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# APPLICATIONS OF CR GEOMETRY TO REPRESENTATIONS OF SU$(p,q)$

WEI WANG
Department of Mathematics, Zhejiang University

Conformal geometry on pseudo-Riemannian manifolds can be applied to the representation theory of the group SO$(p,q)$ (cf. [3] [11] [12] [13] [14] and references therein). Kostant used the conformal invariance of the vanishing of scalar curvature on 6 dimensional manifolds to explore the minimal representation of SO$(4,4)$ in [14]. Recently, T. Kobayashi and B. Orsted [11] [12] [13] gave a geometric and intrinsic model of the minimal irreducible unitary representation $\varpi^{p,q}$ of SO$(p,q)$ on $S^{p-1}\times S^{q-1}$ and on various pseudo-Riemannian manifolds which are conformally equivalent, by using the Yamabe operator. They also gave branching formulae and unitarization of various models. Here we use CR geometry to realize representations of SU$(p,q)$.

## 1. Preliminaries on CR Geometry

Let $M$ be a real $(2n+1)$-dimensional orientable $C^\infty$ manifold. A *CR structure* on $M$ is a $n$-dimensional complex subbundle $T_{1,0}M$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $T_{1,0}M\cap T_{0,1}M=\{0\}$, where $T_{0,1}M=\overline{T_{1,0}M}$, and the integrability condition: $[Z_1,Z_2]\in C^\infty(M,T_{1,0}M)$ whenever $Z_1,Z_2\in C^\infty(M,T_{1,0}M)$. $T_{1,0}M$ is usually called the *complex tangential space.* Set

$$H=\mathrm{Re}\{T_{1,0}M\oplus T_{0,1}M\}, \tag{1.1}$$

the $2n$-dimensional *real horizontal subbundle* of $TM$. $H$ carries a *complex structure* $J:H\longrightarrow H$ satisfying $J^2=-\mathrm{id}_H$ and $T_{1,0}M=\ker(J-i\cdot\mathrm{id}_{\mathbb{C}H})$, $T_{0,1}M=\ker(J+i\cdot\mathrm{id}_{\mathbb{C}H})$. When $M$ is the boundary of a domain in a complex manifold $W$, it has an *induced CR structure* from the complex structure of $W$ defined by

$$T_{1,0}M=\mathbb{C}TM\cap T_{1,0}W, \tag{1.2}$$

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if $\dim(T_{1,0}M)_x$ =const. for each $x \in M$, where $T_{1,0}W$ is the holomorphic tangential space of complex manifold $W$.

A mapping $f : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is called a *Cauchy-Riemann mapping* (or *CR mapping* ) if

$$f_* T_{1,0}M_1 \subset T_{1,0}M_2, \tag{1.3}$$

where $f_*$ is the tangential mapping of $f$. If $f$ is invertible, $f$ and $f^{-1}$ are both CR mappings, $f$ is called a *CR diffeomorphism.*

Let $\theta$ be a 1-form on $M$ such that

$$\ker\theta = H. \tag{1.4}$$

We require $\theta$ to be a *contact form* , i.e. $\theta \wedge (d\theta)^n$ is non-vanishing on $M$. Such $\theta$ is called a *pseudohermitian structure* on $(M, T_{1,0}M)$. We call the triple $(M, T_{1,0}M, \theta)$ a *pseudohermitian manifold.* $\theta$ plays the role of metric $g$ in pseudo-Riemannian geometry.

We say $\tilde{\theta}$ is *conformal* to $\theta$ if

$$\tilde{\theta} = \phi^2\theta \tag{1.5}$$

for some non-vanishing smooth function $\phi$ on $M$. A CR mapping between two pseudohermitian manifolds, $f : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$, is called *conformal* if $f^*\theta_2 = \phi^2\theta_1$ for some non-vanishing smooth function $\phi$ on $M_1$.

We can define a Hermitian form on $T_{1,0}M$ associated to a pseudohermitian structure $\theta$ by

$$L_\theta(V, \overline{W}) = -id\theta(V \wedge \overline{W}), \tag{1.6}$$

which is called *the Levi form* of $\theta$.

If the Levi form has $k$ positive eigenvalues and $n-k$ negative eigenvalues, $(M, T_{1,0}M, \theta)$ is said to be *strictly k-pseudoconvex.* The inner product $L_\theta(\cdot,\cdot)$ determines a dual form $L_\theta^*(\cdot,\cdot)$ on $H^*$. $L_\theta^*(\cdot,\cdot)$ can be naturally extended to $T^*M$.

