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On unitarizability of certain lowest $(\mathfrak{g}, K)$-modules

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1 Introduction

Let $G_{\mathbb{C}}$ be a connected simply connected complex simple Lie group and G a connected noncompact simple real form of $G_{\mathbb{C}}$. We denote the Lie algebras of G and $G_{\mathbb{C}}$ respectively by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. Let θ be the Cartan involution of G and K the maximal compact subgroup of G. Then G is inner if θ belongs to the adjoint group of K. We shall assume G is inner. Then K contains a Cartan subgroup B of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the eigenspace of θ with the eigenvalue -1. Let $\hat{K}$ be the unitary dual of K and (π, V) a finite generated $\mathfrak{g}$-module. Then (π, V) is said to be a $(\mathfrak{g}, K)$-module if $\dim Hom_K(V_\sigma, V)$ are finite for all $(\sigma, V_\sigma) \in \hat{K}$, where $Hom_K(V_\sigma, V)$ is the space of all K-homomorphisms of V_σ to V. Let $\mathfrak{b}$ be the Lie algebra of B and Σ the root system of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$, where $\mathfrak{g}_{\mathbb{C}}$ the Lie algebra of $G_{\mathbb{C}}$. The root system Σ_K of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ is a subset of Σ. Let P and P_K be respectively the positive root systems of Σ and Σ_K. We assume $P_K \subset P$. Let P_n be the set of all noncompact roots in P, and assume the simple root system Ψ of P has exactely one noncompact root. For a $(\mathfrak{g}, K)$-module (π, V) we denote by Γ_π the set of all P_K-dominant integral forms ν on $\mathfrak{b}_{\mathbb{C}}$ satisfying $\dim Hom_K(V_\nu, V) \neq 0$, where (π_ν, V_ν) is a unitary simple K-module with the highest weight ν. A simple $(\mathfrak{g}, K)$-module (π, V) is a lowest module if there exists a P-dominant integral form μ on $\mathfrak{b}_{\mathbb{C}}$ such that $\dim Hom_K(V_\mu, V) = 1$ and $\mu - \omega \notin \Gamma_\pi$ for all $\omega \in P_n$. (π, V) is said to be a lowest $(\mathfrak{g}, K)$-module with a P-dominant data μ. Our object of this note is to give a necessary condition for the unitarizable lowest $(\mathfrak{g}, K)$-module with a P-dominant nonzero data μ under the assumption : $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ with $\dim \mathfrak{k}^* \leq 3$. Let us state our main result after the following preparations. Let $(\pi_\nu, V_\nu) \in \hat{K}$, and consider a tensor K-module $\mathfrak{p}_{\mathbb{C}} \otimes V_\nu$, where $\mathfrak{p}_{\mathbb{C}}$ is the complexification of $\mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$. Let Γ_K be the set of all P_K-dominant integral forms on $\mathfrak{b}_{\mathbb{C}}$ and Σ_n the set of all noncompact roots in Σ. For $\omega \in \Sigma_n$ we define a projection opertator $P_{\nu+\omega}$ on $\mathfrak{p}_{\mathbb{C}} \otimes V_\nu$ by

$$P_{\mu+\omega}(X \otimes v) = \begin{cases} \int_K (Ad \otimes \pi_\nu)(k)(X \otimes v)\chi_{\nu+\omega}(k^{-1})dk & \text{if } \nu+\omega \in \Gamma_K \\ 0 & \text{if } \nu+\omega \notin \Gamma_K \end{cases}$$

, where $X \in \mathfrak{p}_{\mathbb{C}}, v \in V_\nu$, dk is the Haar measure on K normalized as $\int_K dk = 1$ and $\chi_{\nu+\omega}(k) = (\dim V_{\nu+\omega}) trace \pi_{\nu+\omega}(k)$.

