(x) := \begin{cases} u_\varepsilon'(x) & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\ 0 & \text{in } (0,1) \setminus (\zeta_{2k-2}, \zeta_{2k-1}). \end{cases}$$

Then $\{w_k\}_{k=1}^{m}$ is a family of linearly independent functions in $H^1(0,1)$ and $(w_j, w_k)_{L^2(0,1)} = 0$ for $j \neq k$. Note that $w_k$ satisfy

$$\varepsilon^2 w_k'' + f_u(x, u_\varepsilon) w_k + f_x(x, u_\varepsilon) = 0 \qquad \text{in } (\zeta_{2k-2}, \zeta_{2k-1}). \tag{4.4}$$

Taking $L^2(\zeta_{2k-2}, \zeta_{2k-1})$-inner product of (4.4) with $w_k$, we get

$$\mathscr{H}(w_k) = -\int_{\zeta_{2k-2}}^{\zeta_{2k-1}} a'(x) u_\varepsilon(x)(1 - u_\varepsilon(x)) u_\varepsilon'(x) dx.$$

Since $a$ is monotone increasing in $(z_1 - M\varepsilon|\log\varepsilon|, z_1 + M\varepsilon|\log\varepsilon|)$, it is easy to see

$$\mathscr{H}(w_k) < 0 \tag{4.5}$$

for $k = 2, \ldots, m-1$.

It should be noted that $a'(x)$ is not necessarily positive in $(\zeta_0, \zeta_1)$ and $(\zeta_{2m-2}, \zeta_{2m-1})$. However, we can show that both $\mathscr{H}(w_1)$ and $\mathscr{H}(w_m)$ are negative without the monotonicity condition of $a$. For the proofs, see [9].

Thus $\mathscr{H}(w_k) < 0$ for every $k = 1, 2, \ldots, m$. This fact together with Proposition 3.2 implies $\lambda_m < 0$. □

We now show Lemma 4.7. For this purpose, we will introduce auxiliary eigenvalue problems as follows:

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi = \lambda\phi & \text{in } J_k^+ := (\zeta_{2k-2}, \zeta_{2k-1}), \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0, & k = 1, 2, \ldots, m, \end{cases} \tag{4.6}$$

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi = \lambda\phi & \text{in } J_k^- := (\zeta_{2k-1}, \zeta_{2k}), \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0 & k = 1, 2, \ldots, m-1. \end{cases} \tag{4.7}$$

It should be noted that $u_\varepsilon'$ is positive in $J_k^+$, while $u_\varepsilon'$ is negative in $J_k^-$. We denote the $j$-th eigenvalue of (4.6) (resp. (4.7)) by $\lambda_j(J_k^+)$ for $k = 1, 2, \ldots, m$ (resp. $\lambda_j(J_k^-)$ for $k = 1, 2, \ldots, m-1$).

For (4.6) and (4.7), we can show the following two lemmas:

**Lemma 4.8.** *For each $k = 1, 2, \ldots, m$, it holds that*

$$\lambda_1(J_k^+) < 0 < \lambda_2(J_k^+).$$

**Lemma 4.9.** *For each $k = 1, 2, \ldots, m-1$, it holds that*

$$\lambda_1(J_k^-) > 0.$$

Before giving proofs of Lemmas 4.8 and 4.9, we will prove Lemma 4.7, which is essential in our analysis.

*Proof of Lemma 4.7.* Let $\phi_{1,k}^+$ be the first eigenfunction of (4.6) and set

$$\mathscr{H}_k^{\pm}(\phi) := \int_{J_k^{\pm}} \left\{\varepsilon^2|\phi'(x)|^2 - f_u(x, u_\varepsilon(x))|\phi(x)|^2\right\}dx.$$

For each $k = 1, 2, \ldots, m$, take any $w_k \in H^1(J_k^+) \setminus \{0\}$ satisfying

$$\int_{J_k^+} w_k(x)\phi_{1,k}^+(x)dx = 0.$$

Then, it follows from Lemma 4.8 that

$$\lambda_2(J_k^+)\int_{J_k^+} |w_k(x)|^2 dx \le \mathscr{H}_k^+(w_k).$$

We extend $\phi_{1,k}^{+}$ to $\psi_k \in L^2(0,1)$ by

$$\psi_k(x) := \begin{cases} \phi_{1,k}^{+} & \text{in } J_k^{+}, \\ 0 & \text{in } (0,1) \setminus J_k^{+}. \end{cases} \tag{4.8}$$

