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Optimal non-projective ternary linear codes

Mito Takenaka[a](竹中みと), Kei Okamoto[b](岡本けい),
Tatsuya Maruta[b] (丸田辰哉)

[a]*Department of Applied Mathematics*
Osaka Women's University
(大阪女子大学大学院 理学研究科)
[b]*Department of Mathematics and Information Sciences*
Osaka Prefecture University
(大阪府立大学大学院 理学系研究科)

Abstract

We prove the existence of a $[406,6,270]_3$ code and the nonexistence of linear codes with parameters $[458,6,304]_3$, $[467,6,310]_3$, $[471,6,313]_3$, $[522,6,347]_3$. These yield that $n_3(6,d)=g_3(6,d)$ for $268\leq d\leq 270$, $n_3(6,d)=g_3(6,d)+1$ for $d\in\{280-282,304-306,313-315,347,348\}$, $n_3(6,d)=g_3(6,d)$ or $g_3(6,d)+1$ for $298\leq d\leq 301$ and $n_3(6,d)=g_3(6,d)+1$ or $g_3(6,d)+2$ for $310\leq d\leq 312$, where $n_q(k,d)$ denotes the minimum length n for which an $[n,k,d]_q$ code exists and $g_q(k,d)=\sum_{i=0}^{k-1}\lceil d/q^i\rceil$.

1. Introduction

Let $V(n,q)$ denote the vector space of n-tuples over $\mathrm{GF}(q)$, the Galois field of order q. A q-ary linear code $\mathcal{C}$ of length n and dimension k is a k-dimensional subspace of $V(n,q)$. The Hamming distance $d(\boldsymbol{x},\ \boldsymbol{y})$ between two vectors $\boldsymbol{x},\boldsymbol{y}\in V(n,q)$ is the number of nonzero coordinate positions in $\boldsymbol{x}-\boldsymbol{y}$. Now the minimum distance of a linear code $\mathcal{C}$ is defined by $d(\mathcal{C})=\min\{d(\boldsymbol{x},\boldsymbol{y})\mid \boldsymbol{x},\boldsymbol{y}\in\mathcal{C},\ \boldsymbol{x}\neq\boldsymbol{y}\}$ which is equal to the minimum weight of $\mathcal{C}$ defined by $wt(\mathcal{C})=\min\{wt(\boldsymbol{x})\mid \boldsymbol{x}\in\mathcal{C},\ \boldsymbol{x}\neq 0\}$, where $\mathbf{0}$ is the all-0-vector and $\mathrm{wt}(\boldsymbol{x})=d(\boldsymbol{x},\mathbf{0})$ is the weight of $\boldsymbol{x}$. A q-ary linear code of length n, dimension k and minimum distance d is referred to as an $[n,k,d]_q$ code. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. A $k \times n$ matrix having as rows the vectors of a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$.

A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists ([13]). An $[n,k,d]_q$ code is called *optimal* if $n = n_q(k,d)$. There is a natural lower bound on $n_q(k,d)$, the so-called Griesmer bound ([8],[25]):

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k. For ternary linear codes, $n_3(k,d)$ is known for $k \leq 5$ for all d. As for the case $k = 6$, the value of $n_3(6,d)$ is unknown for many integers d ([1],[4],[5],[9],[10], [17],[20],[22]). See [2] or [24] for the updated table of $n_q(k,d)$ for some small q. A linear code $\mathcal{C}$ with a generator matrix G is called *projective* if any two columns of G are independent, equivalently, if the dual code of $\mathcal{C}$ has the minimum distance > 2.