In [19], Webster showed that there exists a natural connection on the bundle $T_{1,0}M$ adapted to a pseudohermitian structure $\theta$. For a pseudohermitian structure $\theta$ on a strictly

$k$-pseudoconvex CR manifold $(M, T_{1,0}M, \theta)$, there is a unique vector field $T$, which is transversal to $H$, defined by

$$\theta(T) = 1, \qquad d\theta(T \wedge \cdot) = 0. \tag{1.7}$$

Let $\theta^\alpha$ be *an admissible coframe*, i.e. $(1,0)$–forms $\theta^\alpha$ form a basis for $T^*_{1,0}$ such that $\theta^\alpha(T) = 0$ for all $\alpha = 1, \cdots, n$. The integrability condition implies

$$d\theta = i g_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{1.8}$$

for some Hermitian matrix of functions $(g_{\alpha\overline{\beta}})$, which is nondegenerate and has $k$ positive eigenvalues and $n-k$ negative eigenvalues if $(M, T_{1,0}M, \theta)$ is strictly $k$-pseudoconvex. Webster showed that there are uniquely determined 1-forms ${\omega_\alpha}^\beta$ and $\tau^\beta$ on $M$ satisfying

$$\begin{cases} d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta \\ \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dg_{\alpha\overline{\beta}} \\ \tau_\alpha \wedge \theta^\alpha = 0, \end{cases} \tag{1.9}$$

where we use $(g_{\alpha\overline{\beta}})$ to raise and lower indices, e.g. $\omega_{\alpha\overline{\beta}} = {\omega_\alpha}^\gamma g_{\gamma\overline{\beta}}$. Let

$${\Omega_\beta}^\alpha = d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha. \tag{1.10}$$

Webster showed that ${\Omega_\beta}^\alpha$ could be written as

$${\Omega_\beta}^\alpha = {R_\beta}^\alpha{}_{\rho\overline{\sigma}} \theta^\rho \wedge \theta^{\overline{\sigma}} + {W_\beta}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\overline{\rho}} \theta^{\overline{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha \tag{1.11}$$

The *Webster-Ricci tensor* of $(M, T_{1,0}M, \theta)$ has components $R_{\alpha\overline{\beta}} = {R_\rho}^\rho{}_{\alpha\overline{\beta}}$. The *Webster scalar curvature* is

$$R_\theta = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}. \tag{1.12}$$

*The CR Yamabe problem* is to find a contact form $\tilde{\theta} = u^2\theta, u > 0$, which is conformal to the given contact form $\theta$, such that $R_{\tilde{\theta}} \equiv$constant. This problem is considered by Lee and Jerison [9] for strictly pseudoconvex CR manifolds and completely solved recently by N. Gamara and R. Yacoub [6] [7].

A pseudohermitian manifold $(M, T_{1,0}M, \theta)$ has a natural volume form

$$\psi_\theta = (-1)^{n-k}\theta \wedge (d\theta)^n, \tag{1.13}$$

which is nowhere vanishing because $M$ is strictly $k$-pseudoconvex. It induces an $L^2$ inner product on functions

$$\langle u, v\rangle_\theta = \int_M u\overline{v}\psi_\theta, \tag{1.14}$$

and an $L^2$ inner product on sections of $H^*$,

$$\langle \omega, \eta\rangle_\theta = \int_M L_\theta^*(\omega, \eta)\psi_\theta. \tag{1.15}$$

For $u \in C^\infty(M)$, we define a section $d_b u$ of $H^*$ by

$$d_b u = pr \circ du, \tag{1.16}$$

where $pr : T^*M \longrightarrow H^*$ is the restriction map. We can define *the SubLaplacian* $\Box_\theta$ associated to a strictly $k$-pseudoconvex contact form $\theta$ by

$$\langle \Box_\theta u, v\rangle_\theta = \frac{1}{2}\langle d_b u, d_b v\rangle_\theta. \tag{1.17}$$

Since evidently, $|\theta|_\theta = 0$, $L_\theta^*(\cdot,\cdot)$ is degenerate on $T^*M$ and so the operator $\Box_\theta$ is a degenerate ultrahyperbolic operator.