Theorem 1.1

Let $\mathfrak{g}$ be an inner type noncompact real simple Lie algebra. We choose a positive root system P satisfying $P_K \subset P$ and its simple root system Ψ has exactly one noncompact root. Assume $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ with $\dim \mathfrak{k}^* \leq 3$. If (π, V) is a unitarizable lowest $(\mathfrak{g}, K)$- module with a P-dominant nonzero data μ, then

$$\frac{2(\mu + \rho_K - \rho_n, \omega)}{|\omega|^2} \geq -1 \text{ for all } \omega \in P_n$$

satisfying $P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes v_\mu) \neq 0$, where ρ_K (resp. ρ_n) is one half the sum of all roots in P_K (resp. P_n).

This theorem is proved by solving a system of linear equation associated with the Clebsch-Gordan coefficients of the tensor K- module $\mathfrak{p}_{\mathbb{C}} \otimes V_\mu$. This method is treated by the papers of V. Bargmann [1] for $SL(2, \mathbb{R})$, L.H.Thomas [9] and J. Deximier [2] for De Sitter group (see for the related works T. Hirai [4], A.U. Klimyk and U. A. Shirokov [5]). Let Ω be the Casimir operator on G and (π, V) a lowest $(\mathfrak{g}, K)$-module with a P- dominant nonzero data μ. Then Ω acts on V as the scalar $|\mu + \rho_K - \rho_n|^2 - |\rho|^2$, where $\rho = \rho_K + \rho_n$. If $\dim \mathfrak{k}^* = 1$, then $(\mathfrak{g}, \mathfrak{k})$ is a hermitian symmetric pair. In [8] R. Parthasarathy gives a criterion for the necessary and sufficient condition for the unitarizability of the lowest $(\mathfrak{g}, K)$-module with a P-dominant data μ under the assumptions: $(\mathfrak{g}, \mathfrak{k})$ is hermitian and $\mu + \rho_K - \rho_n$ is P-regular. If $\dim \mathfrak{k}^* = 3$, then $\mathfrak{g}$ is one of the Lie algebras $\mathfrak{sp}(n, 1)$, EII, EVI, EIX, FI and G_2 (see Table II, p354, [3]). The detailed proof of the theorem will be appear elsewhere.

In the following I summarize an outline of the proof of the main theorem.

2 Linear equation of lowest $(\mathfrak{g}, K)$-module

Let P be a postive root system of Σ containing P_K. Throughout of this note we assume the simple root system Ψ of P has exactly one noncompact root. Let $\alpha \in \Sigma$ and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} : ad(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h}_{\mathbb{C}}\}$. Let $\phi(X, Y)$ be the Killing form on $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{b}$ the compact real form of $\mathfrak{g}_{\mathbb{C}}$. We choose $X_\alpha \in \mathfrak{g}_\alpha$ satisfying

$$X_\alpha - X_{-\alpha}, \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_u \text{ and } \phi(X_\alpha, X_{-\alpha}) = 1. \qquad (1)$$

Then $\phi(H, H_\alpha) = \alpha(H)$ for $H \in \mathfrak{b}_{\mathbb{C}}$, where $H_\alpha = ad(X_\alpha)X_{-\alpha}$. Let τ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_u$. We define a bilinear form (X, Y) on $\mathfrak{p}_{\mathbb{C}}$ by $(X, Y) = -\phi(X, \tau(Y)), X, Y \in \mathfrak{p}_{\mathbb{C}}$. Then (X, Y) is a positive definite hermitian form on $\mathfrak{p}_{\mathbb{C}}$. Moreover for $\alpha, \beta \in \Sigma_n$, $(X_\alpha, X_\beta) = \delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ is Kronecker's delta. Let (π, V) be a lowest $(\mathfrak{g}, K)$-module with a P-dominant data μ. We define a K-homomorphism φ of $V_\mu \oplus (\mathfrak{p}_{\mathbb{C}} \otimes V_\mu) \oplus (\mathfrak{p}_{\mathbb{C}} \otimes \mathfrak{p}_{\mathbb{C}} \otimes V_\mu)$ to V by

$$\begin{aligned} \varphi(v) = v, \ \varphi(X \otimes v) &= \pi(X)v \\ \varphi(X \otimes Y \otimes v) &= \pi(X)\pi(Y)v, \end{aligned}$$

where $X, Y \in \mathfrak{p}_{\mathbb{C}}$, $v \in V_\mu$.