We concentrate ourselves to find optimal ternary linear codes of dimension 6 with the minimum distance $d > 243$, which are necessarily non-projective. For $d \geq 244$, it is only known ([20]) that $n_3(6,d) = g_3(6,d) + 1$ for $349 \leq d \leq 351$ and that $n_3(6,d) = g_3(6,d)$ for $d \geq 352$. The existence of an $[n_1,k,d_1]_q$ code and an $[n_2,k,d_2]_q$ code trivially implies the existence of an $[n_1+n_2,k,d_1+d_2]_3$ code. For example, one can get a $[372,6,246]_3$ code from a $[56,6,36]_3$ code and a $[316,6,210]_3$ code. Similarly one can get $[g_3(6,d),6,d]_3$ codes for $d \in \{244-252, 271-279, 322-330, 334-336\}$, $[g_3(6,d)+1,6,d]_3$ codes for $d \in \{253-270, 331-333, 337-351\}$ and $[g_3(6,d)+2,6,d]_3$ codes for $280 \leq d \leq 315$ from the known $n_3(6,d)$ table. We also have $[g_3(6,d),6,d]_3$ codes for $316 \leq d \leq 321$ by Theorem 2.1 in [13] and a $[474,6,315]_3$ code by Theorem 4.5 in [12] from a $[158,5,105]_3$ code. On the other hand, the nonexistence of $[n,5,d]_3$ codes for $(n,d) \in \{(143,94), (144,95), (145,96), (147,97), (148,98), (149,99)\}$ implies $n_3(6,d) \geq g_3(6,d)+1$ for $280 \leq d \leq 297$, for the residual code (see [13]) of each $[g_3(6,d),6,d]_3$ code with respect to a codeword with weight d can not exist. Hence we obtain the following.

Theorem 1.1.
(1) $n_3(6,d) = g_3(6,d)$ for $d \in \{244-252, 271-279, 316-330, 334-336\}$ and for $d \geq 352$.
(2) $n_3(6,d) = g_3(6,d)+1$ for $349 \leq d \leq 351$.
(3) $n_3(6,d) = g_3(6,d)$ or $g_3(6,d)+1$ for $d \in \{253-270, 313-315, 331-333, 337-348\}$.
(4) $n_3(6,d) = g_3(6,d)+1$ or $g_3(6,d)+2$ for $280 \leq d \leq 297$.
(5) $g_3(6,d) \leq n_3(6,d) \leq g_3(6,d)+2$ for $298 \leq d \leq 312$.

We improve Theorem 1.1 for $d \in \{268-270, 280-282, 298-301, 304-306, 310-315, 347, 348\}$ as follows.

Theorem 1.2. (1) $n_3(6,d) = g_3(6,d)$ for $268 \le d \le 270$.
(2) $n_3(6,d) = g_3(6,d)+1$ for $d \in \{280-282, 304-306, 313-315, 347, 348\}$.
(3) $n_3(6,d) = g_3(6,d)$ or $g_3(6,d)+1$ for $298 \le d \le 301$.
(4) $n_3(6,d) = g_3(6,d)+1$ or $g_3(6,d)+2$ for $310 \le d \le 312$.

To prove Theorem 1.2, we need to show the following theorems.

Theorem 1.3. *There exist a* $[406, 6, 270]_3$ *code.*

Theorem 1.4. *There exists no* $[g_3(6,d), 6, d]_3$ *code for* $d = 304, 310, 313, 347$.

We prove Theorem 1.4 in Section 4 and Theorems 1.3 and 1.2 in Section 5.

2. Preliminaries

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathrm{GF}(q)$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r,q)$. 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes* respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r,q)$ and denote by θ_j the number of points in a j-flat, i.e.

$$\theta_j = (q^{j+1}-1)/(q-1).$$

Let $\mathcal{C}$ be an $[n,k,d]_q$ code which does not have any coordinate position in which all the codewords have a zero entry. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted also by $\mathcal{C}$. We see linear codes from this geometrical point of view. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{C}$ and let C_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. For any subset S of Σ we define *the multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of points in T for a subset T of Σ. When the code is *projective*, i.e. when $\gamma_0 = 1$, the multiset $\mathcal{C}$ forms an n-set in Σ and the above $m_{\mathcal{C}}(S)$ is equal to $|\mathcal{C} \cap S|$. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane*, a *t-solid* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that

$$\begin{aligned} n &= m_{\mathcal{C}}(\Sigma), \\ n-d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n,k,d]_q$ code in the natural manner. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. Clearly we have $\gamma_{k-2} = n-d,\ \gamma_{k-1} = n$.