**Proposition 1.1.** (Proposition 4.10 in [15]) *If* $u \in C_0^\infty$, *then,*

$$\Box_\theta u = -u_\alpha{}^\alpha - u_{\overline{\alpha}}{}^{\overline{\alpha}}. \tag{1.18}$$

Define a product on $\mathbf{C}^{n+2}$ by

$$(\zeta, \xi)_{p,q} = \sum_{j=0}^{n+1} \varepsilon_j \zeta_j \overline{\xi}_j, \tag{1.19}$$

where $n + 2 = p + q$, and

$$\varepsilon_j = \begin{cases} 1, & \text{for } j = 0, 1, \cdots, p-1, \\ -1, & \text{for } j = p, \cdots, p+q-1. \end{cases} \tag{1.20}$$

We denote $(\zeta, \zeta)_{p,q}$ by $|\zeta|_{p,q}^2$ for $\zeta \in \mathbf{C}^{n+2}$. Similarly, we define a product on $\mathbf{C}^n$ by

$$(z, w)_{p-1,q-1} = \sum_{\alpha=1}^{n} \varepsilon_\alpha z_\alpha \overline{w}_\alpha. \tag{1.21}$$

We also denote $(z, z)_{p-1,q-1}$ by $|z|_{p-1,q-1}^2$ for $z \in \mathbf{C}^n$.

The simplest CR manifold is *the Heisenberg group* $\mathbb{H}^{p-1,q-1}$, whose underlying manifold is $\mathbf{C}^{p+q-2} \times \mathbf{R}$, with coordinates $(z,t)$. Its multiplication is given by

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\mathrm{Im}(z,z')_{p-1,q-1}). \tag{1.22}$$

The vector fields

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} + i\varepsilon_\alpha \overline{z}_\alpha \frac{\partial}{\partial t}, \tag{1.23}$$

$\alpha = 1, \cdots, n$, are left invariant vector fields on $\mathbb{H}^{p-1,q-1}$. The *standard CR structure* on the Heisenberg group $\mathbb{H}^{p-1,q-1}$ is given by the subbundle

$$T_{1,0}\mathbb{H}^{p-1,q-1} = \mathrm{span}_{\mathbf{C}}\{Z_1, \cdots, Z_n\}. \tag{1.24}$$

Let

$$\theta_{\mathbb{H}^{p-1,q-1}} = dt + \sum_{\alpha=1}^{n} i\varepsilon_\alpha (z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha) \tag{1.25}$$

be the *standard contact form* on $\mathbb{H}^{p-1,q-1}$.

$$\Box_{\theta_{\mathbb{H}^{p-1,q-1}}} = -\frac{1}{2}\sum_{\alpha=1}^{p-1}(Z_\alpha\overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha) + \frac{1}{2}\sum_{\alpha=p}^{p+q-2}(Z_\alpha\overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha). \tag{1.26}$$

Let us consider a real hypersurface $Q'_{p,q}$ in $\mathbf{C}^{n+1}$ defined by equation

$$\mathrm{Im} z_0 = |z|^2_{p-1,q-1}, \qquad z \in \mathbf{C}^n, \quad z_0 \in \mathbf{C}, \tag{1.27}$$

which is the boundary of the *Siegel upper half space*

$$\mathcal{S} = \left\{(z_0, z) \in \mathbf{C} \times \mathbf{C}^n; \mathrm{Im}\ z_0 > |z|^2_{p-1,q-1}\right\}. \tag{1.28}$$

The *Cayley transformation* $C$ is defined by

$$w_0 = \frac{z_0 - i}{z_0 + i}, \qquad w_\alpha = \frac{2z_\alpha}{z_0 + i}, \tag{1.29}$$

which transforms the hypersurface $Q'_{p,q}$ into the hyperquadric $Q_{p,q}$,

$$Q_{p,q} = \left\{w = (w_0, w'); w_0 \in \mathbf{C}, w' \in \mathbf{C}^n, |w_0|^2 + |w'|^2_{p-1,q-1} = 1\right\}. \tag{1.30}$$