For any $w \in X[\psi_1, \psi_2, \ldots, \psi_m]$, it follows from (4.8) that

$$(w, \psi_k)_{L^2(0,1)} = \int_{J_k^+} w(x)\phi_{1,k}^{+}(x)dx = 0.$$

Hence we have

$$\mathscr{H}_k^{+}(w) \geq \lambda_2(J_k^{+}) \int_{J_k^+} |w_k(x)|^2 dx > 0.$$

On the other hand, Lemma 4.9 yields

$$0 < \lambda_1(J_k^{-}) \int_{J_k^-} |w(x)|^2 dx \leq \mathscr{H}_k^{-}(w),$$

for $k = 1, 2, \ldots, m-1$. Therefore, one can see that

$$\begin{aligned} \mathscr{H}(w) &= \sum_{k=1}^{m} \mathscr{H}_k^{+}(w) + \sum_{k=1}^{m-1} \mathscr{H}_k^{-}(w) \\ &\geq \sum_{k=1}^{m} \lambda_2(J_k^{+}) \int_{J_k^+} |w(x)|^2 dx + \sum_{k=1}^{m-1} \lambda_1(J_k^{-}) \int_{J_k^-} |w(x)|^2 dx \\ &\geq \lambda^* \int_0^1 |w(x)|^2 dx, \end{aligned}$$

where

$$\lambda^* := \min \left\{ \min_{k=1,2,\ldots,m} \lambda_2(J_k^{+}), \min_{k=1,2,\ldots,m-1} \lambda_1(J_k^{-}) \right\} > 0.$$

Thus we can conclude by Proposition 3.2 that

$$\lambda_{m+1} = \sup_{\psi_1,\ldots,\psi_m} \inf_{w \in X[\psi_1,\ldots,\psi_m]} \frac{\mathscr{H}(w)}{||w||_{L^2(0,1)}} \geq \lambda^* > 0.$$

□

We next discuss Lemmas 4.8 and 4.9. However, their proofs require quite lengthly argument. So we will only give the outline of proofs. For the complete proofs, see [9].

*Outline of the proof of Lemma 4.8.* By virtue of Propositions 3.1, 3.2 and 3.4, it suffices to show the existence of a pair of functions $A \in C(J_k^+)$ and $w \in C^2(J_k^+)$ with the following properties:
(i) $A$ and $w$ satisfy the following equation:

$$\begin{cases} -\varepsilon^2 w'' + A(x)w = 0 & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \\ w'(\zeta_{2k-2}) = w'(\zeta_{2k-1}) = 0, & \\ -f_u(x, u_\varepsilon) \geq A(x) & \text{in } (\zeta_{2k-2}, \zeta_{2k-1}), \end{cases} \tag{4.9}$$

(ii) $w$ has only one zero point in $(\zeta_{2k-2}, \zeta_{2k-1})$.

Take a small number $\delta > 0$ and let $g$ be a smooth function satisfying

$$g(x) = \begin{cases} 1 & \text{for } |x| \leq \delta, \\ 0 & \text{for } |x| \geq 2\delta, \end{cases}$$

and $|g(x)| \leq 1$ for any $x \in \mathbb{R}$. We introduce a cut-off function $\rho$ by

$$\rho(x) := g\left(\frac{x - z_{2k-1}}{\varepsilon}\right) \qquad \text{in } J_k^+.$$

Furthermore, let $\varphi$ be a $C^2$-function which satisfying

$$\begin{cases} -\varepsilon^3\varphi'' - (1/2 - a(x) + 2a(x)u_\varepsilon - u_\varepsilon^2)\varphi \\ \qquad +(u_\varepsilon^2 - u_\varepsilon + 1/2)(1/2 - a(x)) = 0 \quad \text{in } (z_{2k-1} - 2\varepsilon\delta, z_{2k-1} + 2\varepsilon\delta), \\ \varphi(z_{2k-1} - 2\varepsilon\delta) = \varphi(z_{2k-1} + 2\varepsilon\delta) = 0, \\ \sup\{|\varphi(x)|\,;\, x \in (z_{2k-1} - 2\varepsilon\delta, z_{2k-1} + 2\varepsilon\delta)\} = O(|\log\varepsilon|). \end{cases} \tag{4.10}$$

We should note that such $\varphi$ can be constructed by super and subsolution method.