Lemma 2.1 ([22]). (1) *Let Π be an $(s-1)$-flat in Σ, $2 \le s \le k-1$, with $m_{\mathcal{C}}(\Pi) = w$. For any $(s-2)$-flat δ in Π, we have*

$$m_{\mathcal{C}}(\delta) \le \gamma_{s-1} - \frac{n-w}{\theta_{k-s}-1}.$$

In particular for $0 \le j \le k-3$,

$$\gamma_j \le \gamma_{j+1} - \frac{n-\gamma_{j+1}}{\theta_{k-2-j}-1}.$$

(2) *Let δ_1 and δ_2 be distinct t-flats in a fixed $(t+1)$-flat Δ in Σ, $1 \le t \le k-2$. Then*

$$m_{\mathcal{C}}(\delta_1) + m_{\mathcal{C}}(\delta_2) \ge m_{\mathcal{C}}(\Delta) - (q-1)\gamma_t + q{\cdot}m_{\mathcal{C}}(\delta_1 \cap \delta_2).$$

When $\mathcal{C}$ attains the Griesmer bound, γ_j's are uniquely determined as follows.

Lemma 2.2 ([19]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code attaining the Griesmer bound. Then it holds that*

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \ \text{ for } 0 \le j \le k-1.$$

By Lemma 2.2 every $[n,k,d]_q$ code attaining the Griesmer bound is projective if $d \le q^{k-1}$. Denote by a_i the number of hyperplanes Π in Σ with $m_{\mathcal{C}}(\Pi) = i$ and by λ_s the number of s-points in Σ. Note that we have $\lambda_2 = \lambda_0 + n - \theta_{k-1}$ when $\gamma_0 = 2$. The list of a_i's is called the *spectrum* of $\mathcal{C}$. Simple counting arguments yield the following.

Lemma 2.3. (1) $\sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}$. (2) $\sum_{i=1}^{\gamma_{k-2}} i a_i = n\theta_{k-2}$.

(3) $\sum_{i=2}^{\gamma_{k-2}} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s$.

Lemma 2.4 ([22]). *Let Π be an i-hyperplane through a t-secundum δ with $t = \gamma_{k-3}(\Pi)$. Then*

(1) $t \leq \gamma_{k-2} - \dfrac{n-i}{q} = \dfrac{i + q\gamma_{k-2} - n}{q}$.

(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \geq i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

(3) $t = \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \geq i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor + 1$ *does not exist.*

(4) Let c_j be the number of j-hyperplanes through δ other than Π. Then the following equality holds:

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \tag{2.1}$$

(5) For a γ_{k-2}-hyperplane Π_0 with spectrum $(\tau_0, \cdots, \tau_{\gamma_3})$, $\tau_t > 0$ *holds if* $i+q\gamma_{k-2}-n-qt < q$.

The code obtained by deleting the same coordinate from each codeword of C is called a *punctured code* of C. If there exists an $[n+1, k, d+1]_q$ code C' which gives C as a punctured code, C is called *extendable* (to C') and C' is an *extension* of C.

Let C be an $[n, k, d]_q$ code with $k \geq 3$, $\gcd(q, d) = 1$. Define

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv 0, d \pmod q} A_i,$$

where the notation $x|y$ means that x is a divisor of y. The pair (Φ_0, Φ_1) is called the *diversity* of C ([21]).

Theorem 2.5 ([14]). *Let C be an $[n, k, d]_q$ code with diversity (Φ_0, Φ_1), $gcd(q, d) = 1$, $k \geq 3$. Then C is extendable if $\Phi_1 = 0$.*

See [23] for the extendability of ternary linear codes in detail. Note that $a_i = A_{n-i}/(q-1)$ for $0 \leq i \leq \gamma_{k-2}$. Hence the above diversity is given as

$$\Phi_0 = \sum_{i \equiv n \pmod 3} a_i, \quad \Phi_1 = \sum_{i \not\equiv n, n-d \pmod 3} a_i.$$

The following is known as the Ward's divisibility theorem.