Now introduce homogeneous coordinates $\zeta_j$, $j = 0, \cdots, n+1$. By equations

$$z_j = \frac{\zeta_j}{\zeta_{n+1}}, \qquad j = 0, \cdots, n+1, \tag{1.31}$$

$\mathbf{C}^{n+1}$ is embedded as an open subset of the complex projective space $\mathbf{CP}^{n+1}$ of dimension $n+1$. In the homogeneous coordinates, $Q_{p,q}$ is embedded as an open subset of the *projective hyperquadric*

$$\overline{Q}_{p,q} = \left\{\zeta = (\zeta_0, \cdots, \zeta_{n+1}) \in CP^{n+1}; |\zeta|^2_{p,q} = 0\right\}. \tag{1.32}$$

Projective hyperquadric $\overline{Q}_{p,q}$ is the compactification of $Q_{p,q}$ in $\mathbf{CP}^{n+1}$. The hypersurface $Q'_{p,q}$ and the projective hyperquadric $\overline{Q}_{p,q}$ have induced CR structures by (1.2) from complex manifolds $\mathbf{C}^{n+1}$and $\mathbf{CP}^{n+1}$, respectively.

$\mathrm{SU}(p,q)$ is the group of unimodular transformations preserving the Hermitian form (1.19). Its center $K$ consists of $n+2$ transformations. Then $\mathrm{SU}(p,q)/K$ acts on $\overline{Q}_{p,q}$ effectively and $\mathrm{PU}(p,q) = \mathrm{SU}(p,q)/K$. It is well known that $\mathrm{Aut}_{CR}\overline{Q}_{p,q} = \mathrm{PU}(p,q)$ [4].

| Pseudo-Riemannian geometry | CR geometry |
|---|---|
| *A metric* $g$ | *A contact form* $\theta$ |
| *conformal* $\tilde{g} = \phi^2 g$ | $\tilde{\theta} = \phi^2\theta$ |
| *pseudo-Riemannian connection* | *Webster connection* |
| *the Laplacian* $\Box_g$ | *the SubLaplacians* $\Box_\theta$ |
| $SO(p,q)$ | $\mathrm{SU}(p,q)$ |
| *the flat model* $\mathbb{R}^{p-1,q-1}$ | $\mathbb{H}^{p-1,q-1}$ |
| $S^{p-1} \times S^{q-1}$ | *the projective hyperquadric* $\overline{Q}_{p,q}$ |
| *the Yamabe operator* | *the CR Yamabe operator* |
| $\vdots$ | $\vdots$ |

## 2. Representations realized as conformal CR diffeomorphisms

Let $Q = \dim M + 1 = 2n+2$, the *homogeneous dimension* of $M$. The following transformation formula is due to Lee.

**Proposition 2.1.** *Let $(M, T_{1,0}M, \theta)$ be a pseudohermitian manifold with* $\dim M = 2n+1$. *The Webster scalar curvature $R_{\tilde{\theta}}$ associated with the pseudohermitian structure $\tilde{\theta} = u^{\frac{4}{Q-2}}\theta$ satisfies*

$$b_n\Box_\theta u + R_\theta u = R_{\tilde{\theta}} u^{\frac{Q+2}{Q-2}}, \tag{2.1}$$

*where* $b_n = 2 + \frac{2}{n}$.

The following is a transformation formula for the SubLaplacians under a conformal CR transformation.

**Proposition 2.2.** *Let $(M_1, T_{1,0}M_1)$ and $(M_2, T_{1,0}M_2)$ be two CR manifolds with strictly k-psedoconvex pseudohermitian structure $\theta_1$ and $\theta_2$, respectively. Suppose $\Phi : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is a CR diffeomorphism with $\Phi^*\theta_2 = u^{\frac{4}{Q-2}}\theta_1$ for some positive smooth function $u$ on $M_1$. Then*

$$\Box_{\theta_1}(u \cdot \Phi^* f) - u^{\frac{Q+2}{Q-2}}\Phi^*\left(\Box_{\theta_2} f\right) = \Box_{\theta_1} u \cdot \Phi^* f, \tag{2.2}$$

*for any smooth real function $f$ on $M_2$.*

Now define the *CR Yamabe operator* to be

$$\widetilde{\Box}_\theta = \Box_\theta + \frac{1}{b_n} R_\theta, \tag{2.3}$$

where $b_n = 2 + \frac{2}{n}$, $R_\theta$ is the Webster scalar curvature (1.12). The transformation formula for the CR Yamabe operator is a consequence of Corollary 2.1 and Proposition 2.2 as follows.