2.1 Lemma *Let (π, V) be a lowest $(\mathfrak{g}, K)$-module with a P- dominant data μ. Then for $\omega \in \Sigma_n$, $X, Y \in \mathfrak{p}_{\mathbb{C}}$ and $v \in V_\mu$ we have*

$$\begin{aligned} \varphi(P_\mu(X \otimes Y \otimes v)) &= P_\mu(\pi(X)\pi(Y)v), \\ \varphi(P_\mu(X \otimes P_{\mu+\omega}(Y \otimes v))) &= P_\mu(\pi(X)P_{\mu+\omega}(\pi(Y)v)). \end{aligned}$$

Let $\omega \in \Sigma_n$, and assume $P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu) \neq 0$. Then the K-module $P_\mu(\mathfrak{p}_{\mathbb{C}} \otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu))$ is simple. Moreover (see Corollary 4.5, [7]) there exist a unit vector $v_\omega(\mu)$ and a positive constant $c(\mu; \omega)$ such that

$$P_\mu(X_{-\gamma} \otimes P_{\mu+\omega}(X_\gamma \otimes v(\mu))) = c(\mu; \omega)|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 v_\omega(\mu) \qquad (2)$$

for all $\gamma \in P_n$, where $v(\mu)$ is the highest weight vector of V_μ normalized as $|v(\mu)| = 1$. We remark that $v_\omega(\mu)$ is the highest weight vector of $P_\mu(\mathfrak{p}_{\mathbb{C}} \otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu))$. We put

$$\mathcal{S}(\mu; P_n) = \{\omega \in P_n; P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu) \neq 0\}. \qquad (3)$$

Let us enumerate the sets P_n and $\mathcal{S}(\mu; P_n)$ respectively by

$$\begin{aligned} P_n &= \{\gamma_1, \gamma_2, ..., \gamma_N\}, \ \gamma_1 > \gamma_2 > ... > \gamma_N, \\ \mathcal{S}(\mu; P_n) &= \{\omega_1, \omega_2, ..., \omega_k\}, \ \omega_1 > \omega_2 > ... > \omega_k. \end{aligned}$$

We define two matrices $A_0(\lambda)$ and $B_0(\lambda)$ respectively by

$$\begin{aligned} A_0(\lambda) &= (|P_{\mu+\omega_j}(X_{\gamma_i} \otimes v(\mu))|^2), && (4) \\ B_0(\lambda) &= (|P_{\mu+\omega_j}(X_{-\gamma_i} \otimes v(\mu)|^2), && (5) \end{aligned}$$

where $\lambda = \mu + \rho_K$. By Lemma 4.3 and Theorem 5.5 in [6] $|P_{\mu+\omega_j}(X_{\pm\gamma} \otimes v(\mu))|^2$ is a rational function in λ.

Theorem 2.1
Let (π, V) be a lowest simple $(\mathfrak{g}, \mathfrak{k})$-module with a P-dominant data μ. Define $A_0(\lambda)$ and $B_0(\lambda)$ by (2.4) and (2,5). We put

$$\begin{aligned}\mathbf{x} &= {}^t(x_1, x_2, \ldots, x_k) \text{ and}\\ \mathbf{b}(\lambda) &= {}^t((\mu, \gamma_1), (\mu, \gamma_2), \ldots, (\mu, \gamma_N)),\end{aligned}$$

where x_i is defined by $x_i = -\varphi(c(\mu; \omega_i)v_{\omega_i}(\mu))$. Then we have

$$(A_0(\lambda) - B_0(\lambda))\mathbf{x} = \mathbf{b}(\lambda).$$

Proof. Let $\gamma \in \Sigma_n$. Since $\mu - \omega \notin \Gamma_\pi$ for $\omega \in P_n$, Lemma 2.1 and (2.3) imply

$$\begin{aligned}&\varphi(P_\mu(X_{-\gamma} \otimes X_\gamma \otimes v(\mu)))\\ &\quad= \sum_{\omega\in\Sigma_n} \varphi(P_\mu(X_{-\gamma} \otimes P_{\mu+\omega}(X_\gamma \otimes v(\mu))))\\ &\quad= \sum_{j=1}^{k} \varphi(P_\mu(X_{-\gamma} \otimes P_{\mu+\omega_j}(X_\gamma \otimes v(\mu))))\\ &\quad= \sum_{j=1}^{k} |P_{\mu+\omega_j}(X_\gamma \otimes v(\mu))|^2 \varphi(c(\mu; \omega_j)v_{\omega_j}(\mu)).\end{aligned}$$