Theorem 2.6 ([26]). *Let C be an $[n, k, d]_p$ code, p a prime, attaining the Griesmer bound. If $p^e|d$, then p^e is a divisor of all nonzero weights of C.*

3. The spectra of some ternary linear codes of dimension $k \le 5$

We supply the results about the possibilities of spectra for some ternary linear codes of dimension $k \le 5$ which we need to prove Theorem 1.4 in the next section.

An f-set F in $\mathrm{PG}(r,q)$ satisfying

$$m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{r-1}\}$$

is called an $\{f,m;r,q\}$-*minihyper*. When an $[n,k,d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points C_0 forms a $\{\theta_{k-1} - n, \theta_{k-2} - (n-d); k-1, q\}$-minihyper, where $\theta_j = (q^{j+1}-1)/(q-1)$. The following lemma can be obtained from the classification of some minihypers by Hamada [11].

Lemma 3.1. (1) *The spectrum of a* $[80,5,53]_3$ *code is* $(a_0, a_{26}, a_{27}) = (1,40,80)$.
(2) *The spectrum of a* $[81,5,54]_3$ *code is* $(a_0, a_{27}) = (1,120)$.
(3) *The spectrum of a* $[104,5,69]_3$ *code is* $(a_{26}, a_{32}, a_{35}) = (4,13,104)$.
(4) *The spectrum of a* $[107,5,71]_3$ *code is* $(a_{26}, a_{27}, a_{35}, a_{36}) = (1,3,39,78)$.
(5) *The spectrum of a* $[108,5,72]_3$ *code is* $(a_{27}, a_{36}) = (4,117)$.
(6) *The spectrum of a* $[113,5,75]_3$ *code is* $(a_{32}, a_{35}, a_{38}) = (1,24,96)$.
(7) *The spectrum of a* $[116,5,77]_3$ *code is* $(a_{35}, a_{36}, a_{38}, a_{39}) = (4,9,36,72)$.
(8) *The spectrum of a* $[117,5,78]_3$ *code is* $(a_{36}, a_{39}) = (13,108)$.

Since a $[\theta_{k-1} - e, k, q^{k-1} - e]_3$ code $(0 \le e \le 2)$ is projective, the set of 0-points C_0 consists of e points. Hence the following lemma follows.

Lemma 3.2. Assume $k \ge 3$ and put $u = \theta_{k-2}$.
(1) *The spectrum of a* $[\theta_{k-1} - 2, k, q^{k-1} - 2]_3$ *code is*

$$(a_{u-2}, a_{u-1}, a_u) = (\theta_{k-3}, (\theta_{k-1} - \theta_{k-3})/2, (\theta_{k-1} - \theta_{k-3})/2).$$

(2) *The spectrum of a* $[\theta_{k-1} - 1, k, q^{k-1} - 1]_3$ *code is* $(a_{u-1}, a_u) = (\theta_{k-2}, q^{k-1})$.
(3) *The spectrum of a* $[\theta_{k-1}, k, q^{k-1}]_3$ *code is* $a_u = \theta_{k-1}$.

The following lemma relies upon the classification of some optimal ternary linear codes of small length by van Eupen and Lisoněk [7].

Lemma 3.3 ([7]). (1) *The spectrum of a* $[8,3,5]_3$ *code is* $(a_0, a_2, a_3) = (1,4,8)$.
(2) *The spectrum of a* $[9,3,6]_3$ *code is* $(a_0, a_3) = (1,12)$.
(3) *The spectrum of a* $[14,3,9]_3$ *code is either* $(a_4, a_5) = (9,4)$, $(a_2, a_5) = (3,10)$ *or* $(a_3, a_4, a_5) = (3,3,7)$.
(4) *The spectrum of a* $[18,3,12]_3$ *code is* $(a_0, a_6) = (1,12)$ *or* $(a_3, a_6) = (2,11)$.