**Proposition 2.3.** *Under the same assumption as in proposition 2.2, we have that*

$$\widetilde{\Box}_{\theta_1}(u \cdot \Phi^* f) = u^{\frac{Q+2}{Q-2}}\Phi^*\left(\widetilde{\Box}_{\theta_2} f\right), \tag{2.4}$$

*for any smooth function $f$ on $M_2$.*

Suppose $(M_1, T_{1,0}M_1, \theta_1)$ and $(M_2, T_{1,0}M_2, \theta_2)$ are two pseudohermitian manifolds of homogeneous dimension $Q$. Let conformal CR mapping $\Phi : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$ be a local diffeomorphism such that

$$\Phi^*\theta_2 = \Omega^2\theta_1, \tag{2.5}$$

for some positive function $\Omega$ on $M_1$. We can define *twisted pull back*

$$\Phi^*_\lambda : C^\infty(M_2) \longrightarrow C^\infty(M_1), \qquad f \longmapsto \Omega^\lambda(\Phi^* f). \tag{2.6}$$

Let $G$ be a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$. We write the action of $h \in G$ as $L_h : (M, T_{1,0}M, \theta) \longrightarrow$

$(M, T_{1,0}M, \theta)$, $x \longmapsto L_h x$. There exists a positive valued function $\Omega(h,x)$ for $h \in G$ and $x \in M$ such that

$$L_h^*\theta = \Omega(h,\cdot)^2\theta. \tag{2.7}$$

We have the *cocycle formula* for $\Omega(\cdot,\cdot)$.

**Proposition 2.4.** *For $h_1, h_2 \in G$ and $x \in M$, we have*

$$\Omega(h_1h_2, x) = \Omega(h_1, L_{h_2}x)\Omega(h_2, x). \tag{2.8}$$

Now for $\lambda \in \mathbf{C}$, we can define a representation $\varpi_\lambda$ of the group $G$ on $C^\infty(M)$ as follows. For $h \in G$, $f \in C^\infty(M)$ and $x \in M$, let

$$\left(\varpi_\lambda(h^{-1})f\right)(x) = \Omega(h,x)^\lambda f(L_h x). \tag{2.9}$$

Proposition 2.4 assures that $\varpi_\lambda(h_1h_2) = \varpi_\lambda(h_1)\varpi_\lambda(h_2)$, i.e., $\varpi_\lambda$ is a representation of $G$. Thus, $\tilde{\Box}_\theta f = 0$ if and only if $\tilde{\Box}_\theta \left(\Omega^{\frac{Q-2}{2}}\tilde{\Phi}^* f\right) = 0$. In summary, we have the following theorem.

**Theorem 2.5.** *Suppose $G$ is a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$ of homogeneous dimension $Q$. Then,*

*(1) the CR Yamabe operator $\tilde{\Box}_\theta$ is an intertwining operator from $\varpi_{\frac{Q-2}{2}}$ to $\varpi_{\frac{Q+2}{2}}$.*

*(2) The kernel $\ker\tilde{\Box}_\theta$ is a subrepresentation of $G$ through $\varpi_{\frac{Q-2}{2}}$.*

## 3. The CR Yamabe operator on the hypersurface $Q'_{p,q}$

Let $\xi \longrightarrow [\xi]$ denote the canonical projection of $\mathbf{C}^{n+2}\backslash\{0\}$ into the complex projective space $\mathbf{C}P^{n+1}$. It is easy to see that the transformation

$$I(z_0, z_1, \cdots, z_n) = \left[\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2}\right], \tag{3.1}$$

maps the hypersurface $Q'_{p,q}$ defined by (1.27) into the projective hyperquadric $\overline{Q}_{p,q}$ (1.32). Define a 1-form

$$\theta = \frac{\sum_{j=0}^{n+1} i\varepsilon_j(\xi_j d\bar{\xi}_j - \bar{\xi}_j d\xi_j)}{\sum_{j=0}^{p-1}|\xi_j|^2}, \tag{3.2}$$

on $\mathbf{C}^{n+2}\setminus\{\xi\in\mathbf{C}^{n+2};\xi_0=\cdots=\xi_{p-1}=0\}$. It induces a 1-form on the projective hyperquadric $\overline{Q}_{p,q}$ in (1.32). We denote it by $\theta_{\overline{Q}_{p,q}}$. The hyperquadric $Q_{p,q}$ in (1.30) has a contact form

$$\theta_{Q_{p,q}}=\sum_{\alpha=0}^{n} i\varepsilon_\alpha(z_\alpha d\overline{z}_\alpha-\overline{z}_\alpha dz_\alpha), \tag{3.3}$$