This implies that

$$\begin{aligned}&\varphi(P_\mu(X_{-\gamma_i} \otimes X_{\gamma_i} \otimes v(\mu) - X_{\gamma_i} \otimes X_{-\gamma_i} \otimes v(\mu)))\\ &\quad= \sum_{j=1}^{k} \{|P_{\mu+\omega_j}(X_{\gamma_i} \otimes v(\mu)|^2 - |P_{\mu+\omega_j}(X_{-\gamma_i} \otimes v(\mu))|^2\}(-x_j).\end{aligned}$$

Since

$$\begin{aligned}&\varphi(P_\mu(X_{-\gamma_i} \otimes X_{\gamma_i} \otimes v(\mu) - X_{\gamma_i} \otimes X_{-\gamma_i} \otimes v(\mu)))\\ &\quad= P_\mu([\pi(X_{-\gamma_i}, \pi(X_{\gamma_i})]v(\mu))\\ &\quad= -(\mu, \gamma_i)v(\mu),\end{aligned}$$

we have $(A_0(\lambda) - B_0(\lambda))\mathbf{x} = \mathbf{b}(\lambda)$.

Let (π, V) be a finite generated $(\mathfrak{g}, \mathfrak{k})$-module. Then (π, V) is unitarizable if there exists a positive definite hermitian form (v, w) on V such that

$$(\pi(X)v, w) + (v, \pi(X)w) = 0 \text{ for } X \in \mathfrak{g} \text{ and } v, w \in V.$$

2.2 Lemma *Let (π, V) be a unitarizable lowest $(\mathfrak{g}, \mathfrak{k})$-module with a P-dominant data μ. Define $\mathbf{x}$ as in the above theorem. Then $x_i \geq 0$ for $i, 1 \leq i \leq k$.*

Proof. By the choice of $X_\omega \in \mathfrak{g}_\omega$ (see (2.1)) , we have $(\pi(X_\omega)v, w) = -(v, \pi(X_{-\omega})w)$ for $v, w \in V$. Then by Lemma 2.1 and (2.2)

$$\begin{aligned} -x_i|P_{\mu+\omega_i}(X_{\omega_i} \otimes v(\mu))|^2 &= |P_{\mu+\omega_i}(X_{\omega_i} \otimes v(\mu))|^2(\varphi(c(\mu;\omega_i)v_{\omega_i}(\mu)), v(\mu)) \\ &= (\varphi(X_{-\omega_i} \otimes P_{\mu+\omega_i}(X_{\omega_i} \otimes v(\mu))), v(\mu)) \\ &= (P_\mu(\pi(X_{-\omega_i})P_{\mu+\omega_i}(\pi(X_{\omega_i})v(\mu)); v(\mu)) \\ &= (\pi(X_{-\omega_i})P_{\mu+\omega_i}(\pi(X_{\omega_i}v(\mu))), v(\mu)) \\ &= -(P_{\mu+\omega_i}(\pi(X_{\omega_i})v(\mu)), P_{\mu+\omega_i}(\pi(X_{\omega_i})v(\mu))) \\ &\leq 0. \end{aligned}$$

Since $P_{\mu+\omega_i}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu) \neq 0$, $|P_{\mu+\omega_i}(X_{\omega_i} \otimes v(\mu))| > 0$ (see Corollary 3.5, [6]), and hence the lemma follows.