(5) *The spectrum of a* $[20,3,13]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{2,3,4,5,6,7\}$.
(6) *The spectrum of a* $[10,4,6]_3$ *code is* $(a_1, a_4) = (10,30)$.
(7) *The spectrum of a* $[19,4,12]_3$ *code is* $(a_1, a_4, a_7) = (1,9,30)$.
(8) *The spectrum of a* $[27,4,35]_3$ *code is* $(a_0, a_9) = (1,39)$.
(9) *The spectrum of a* $[32,4,21]_3$ *code is* $(a_8, a_{13}) = (8,32)$.
(10) *The spectrum of a* $[35,4,23]_3$ *code is* $(a_8, a_9, a_{11}, a_{12}) = (1,3,12,24)$.
(11) *The spectrum of a* $[36,4,24]_3$ *code is* $(a_9, a_{12}) = (4,36)$.

Lemma 3.4. *The spectrum of a* $[41,4,27]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{11,12,13,14\}$.

Lemma 3.5. (1) *The spectrum of a* $[52,4,34]_3$ *code satisfies*

$$a_i = 0 \text{ for all } i \notin \{0,7,8,9,16,17,18\}.$$

(2) *The spectrum of a* $[53,4,35]_3$ *code is one of the following:*

(a) $(a_0, a_{17}, a_{18}) = (1,13,26)$, (b) $(a_8, a_9, a_{17}, a_{18}) = (1,1,12,26)$, (c) $(a_9, a_{17}, a_{18}) = (2,13,25)$.

Lemma 3.6. *The spectrum of a* $[59,4,39]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{8,11,14,17,20\}$.

Lemma 3.7. *The spectrum of a* $[122,5,81]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{38,39,40,41\}$.

The following lemma is due to Landjev [18].

Lemma 3.8 ([18]). (1) *The spectrum of a* $[50,4,33]_3$ *code is one of the following:*

(a) $(a_8, a_{14}, a_{17}) = (2,4,34)$, (b) $(a_{11}, a_{14}, a_{17}) = (2,6,32)$, (c) $(a_{14}, a_{17}) = (11,30)$.

(2) *Every* $[49,4,32]_3$ *code is extendable, so* $a_i = 0$ for all $i \notin \{7,8,10,11,13,14,16,17\}$.

Lemma 3.9. *The spectrum of a* $[154,5,102]_3$ *code satisfies* $a_i = 0$ *for all* $i \notin \{25,46,49,52\}$.

Lemma 3.10. (1) *The spectrum of a* $[158,5,105]_3$ *code is* $(a_{26}, a_{50}, a_{53}) = (2,13,106)$.
(2) *Every* $[157,5,104]_3$ *code is extendable.*

We omit the proof of Lemmas 3.1–3.10 here.

Lemma 3.11. (1) *The spectrum of a* $[176,5,117]_3$ *code is either*
(a) $(a_{32}, a_{50}, a_{59}) = (1,8,112)$ *or*
(b) $(a_{41}, a_{50}, a_{59}) = (a, 11-2a, 110+a)$ *for some a with* $0 \le a \le 5$.

(2) *Every* $[175,5,116]_3$ *code is extendable.*

Proof. (1) See [20].
(2) Let $\mathcal{C}$ be a $[175,5,116]_3$ code. Then γ_3-solid has no j-solid for $j<8$ by Lemma 3.6, so $a_i=0$ for all $i<22$ by Lemma 2.1. Hence, by Lemma 2.4, we have

$$a_i=0 \text{ for all } i\notin\{31,32,40,41,49,50,58,59\},$$

which implies that $\mathcal{C}$ is extendable by Theorem 2.5. □

4. Proof of Theorem 1.4

Theorem 4.1. *There exists no* $[458,6,304]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[458,6,304]_3$ code. Then a γ_4-hyperplane has no j-solid for $j<25$ by Lemma 3.9, so $a_i=0$ for all $i<71$ by Lemma 2.1. Hence

$$a_i=0 \text{ for all } i\notin\{80,81,104,107,108,113,116,117,119\text{–}122,134,135,136,152,153,154\}.$$