(here we use variables $z_\alpha$ instead of $w_\alpha$, $\alpha=0,\cdots,n$, in the definition of $Q_{p,q}$ in (1.30)) and the hypersurface $Q'_{p,q}$ in (1.27) has a contact form

$$\theta_{Q'_{p,q}}=\sum_{\alpha=1}^{n} i\varepsilon_\alpha(z_\alpha d\overline{z}_\alpha-\overline{z}_\alpha dz_\alpha)+\frac{1}{2}(d\overline{z}_0+dz_0). \tag{3.4}$$

Contact forms (3.3) and (3.4) are actually

$$i(\overline{\partial}-\partial)r \tag{3.5}$$

for corresponding defining functions $r$ of $Q_{p,q}$ and $Q'_{p,q}$, respectively.

**Proposition 3.1.**

$$I^*\theta_{\overline{Q}_{p,q}}=\frac{1}{\frac{1}{4}|z_0-i|^2+\sum_{j=1}^{p-1}|z_j|^2}\theta_{Q'_{p,q}}, \tag{3.6}$$

*on the hypersurface* $Q'_{p,q}$.

**Proposition 3.2.** *Let* $S_0=\sum_{j=0}^{n+1}a_j|\xi_j|^2$ *with* $a_j=\varepsilon_j$ *or* $0$, *but* $a_0=1$ *and* $a_{n+1}=0$. *Then the function*

$$S(z_0,z)=S_0\left(\frac{z_0-i}{2},z,\frac{z_0+i}{2}\right) \tag{3.7}$$

*on hypersurface* $Q'_{p,q}$ *satisfies where it is positive*

$$\tilde{\Box}_{\theta_{Q'_{p,q}}}S^{-\frac{Q-2}{4}}=\frac{n+1}{2}\left(\sum_{j=1}^{n}2a_j\varepsilon_j-n\right)S^{-\frac{Q+2}{4}}, \tag{3.8}$$

*where* $Q=2n+2$.

**Corollary 3.3.** *The scalar curvature of the projective hyperquadric* $\overline{Q}_{p,q}$ *with contact form* $\theta_{\overline{Q}_{p,q}}$ *is* $\frac{n+1}{2}(p-q)$.

## 4. Representations on the projective hyperquadric $\overline{Q}_{p,q}$

**Proposition 4.1.** *For* $g \in \mathrm{SU}(p,q)$ *and* $z \in Q_{p,q}$, *we have*

$$g^*\theta_{Q_{p,q}}(z) = \frac{1}{|g(z,1)_{n+1}|^2}\theta_{Q_{p,q}}(z). \tag{4.1}$$

Define the *light cone* to be

$$\Xi := \{\xi \in \mathbf{C}^{n+2}; |\xi|_{p,q} = 0\}\backslash\{0\}, \tag{4.2}$$

and

$$\Sigma := \left\{\xi \in \mathbf{C}^{n+2}; \sum_{j=0}^{p-1} |\xi_j|^2 = \sum_{j=p}^{p+q-1} |\xi_j|^2 = 1\right\} \simeq S^{2p-1} \times S^{2q-1}. \tag{4.3}$$

The multiplicative group $\mathbf{R}_+^\times$ acts on $\Xi$ as a dilation and the quotient space $\Xi/\mathbf{R}_+^\times$ is identified with $\Sigma$. By definition, $\Xi/\mathbf{C}^\times \simeq \Sigma/S^1 \simeq \overline{Q}_{p,q}$. Because the action of $\mathrm{SU}(p,q)$ on $\mathbf{C}^{n+2}$ commutes with that of $\mathbf{C}^\times$, we can define the action of $\mathrm{SU}(p,q)$ on the quotient space $\Xi/\mathbf{C}^\times$, and also on $\overline{Q}_{p,q}$ through the above diffeomorphism. This action will be denoted by

$$L_h : \overline{Q}_{p,q} \longrightarrow \overline{Q}_{p,q}, \quad \xi \mapsto L_h\xi, \tag{4.4}$$

for $h \in \mathrm{SU}(p,q), \xi \in \overline{Q}_{p,q}$.