3 Solution of $A(\eta)\mathbf{x} = \mathbf{b}(\eta)$

Let p be a nonnegative integer. We define a set Π_p by

$$\begin{aligned} \Pi_0 &= \{\tilde{\phi}\} \text{ for } p = 0, \\ \Pi_p &= \{(\alpha_1, \alpha_2, ..., \alpha_p) : \alpha_i \in P_K\} \text{ for } p > 1, \text{ and put} \\ \Pi &= \bigcup_{p=0}^{\infty} \Pi_p. \end{aligned}$$

For $I = (\alpha_1, \alpha_2, ..., \alpha_p)$, $J = (\beta_1, \beta_2 ..., \beta_q) \in \Pi$ we define $I \star J$ by

$$I \star J = (\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q).$$

By $\star$-operation Π is a semigruop with the identity $\tilde{\phi}$. Let $\omega \in \Sigma_n$ and η a generic point in the dual space $(\sqrt{-1}\mathfrak{b})^*$ of the real vector space $\sqrt{-1}\mathfrak{b}$. For $I \in \Pi$, we define $R(\eta; I)$, $S(\eta; I)$ and $T(\eta; I)$ as follows:

$$R(\eta : \tilde{\phi}) = S(\eta; \tilde{\phi}) = T(\eta; \tilde{\phi}) = 1$$

and for $I = (\alpha_1, \alpha_2, ..., \alpha_p) \in \Pi$,

$$R(\eta; I) = (|\eta + <I>|^2 - |\eta|^2)^{-1}, \tag{6}$$

$$S(\eta; I) = \prod_{J,L \in \Pi, J \star L = I, j \neq \tilde{\phi}} R(\eta; J), \tag{7}$$

$$T(\eta; I) = \prod_{J,L \in \Pi, J \star L = I} R(\eta + <J>; L), \tag{8}$$

where $< I >= \sum_{i=1}^{p} \alpha_i$. Let $U(\mathfrak{k}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{k}_{\mathbb{C}}$. For $I \in \Pi$ we define $Q(I) \in U(\mathfrak{k}_{\mathbb{C}})$ by

$$\begin{aligned} Q(I) &= 1 \text{ for } I = \tilde{\phi}, \\ Q(I) &= X_{-\alpha_1} X_{-\alpha_2} ... X_{-\alpha_p} \text{ for } I = (\alpha_1, \alpha_2, ..., \alpha_p). \end{aligned}$$

The map $I \longrightarrow Q(I)$ is a semigroup homomorhism of Π into $U(\mathfrak{k}_{\mathbb{C}})$. $Q(I)$ acts on $\mathfrak{p}_{\mathbb{C}}$ by $Q(I)X = ad(Q(I))X, X \in \mathfrak{p}_{\mathbb{C}}$. The selfadjoint operator $Q(I)^*$ of $Q(I)$ is defined by

$$(Q(I)X, Y) = (X, Q(I)^* Y), X, Y \in \mathfrak{p}_{\mathbb{C}}.$$

Let $\omega, \gamma \in \Sigma_n$. We put

$$\begin{aligned} a_{\omega}(I) &= 2^{\sharp I} |(Q(I)^* X_{\omega}, X_{\omega + <I>})|^2, I \in \Pi \text{ and} \\ \Pi(\gamma, \omega) &= \{I \in \Pi : (Q(I)^* X_{\gamma}, X_{\omega}) \neq 0\}, \end{aligned}$$

where $\sharp I = p$ for $I = (\alpha_1, \alpha_2, ..., \alpha_p)$. Let $\mathbb{R}(\eta)$ be the field of rational functions in η over the real number field $\mathbb{R}$. For $\omega, \gamma \in \Sigma_n$ we define three rational functions $S(\eta; \gamma, \omega)$, $T(\eta; \gamma, \omega)$ and $f(\eta; \gamma)$ by

$$S(\eta; \gamma, \omega) = \sum_{I \in \Pi(\gamma, \omega)} (-1)^{\sharp I} a_{\gamma}(I) S(\eta + \gamma; I), \tag{9}$$

$$T(\eta; \gamma, \omega) = \sum_{I \in \Pi(\gamma; \omega)} a_{\gamma}(I) T(\eta + \gamma), \tag{10}$$

$$f(\eta; \gamma) = \sum_{I \in \Pi} (-1)^{\sharp I} a_{\gamma}(I) S(\eta; I). \tag{11}$$