by Lemma 2.4. Now, let Π be a 104-hyperplane. Then the spectrum of Π is $(\tau_{26},\tau_{32},\tau_{35})=(4,13,104)$ by Lemma 3.1(3), which contradicts Lemma 3.9 (a γ_4-hyperplane has no j-solid for $j=26,32,35$). Hence $a_{104}=0$. Similarly, we get $a_{107}=a_{108}=a_{113}=a_{122}=0$ by Lemmas 3.1(4)(5)(6), 3.7, 3.9. Hence

$$a_i=0 \text{ for all } i\notin\{80,81,116,117,119-121,134-136,152-154\}.$$

Next, let Π_0 be a 154-hyperplane. Since (2.1) with $i=154$ has no solution for $t=25$ and for $t=49$, the spectrum of Π_0 satisfies $a_i=0$ for all $i\notin\{46,52\}$ by Lemma 3.9. Let Δ be a 52-solid in Π_0. Applying Lemma 2.4 to Π_0, (2.1) with $i=52$ has no solution for $t=0,7,8,9,17$. Hence the spectrum of Δ satisfies $a_i=0$ for all $i\notin\{16,18\}$ by Lemma 3.5(1). Let δ be a 16-plane in Δ. Applying Lemma 2.4 to Δ, (2.1) with $i=16$ has no solution for $t=0,1,2,3,5$. Hence the spectrum of δ satisfies $a_i=0$ for all $i\notin\{4,6\}$. But there exists no $[16,3,10]_3$ code with such spectrum (see [7]), a contradiction. This completes the proof. □

Theorem 4.2. *There exists no* $[467,6,310]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[467,6,310]_3$ code. Then a γ_4-hyperplane has no j-solid for $j<25$ by Lemma 3.10, so $a_i=0$ for all $i<71$ by Lemma 2.1. Hence

$$a_i=0 \text{ for all } i\notin\{74,80,81,104,107,108,113,116,117,119-122,146,152-157\}$$

by Lemma 2.4. Let Π be a 108-hyperplane. Then the spectrum of Π is $(\tau_{27},\tau_{36})=(4,117)$ by Lemma 3.1(5), which contradicts Lemma 3.10 (a γ_4-hyperplane has no 27- nor 36-solid).

Hence $a_{108} = 0$. Similarly, we get $a_{81} = a_{113} = a_{116} = a_{117} = a_{119} = a_{120} = a_{121} = a_{122} = 0$ by Lemmas 3.1(2)(6)(7)(8), 3.2, 3.7, 3.10. Hence

$$a_i = 0 \text{ for all } i \notin \{74, 80, 104, 107, 146, 152 - 157\}.$$

Suppose $a_{80} > 0$ and let Π be a 80-hyperplane. Setting $(i, t) = (80, 27)$, (2.1) has no solution since $c_{157} = 0$ (a 157-hyperplane has no 27-solid), which contradicts the spectrum of Π (Lemma 3.1(1)). Hence $a_{80} = 0$. Similarly we get $a_{104} = a_{107} = 0$ by Lemmas 2.1, 2.4, 3.1(3)(4). Hence

$$a_i = 0 \text{ for all } i \notin \{74, 146, 152 - 157\}.$$

Now, let Π_0 be a 157-hyperplane with the spectrum $(\tau_{25}, \tau_{26}, \cdots, \tau_{53})$. Then $\tau_{25} + \tau_{26} = 2$ by Lemma 3.10. Since all the solutions of (2.1) for $i = 157$ are $(c_{74}, c_{154}, c_{157}) = (1,1,1)$ or $(c_{74}, c_{155}, c_{156}) = (1,1,1)$ for $t = 25$; $(c_{74}, c_{157}) = (1,2)$ for $t = 26$, and so on, we obtain

$$a_{74} \geq \tau_{25} + \tau_{26} = 2.$$

On the other hand, it holds that $a_{74} \leq 1$ by Lemma 2.1, a contradiction. This completes the proof. □