For $a \in \mathbf{C}$, denote by $S^a(\Xi)$ the space of smooth function on $\Xi$ homogeneous of degree $a$, i.e.

$$S^a(\Xi) = \{f \in C^\infty(\Xi); f(t\xi) = t^a f(\xi), \xi \in \Xi, t \in \mathbf{R}_+^\times\}. \tag{4.5}$$

A character $\psi$ of $\mathbf{C}^\times$ has the form

$$\psi(t) = |t|^a \left(\frac{t}{|t|}\right)^m, \tag{4.6}$$

for some $a \in \mathbf{C}^\times, m \in \mathbf{Z}$, which can be formally written as

$$\psi(t) = \psi^{\alpha,\beta}(t) = t^\alpha \bar{t}^\beta, \tag{4.7}$$

with $\alpha + \beta = a$ and $\alpha - \beta = m$. We see that a pair $(\alpha, \beta)$ can occur if and only if $\alpha - \beta$ is an integer. For such a pair, we define $S^{\alpha,\beta}(\Xi) \subset S^a(\Xi)$ to be the $\psi^{\alpha,\beta}$ eigenspace for $\mathbf{C}^\times$.

Then, we have a decomposition

$$S^a(\Xi) = \sum_{\substack{\alpha+\beta=a,\\ \alpha-\beta\in\mathbf{Z}}} S^{\alpha,\beta}(\Xi). \tag{4.8}$$

Let $\nu : \Xi \longrightarrow \mathbf{R}_+$ be defined by

$$\nu(\xi) = \left(\sum_{j=0}^{p-1} |\xi_j|^2\right)^{\frac{1}{2}} = \left(\sum_{j=p}^{p+q-1} |\xi_j|^2\right)^{\frac{1}{2}}. \tag{4.9}$$

**Proposition 4.2.** *For $g \in \mathrm{SU}(p,q)$ and $\xi \in \overline{Q}_{p,q}$, we have*

$$g^*\theta_{\overline{Q}_{p,q}}(\xi) = \frac{1}{\nu(g(\xi))^2}\theta_{\overline{Q}_{p,q}}(\xi), \tag{4.10}$$

*if we require the coordinates of $\xi$ satisfying $\sum_{j=0}^{p-1}|\xi_j|^2 = 1$.*

**Proposition 4.3.** *$S^{-\frac{\lambda}{2},-\frac{\lambda}{2}}(\Xi)$ is isomorphic to $\left(\varpi_\lambda, C^\infty\left(\overline{Q}_{p,q}\right)\right)$ as $\mathrm{U}(p,q)$ modules.*

Define the representation $(\varpi^{p,q}, V^{p,q})$ to be $(\varpi_{\frac{Q-2}{2}}, \ker\widetilde{\Box}_{\theta_{\overline{Q}_{p,q}}})$.

We can identify $S^{\alpha,\beta}(\Xi)$ with degenerate principal series representations in standard notation (cf. [5]).

**Corollary 4.4.** *$(\varpi^{p,q}, V^{p,q})$ is a subrepresentation of $S^{-\frac{\rho-1}{2},-\frac{\rho-1}{2}}(\Xi)$, or equivalently, of $C^\infty - \mathrm{Ind}_{P\max}^{G}(\chi_0 \otimes \mathbf{C}_{-1})$*

## 5. Basic properties of $(\varpi^{p,q}, V^{p,q})$

There is a natural action of $S^1$ on $\Sigma$ defined by

$$\mu_\sigma : \Sigma \longrightarrow \Sigma, \qquad (\xi_1, \cdots, \xi_{n+1}) \mapsto \left(e^{i\sigma}\xi_1, \cdots, e^{i\sigma}\xi_{n+1}\right), \tag{5.1}$$

for $\sigma \in [0, 2\pi)$. We can define the projection

$$\Pi : \Sigma \simeq S^{2p-1} \times S^{2q-1} \longrightarrow \overline{Q}_{p,q}, \tag{5.2}$$

by $\Pi(\xi_1, \cdots, \xi_{n+1}) = [\xi_1, \cdots, \xi_{n+1}] \in \mathbf{CP}^{n+1}$. Namely, $\Sigma$ is a $S^1$ fiber bundle over the projective hyperquadric $\overline{Q}_{p,q}$. Let

$$\theta_{S^{2p-1}} = i\sum_{j=0}^{p-1} \xi_j d\overline{\xi}_j - \overline{\xi}_j d\xi_j, \qquad \theta_{S^{2q-1}} = i\sum_{j=p}^{n+1} \xi_j d\overline{\xi}_j - \overline{\xi}_j d\xi_j, \tag{5.3}$$

the standard contact forms on spheres $S^{p-1}$ and $S^{q-1}$, respectively. Then,

$$\Pi^*\theta_{\overline{Q}_{p,q}} = \theta_{S^{2p-1}} - \theta_{S^{2q-1}}. \tag{5.4}$$