Then $f(\eta; \gamma) = \sum_{\delta \in \Sigma_n} S(\eta; \gamma, \delta)$. We define two matrices $A(\eta)$ and $B(\eta)$ by

$$A(\eta) = (T(\eta; \gamma_i, \gamma_j) f(\eta + \gamma_j; \gamma_j)), B(\eta) = (T(\eta; -\gamma_i, \gamma_j) f(\eta + \gamma_j; \gamma_j)).$$

Since $\Pi(\gamma_i, \gamma_i) = \{\tilde{\phi}\}$ and $\Pi(\gamma_i, \gamma_j) = \phi$ for i, j $i < j$, $A(\eta)$ is a lower triangular matrix. Let $\mathbf{b}(\eta)$ be the column vector in $\mathbb{R}(\eta)^N$ defined by

$$\mathbf{b}(\eta) =^t ((\eta - \rho_K, \gamma_1), (\eta - \rho_K, \gamma_2), ..., (\eta - \rho_K, \gamma_N)). \tag{12}$$

In §4 we shall give explicitely the solution $\mathbf{x}$ of linear equation $(A(\eta) - B(\eta))\mathbf{x} = \mathbf{b}(\eta)$ under the assumption: $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ with $\dim \mathfrak{k}^* \leq 3$. Let μ be a P-dominant integral form on $\mathfrak{b}_{\mathbb{C}}$ and define $\mathcal{S}(\mu; P_n)$ by (1.3). We remark that if $\mathcal{S}(\mu; P) = P_n$, then $A_0(\lambda) = A(\lambda)$ and $B_0(\lambda) = B(\lambda)$ (see Lemma 4.3 and Theorem 5.5, [6]).

We define for each pair α and β in Σ a complex number $< \alpha, \beta >$ by

$$< \alpha, \beta >= \begin{cases} \phi(ad(X_{\alpha})X_{\beta}, X_{-\alpha-\beta}) & \text{if } \alpha + \beta \in \Sigma \\ 0 & \text{if } \alpha + \beta \notin \Sigma \end{cases}$$

Theorem 3.1
Let $\mathbf{x} = {}^t(x_1, x_2, ..., x_n)$ *be the solution of* $A(\eta)\mathbf{x} = \mathbf{b}(\eta)$. *Then* x_i *is given by*

$$x_i = (\eta, \gamma_i) - \sum_{\alpha \in P_K} | < \alpha, \gamma_i > |^2 - (\rho_K, \gamma_i), \ \textit{for } i, 1 \leq i \leq N.$$

This theorem is proved by using the following three lemmas.

3.1 Lemma *We put* $S(\eta) = (S(\eta; \gamma_i, \gamma_j))$ *and* $T(\eta) = (T(\eta; \gamma_i, \gamma_j))$. *Then* $S(\eta)$ *is the inverse matrix of* $T(\eta)$.

Since $T(\eta)$ is a lower triangular matrix, the inverse matrix of $T(\eta)$ is given explicitely by a direct calculation. Moreover by using Lemma 4.4, [6] we can prove this lemma.

3.2 Lemma *Let* $\gamma, \omega \in \Sigma_n$. *Then we have*

$$\begin{aligned} & \sum_{\alpha \in P_K} | < \alpha, \omega > |^2 + (\rho_K, \omega) + \frac{1}{2}|\omega|^2 \\ = & \sum_{\alpha \in P_K} | < \alpha, \gamma > |^2 + (\rho_K, \gamma) + \frac{1}{2}|\gamma|^2. \end{aligned}$$

This lemma is proved by calculating the scalar operator Ω_K on $\mathfrak{p}_{\mathbf{C}}$, where Ω_K is the Casimir operator on K.