Theorem 4.3. *There exists no* $[471, 6, 313]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[471, 6, 313]_3$ code. Then a γ_4-hyperplane has no j-solid for $j < 26$ by Lemma 3.10(1), so $a_i = 0$ for all $i < 75$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{81, 108, 117, 120, 121, 156 - 158\}$$

by Lemma 2.4. Now, let Π be a 158-hyperplane. Then the spectrum of Π is $(\tau_{26}, \tau_{50}, \tau_{53}) = (2, 13, 106)$ by Lemma 3.10(1), but (2.1) has no solution for $(i, t) = (158, 50)$, a contradiction. This completes the proof. □

Theorem 4.4. *There exists no* $[522, 6, 347]_3$ *code.*

Proof. Let $\mathcal{C}$ be a $[522, 6, 347]_3$ code. Then a γ_4-hyperplane has no j-solid for $j < 31$ by Lemma 3.11, so $a_i = 0$ for all $i < 90$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{90, 91, 108, 117 - 122, 162, 171, 172, 174, 175\},$$

by Lemma 2.4. Let Π be a γ_4-hyperplane. Then (2.1) for $i = 175$ has no solution for $t = 49$, 50, which contradicts that the spectrum of Π satisfies $\tau_{49} + \tau_{50} > 0$ by Lemma 3.11. This completes the proof. □

5. Proof of Theorem 1.2

A linear code $\mathcal{C}$ is *w-weight* if $\mathcal{C}$ has exactly w non-zero weights i with $A_i > 0$. The method finding another code (called *projective dual* in [16]) from a given 2-weight code was first found by van Eupen and Hill [6], see also [3]. We consider the projective dual of a 3-weight code with $\gamma_0 = 2$. Recall that λ_i stands for the number of i-points in Σ =PG$(k-1,q)$ defined from $\mathcal{C}$. Considering $(n-d-2m)$-hyperplanes, $(n-d-m)$-hyperplanes and $(n-d)$-hyperplanes of Σ as 2-points, 1-points and 0-points respectively in the dual space Σ^* of Σ, we obtain the following lemma.

Lemma 5.1. *Let $\mathcal{C}$ be a 3-weight $[n,k,d]_q$ code with $q = p^h$, p prime, $\gamma_0 = 2$, whose spectrum is $(a_{n-d-2m}, a_{n-d-m}, a_{n-d}) = (\alpha, \beta, \theta_{k-1} - \alpha - \beta)$, where $m = p^r$ for some $1 \le r < h(k-2)$ satisfying $m|d$ and $\lambda_i > 0$ $(0 \le i \le 2)$. Then there exists a 3-weight $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $n^* = 2\alpha + \beta$, $d^* = 2\alpha + \beta - nt + \frac{d}{m}\theta_{k-2}$ whose spectrum is $(a_{n^*-d^*-2t}, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_2, \lambda_1, \lambda_0)$, where $t = p^{h(k-2)-r}$.*

Proof of Theorem 1.3. Let $\mathcal{C}$ be a $[14,6,6]_3$ code with a generator matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Then the spectrum of $\mathcal{C}$ is $(a_2, a_5, a_8) = (93, 220, 51)$ and we have $(\lambda_2, \lambda_1, \lambda_0) = (1, 12, 351)$. Applying Lemma 5.1 we get a $[406, 6, 270]_3$ code $\mathcal{C}^*$ with the spectrum $(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$. □

Lemma 5.2 ([15]). *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $[n_1, k, d_1]_q$ and $[n_2, k-1, d_2]_q$ codes respectively and assume that $\mathcal{C}_1$ contains a codeword of weight at least $d_1 + d_2$. Then there exists an $[n_1 + n_2, k, d_1 + d_2]_q$ code.*

Applying Lemma 5.2 to a $[406, 6, 270]_3$ code as $\mathcal{C}_1$ and $[20, 5, 12]_3$, $[47, 5, 30]_3$, $[49, 5, 31]_3$, $[55, 5, 36]_3$ codes as $\mathcal{C}_2$, we get $[426, 6, 282]_3$, $[453, 6, 300]_3$, $[455, 6, 301]_3$, $[461, 6, 306]_3$ codes respectively. Hence Theorem 1.2 follows from Theorems 1.1 and 1.4.

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