Let $\mathcal{H}^{\alpha,\beta}(\mathbf{C}^p)$ denote the space of harmonic polynomials of bi-degree $(\alpha,\beta)$ in $\mathbf{C}^p$, i.e., harmonic polynomials which are homogeneous of degree $\alpha$ in the $z_j$'s and of degree $\beta$ in the $\overline{z}_j$'s

$$L^2(S^{2p-1}) \simeq \sum_{\alpha,\beta=0}^{\infty} \mathcal{H}^{\alpha,\beta}(\mathbf{C}^p). \tag{5.5}$$

For a function $f \in L^2(\overline{Q}_{p,q})$, $\Pi^* f$ is an $L^2$ function in $\Sigma$ invariant under the action of $S^1$. Thus,

$$L^2\left(\overline{Q}_{p,q}\right) \simeq \sum_{m=0}^{\infty} \sum_{\substack{m_1+m_2=m,\\ n_1+n_2=m}} \mathcal{H}^{m_1,n_1}\left(\mathbf{C}^p\right) \boxtimes \mathcal{H}^{m_2,n_2}\left(\mathbf{C}^q\right) \tag{5.6}$$

as Hilbert direct sum. We denote by $C^\infty\left(S^{2p-1}\times S^{2q-1}\right)_0$ the space of $S^1$-invariant functions in $C^\infty\left(S^{2p-1}\times S^{2q-1}\right)$. We can identify $C^\infty\left(\overline{Q}_{p,q}\right)$ with the subspace $C^\infty(S^{2p-1}\times S^{2q-1})_0$ by the mapping $\Pi$.

**Proposition 5.1.** *For $u \in C^\infty\left(S^{2p-1}\times S^{2q-1}\right)_0$, we have*

$$\widetilde{\Box}_{\theta_{\overline{Q}_{p,q}}}\left(u\circ\Pi^{-1}\right) = \widetilde{\Box}_{\theta_{S^{2p-1}}} u - \widetilde{\Box}_{\theta_{S^{2q-1}}} u. \tag{5.7}$$

The Yamabe operator on the projective hyperquadric $\overline{Q}_{p,q}$ is

$$\widetilde{\Box}_{\theta_{\overline{Q}_{p,q}}} = \Box_{\theta_{\overline{Q}_{p,q}}} + \frac{n}{4}(p-q). \tag{5.8}$$

**Proposition 5.2.** *$\mathcal{H}^{\alpha,\beta}(\mathbb{C}^p)$ is an eigenspace of $\Box_{\theta_{S^{2p-1}}}$ on $S^{2p-1}$ with eigenvalue $\frac{1}{4}(\alpha+\beta)(2p-2+\alpha+\beta)-\frac{1}{4}(\alpha-\beta)^2$.*

**Theorem 5.3.** *The underlying $(\mathfrak{g},K)$-module $(\varpi^{p,q})_K$ has the following $K$-type formula*

$$(\varpi^{p,q})_K \simeq \bigoplus_{\substack{m_1+n_1+p=m_2+n_2+q,\\ m_1+m_2=n_1+n_2}} \mathcal{H}^{m_1,n_1}\left(\mathbb{C}^p\right) \boxtimes \mathcal{H}^{m_2,n_2}\left(\mathbb{C}^q\right) \tag{5.9}$$

***Remark*** 5.4. For other rank-1 Lie groups $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ and $\mathrm{F}_4^{-20}$, there exist quaternionic and octanionic CR geometries. For example, we have corresponding Webster connections, corresponding conformal geometry, corresponding Yamabe operators, etc. (cf. [3]). It is interesting to study the representation theories of $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ (more generally, of $\mathrm{Sp}(p,q)$) and $\mathrm{F}_4^{-20}$ by using corresponding conformal geometries.

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Department of Mathematics,
Zhejiang University (Xixi campus),
Zhejiang 310028,
P. R. China
e-mail: wangf@mail.hz.zj.cn