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BROWN-PETERSON COHOMOLOGY OF $BPU(p)$

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Let p be a fixed odd prime and denote by $BP^*(X)$ (resp. $P(m)^*(X)$) the Brown-Peterson cohomology of a space X with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ (resp. $P(m)^* = \mathbb{Z}/p[v_m, v_{m+1}, \cdots]$) where $\deg v_k = -2p^k + 2$. We denote by $PU(n)$ the projective unitary group which is the quotient of the unitary group $U(n)$ by its center S^1. Recall that the cohomologies of $PU(p)$ and exceptional Lie groups F_4, E_6, E_7, E_8 have odd torsion elements. In this paper, we compute the Brown-Peterson cohomologies of classifying spaces BG of these Lie groups G as BP^*-modules using the Adams spectral sequence. Let us write $H^*(X;\mathbb{Z}/p)$ by simply $H^*(X)$ and let $\mathcal{A}$ be the mod p Steenrod algebra.

Our main result is as follows:

Theorem 0.1. *Let (G,p) be one of cases ($G = PU(p), p$) for an arbitrary odd prime p and $G = F_4, E_7$ for $p = 3$, and $G = E_8$ for $p = 5$. Then the E_2-terms of the Adams spectral sequences abutting to $BP^*(BG)$ and $P(m)^*(BG)$ for $m \geq 1$*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(BP), H^*(BG)), \qquad \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(P(m)), H^*(BG))$$

have no odd degree elements.

As an immediate consequence is as follows:

Corollary 0.2. *For (G,p) in Theorem 1.1, the Adams spectral sequence abutting to $BP^*(BG)$ and $P(m)^*(BG)$ in the previous theorem collapse at the E_2-level. In particular $BP^{odd}(BG) = P(m)^{odd}(BG) = 0$.*

Recall $K(m)^*(X) \cong K(m)^* \otimes_{P(m)^*} P(m)^*(X)$ is the Morava K-theory. From above theorem and corollary, we see $K(m)^{odd}(BPU(p)) = 0$. Then we have the following corollary ([Ko-Ya],[Ra-Wi-Ya])

Corollary 0.3. *For (G,p) in Theorem 1.1, the following holds:*
(1) $BP^*(BG)$ is BP^*-flat for $BP^*(BP)$-modules, i.e.,
$BP^*(BG \times X) \cong BP^*(BG) \otimes_{BP^*} BP^*(X)$ for all finite complexes X
(2) $K(n)^*(BG) \cong K(n)^* \otimes_{BP^*} BP^*(BG)$.
(3) $P(n)^*(BG) \cong P(n)^* \otimes_{BP^*} BP^*(BG)$.

We give the BP^*-module structure of $BP^*(BPU(p))$ more explicitly, in this talk.

Theorem 0.4. *There is a BP^*-algebra isomorphism*

$$0 \to BP^*\hat{\otimes} M \to grBP^*(BPU(p)) \to BP^*\hat{\otimes} IN/(f_0, f_1) \to 0$$

where

1. *$M \cong \mathbb{Z}_{(p)}[x_4, x_6, \cdots, x_{2p}]$ as $\mathbb{Z}_{(p)}$-modules (but not $\mathbb{Z}_{(p)} - algebras$).*
2. *$IN \cong \mathbb{Z}_{(p)}[x_{2p+2}, x_{2p(p-1)}]\{x_{2p+2}\}$; the principal ideal of $\mathbb{Z}[x_{2p+2}, x_{2p(p-1)}]$ generated by x_{2p+2}.*

3. *relations* f_0, f_1 *are given with modulo* $(p, v_1, v_2, \cdots)^2$

$$f_0 \equiv v_0 - v_2 x_{2p+2}^{p-1} + \cdots, \quad f_1 \equiv v_1 - v_2 x_{2p(p-1)} + \cdots.$$

Remark 0.5. *In the above theorem, suffix i of* x_i *means its degree.* $BP^*(BPU(p))$ *does not contain the subalgebra* $BP^*\hat{\otimes}\mathbb{Z}_{(p)}[x_4, \dots, x_{2p}]$*, but contains a subalgebra which is isomorphic as* BP^**-modules to the above* BP*-subalgebra.*

For an algebraic group G over $\mathbb{C}$, Totaro defines its Chow ring [To] and conjectures that $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(BG)_{(p)}$. Recall that $PGL(p, \mathbb{C})$ is the algebraic group over $\mathbb{C}$ corresponding the Lie group $PU(p)$.

Theorem 0.6. *There is the isomorphism*

$$BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(BGL(p, \mathbb{C}))_{(p)}.$$

Hence there is the additive isomorphism

$$CH^*(BGL(p, \mathbb{C}))_{(p)} \cong \mathbb{Z}_{(p)}[x_4, x_6, \cdots, x_{2p}] \oplus \mathbb{F}_p[x_{2p+2}, x_{2p(p-1)}]\{x_{2p+2}\}.$$

Remark. Recently Vistoli [Vi] also determined the additive structure of the Chow ring and integral cohomology of $BPGL(p, \mathbb{F}_p)$ by using stratified methods of Vessozi. Moreover he shows that for $G = PGL(p, \mathbb{C})$

$$H^*(G; \mathbb{Z}) \to H^*(BT; \mathbb{Z})^{W_G(T)}$$

is epic.