We are ready to define $w$ and $A$ by

$$w(x) := u_\varepsilon(x) - \frac{1}{2} + \varepsilon\rho(x)\varphi(x)$$

and

$$A(x) := -\frac{\varepsilon^2 w''(x)}{w(x)}.$$

Then one can prove by direct calculations that $A$ and $w$ fulfill properties (i) and (ii). □

*Outline of the proof of Lemma 4.9.* For each $k = 1, 2, \ldots, m-1$, we consider the following eigenvalue problem.

$$\begin{cases} -\varepsilon^2\phi'' - f_u(x, u_\varepsilon)\phi + \dfrac{e^{-1/\varepsilon}}{\psi}\phi = \mu\phi & \text{in } J_k^-, \\ \phi'(\zeta_{2k-1}) = \phi'(\zeta_{2k}) = 0, \end{cases} \tag{4.11}$$

where $\psi$ is a $C^2$-function satisfying

$$\begin{cases} \varepsilon^2\psi'' + f_u(x, u_\varepsilon)\psi - e^{-1/\varepsilon} = 0 & \text{in } J_k^-, \\ \psi'(\zeta_{2k-1}) = \psi'(\zeta_{2k}) = 0, & \\ \psi < 0 & \text{in } J_k^-. \end{cases} \tag{4.12}$$

The existence of such $\psi$ is not trivial. However, if (4.12) has a solution $\psi$, then $\psi$ is an eigenfunction corresponding to zero eigenvalue of (4.11). Clearly, 0 is the first eigenvalue of (4.11) because $\psi$ does not change its sign in $J_k^-$. Furthermore, the third term of the first equation of (4.12) is negative. Hence, Proposition 3.4 enables us to derive $\lambda_1(J_k^-) > 0$. Therefore, we have only to show the existence of a solution of (4.12).

We will take a super and subsolution method to solve (4.12). Set

$$\overline{\psi}(x) := 0 \qquad \text{in } J_k^-;$$

clearly $\overline{\psi}$ is a supersolution of (4.12).

We will construct a subsolution of (4.12). We only discuss for $x \geq \xi_{2k}$ because the argument for $x \leq \xi_{2k}$ is essentially the same. It should be noted that there exists a positive constants $\kappa$ and $P$ such that

$$f_u(x, u_\varepsilon(x)) \leq -P \quad \text{in } (\xi_{2k} + \kappa\varepsilon, \zeta_{2k}) \tag{4.13}$$

when $\varepsilon$ is sufficiently small. We set $\theta(z) = q(z)e^z$ with $q(z) = z^2/(z^2+1)$ and introduce

$$\eta(x) = \begin{cases} 0 & \text{in } (\xi_{2k}, \xi_{2k}+\kappa\varepsilon), \\ \varepsilon^{K_1}\theta\left(\dfrac{K_2(x-\xi_{2k}-\kappa\varepsilon)}{\varepsilon}\right) & \text{in } (\xi_{2k}+\kappa\varepsilon, \zeta_{2k}]. \end{cases} \tag{4.14}$$

Here, $K_1$ is a sufficiently large positive number and $K_2$ is a positive constant satisfying $(1+\gamma)K_2^2 < P$ with small $\gamma > 0$. We define

$$\underline{\psi}(x) := u_\varepsilon'(x) - \eta(x) \quad \text{in } [\xi_{2k}, \zeta_{2k}]$$

and

$$z^* := \inf\{x \in [\xi_{2k}, \zeta_{2k}]\,;\, \underline{\psi}'(x) = 0\}.$$

If $z^* \le \zeta_{2k}$, then it is easy to show that $\underline{\psi}$ is a subsolution of (4.12) by direct calculation. On the other hand, if $z^* > \zeta_{2k}$, the argument is somewhat complicated. For details, see [7]

Finally, it is obvious that

$$\underline{\psi} < \overline{\psi} \qquad \text{in } J_k^-.$$

Thus there exists a solution $\psi$ of (4.12) satisfying $\underline{\psi} < \psi < \overline{\psi}$ in $J_k^-$. □

We are ready to show Theorem 4.2.

*Proof of Theorem 4.2.* From the proof of Theorem 4.5, it is sufficient to sum up the number of layers at each multi-layer. Thus the proof is complete. □

# References

[1] S. Ai, X. Chen, and S. P. Hastings, *Layers and spikes in non-homogeneous bistable reaction-diffusion equations*, to appear in Trans. Amer. Math. Soc.

[2] S. B. Angenent, J. Mallet-Paret, and L. A. Peletier, *Stable transition layers in a semilinear boundary value problem*, J. Differential Equations, **67**(1987), 212–242.

[3] J. K. Hale and K. Sakamoto, *Existence and stability of transition layers*, Japan J. Appl. Math., **5**(1988), 367–405.

[4] K. Nakashima, *Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation*, J. Differential Equations, **191**(2003), 234–276.

[5] K. Nakashima, *Stable transition layers in a balanced bistable equation*, Differential Integral Equations, **13**(2000), 1025–1238.

[6] M. Urano, K. Nakashima and Y. Yamada, *Transition layers and spikes for a bistable reaction-diffusion equation*, to appear in Adv. Math. Sci. Appl.

[7] M. Urano, K. Nakashima and Y. Yamada, *Stability of solutions with transition layers for a bistable reaction-diffusion equation*, preprint.