Let $MGL^{*,*}(X)$ be the motivic cobordism ring defined by V.Voevodsky [Vo] and $MGL^{2*,*}(X) = \oplus_i MGL^{2i,i}(X)$.

Corollary 0.7. $MGL^{2*,*}(BPGL(p, \mathbb{C}))_{(p)} \cong MU^*(BPU(p))_{(p)}$.

We prove Theorem 1.1 using the Adams spectral sequence converging to the Browen-Peterson cohomology. The E_1-term of the spectral sequence could be given by

$$\mathbb{F}_p[v_0, v_1, \cdots]\hat{\otimes} H^*(X) \quad with \quad d_1 x = \sum_{k=0}^{\infty} v_k Q_k x$$

where Q_k's are Milnor's operations. By the change-of-rings isomorphism, the E_2-term is

$$\mathrm{Ext}_{\mathcal{A}}(H^*(BP), H^*(X)) \cong \mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p, H^*(X))$$

where $\mathcal{E} = \Lambda(Q_0, Q_1, \cdots)$. The E_∞-term is given by $grBP^*(X)$.

To state the cohomology $H^*(BPU(p))$, we recall the Dickson algebra. Let A_n be an elementary abelian p-group of rank n, and

$$H^*(BA_n) \cong \mathbb{F}_p[t_1, ..., t_n] \otimes \Lambda(dt_1, ..., dt_n) \quad with \ \beta(dt_i) = t_i.$$

The Dickson algebra is

$$D_n = \mathbb{F}_p[t_1, ..., t_n]^{GL(n, \mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,0}, ..., c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL(n, \mathbb{F}_p)$ is also given

$$SD_n = \mathbb{F}_p[t_1, ..., t_n]^{SL(n, \mathbb{F}_p)} \cong D_n\{1, e_n, ..., e_n^{p-2}\} \quad with \ e_n^{p-1} = c_{n,0}.$$

We also recall the Mui's ([Mu]) result by using Q_i by [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)} \cong SD_n/(e_n) \oplus SD_n \otimes \Lambda(Q_0, ..., Q_{n-1})\{u_n\}$$

where $u_n = dt_1 ... dt_n$ and $e_n = Q_0 ... Q_{n-1} u_n$.

Theorem 0.8. *There is the short exact sequence*

$$0 \to M/p \to H^*(BPU(p)) \to N \to 0$$

where M/p is the trivial $\mathcal{E}$-module given in Theorem 1.4 and

$$N = SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \cong \mathbb{F}_p[x_{2p+2}, x_{2(p^2-p)}] \otimes \Lambda(Q_0, Q_1)\{u_2\}$$

identifying $x_{2p+2} = e_2$ and $x_{2(p^2-p)} = c_{2,1}$.

This theorem is proved by using the following facts. The group $G = PU(p)$ has just two conjugacy classes of maximal elementary abelian p-subgroups, one of which is toral and the other is non-toral A of $rank_p = 2$. The cohomology $H^*(BG)$ is detected by this two subgroups. The restriction image to the non-toral subgroup is $i_A^*(H^*PU(p)) \cong H^*(BA)^{SL(2,\mathbb{F}_p)}$. Similar (but not same) facts also hold for the exceptional Lie groups given in Theorem 1.1.

Algebraic main result in this talk is as follows:

Theorem 0.9. *For $m \geq 0$, define $f_0, ..., f_{n-1}$ in $P(m)^* \hat{\otimes} SD_n$ by*

$$d_1 u_n = \sum_{k \geq m} v_k Q_k(u_n) = f_0 Q_0 u_n + \cdots + f_{n-1} Q_{n-1} u_n.$$

Then the sequence $f_0, \ldots, f_{n-1}$ is a regular sequence in $P(m)^* \hat{\otimes} SD_n$.

With the notation in this theorem, we prove that the complex

$$C = (P(m)^* \hat{\otimes} SD_n \otimes \Lambda(Q_0, Q_1, \cdots, Q_{n-1})\{u_n\}, d_1)$$

with the differential $d_1 u_n = \sum_{i=0}^{n-1} f_i Q_i u_n$ is a Koszul complex. This means that

$$H_i(C, d_1) = \begin{cases} P(m)^* \hat{\otimes} SD_n\{e_n\}/(f_0, \cdots, f_{n-1}) & for\ i = 0 \\ 0 \quad for\ i \geq 1. \end{cases}$$

Thus Theorem 1.1 follows from the above theorem.

Remark about the convergence of the Adams spectral sequence. By Theorem 15.6 in Boardman's paper [Bo2], since $H^*(BP)$ is of finite type, the above Adams spectral sequence is conditionally convergent. Moreover, since we prove the above Adams spectral sequence collapses at the E_2-level, by the remark after Theorem 7.1 in [Bo1], the above Adams spectral sequence is strongly convergent, so that we know the Brown-Peterson cohomology up to group extension.

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