3.3 Lemma *Let* $\omega, \gamma \in P_n$ *and* $\eta \in \sqrt{-1}\mathfrak{b}$. *Assume that* $\Pi(\omega; \gamma) \neq \phi$. *Then for* $I \in \Pi(\omega; \gamma)$

$$(\eta, \gamma) = \frac{1}{2}(|\eta + \omega + < I > |^2 - |\eta + \omega|^2) + (\eta, \omega) + \frac{1}{2}(|\omega|^2 - |\gamma|^2).$$

Bearing in mind $\gamma = \omega + < I >$, a direct calculation implies this lemma.

Let us now prove Theorem 3.1. We put $F(\eta) = (f(\eta + \gamma_j; \gamma_j)\delta_{i,j})$. Then $A(\eta) = T(\eta)F(\eta)$. By Lemma 3.1 $F(\eta)\mathbf{x} = T(\eta)^{-1}\mathbf{b}(\eta) = S(\eta)\mathbf{b}(\eta)$. We put $S(\eta)\mathbf{b}(\eta) = {}^t(g_1, g_2, ..., g_N)$. By (3.9) we have

$$g_i = \sum_{j=1}^{N} \{ \sum_{I \in \Pi(\gamma_i; \gamma_j)} (-1)^{\sharp I} a_{\gamma_i}(I) S(\eta + \gamma_i; I)\}\{(\eta, \gamma_j) - (\rho_K, \gamma_j)\}.$$

For a fixed $\omega \in P_n$ we put

$$g = \sum_{\gamma > 0} \{ \sum_{I \in \Pi(\omega; \gamma)} (-1)^{\sharp I} a_{\omega}(I) S(\eta + \omega; I)\}\{(\eta, \gamma) - (\rho_K, \gamma)\}.$$

It is sufficient to prove that

$$g = \{(\eta,\omega) - \sum_{\alpha\in P_K} |<\alpha,\omega>|^2 - (\rho_K,\omega)\} f(\eta+\omega;\omega).$$

By Lemma 3.3

$$\begin{aligned}2g &= \{2(\eta,\omega)+|\omega|^2\}\sum_{I\in\Pi}(-1)^{\sharp I}a_\omega(I)S(\eta+\omega;I)\\ &\quad +\sum_{\gamma>0}\sum_{I\in\Pi(\omega;\gamma)}\{|\eta+\omega+<I>|^2-|\eta+\omega|^2\}(-1)^{\sharp I}a_\omega(I)S(\eta+\omega;I)\\ &\quad -\sum_{\gamma>0}\{|\gamma|^2+2(\rho_K,\gamma)\}\sum_{I\in\Pi(\omega;\gamma)}(-1)^{\sharp I}a_\omega(I)S(\eta+\omega;I).\end{aligned}$$

Let $I\in\Pi(\omega;\gamma)$, and assume $\sharp I\geq 1$. Then there exist $\alpha\in P_K$ and $L\in\Pi$ such that $I = L\star\alpha$. Since

$$a_\omega(I) = a_\omega(L)a_{\omega+<L>}(\alpha) \text{ and } S(\eta+\omega;I) = S(\eta+\omega;L)R(\eta+\omega;I),$$

$$\begin{aligned}&a_\omega(I)S(\eta+\omega;I)\{|\eta+\omega+<I>|^2-|\eta+\omega|^2\}\\ &\qquad = a_\omega(L)S(\eta+\omega;L)2|<\alpha,\omega+<L>>|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&\sum_{\gamma>0}\sum_{I\in\Pi(\omega;\gamma)}\{|\eta+\omega+<I>|^2-|\eta+\omega|^2\}(-1)^{\sharp I}a_\omega(I)S(\eta+\omega;I)\\ &= -\sum_{\gamma_0>\gamma\geq\omega}\sum_{L\in\Pi(\omega;\gamma)}2(-1)^{\sharp L}a_\omega(L)S(\eta+\omega;L)(\sum_{\alpha\in P_K}|<\alpha,\gamma>|^2)\\ &\quad \text{, where } \gamma_0 \text{ is the highest root in } P_n,\\ &= -\sum_{\gamma>0}\sum_{I\in\Pi(\omega,\gamma)}2(-1)^{\sharp I}a_\omega(I)S(\eta+\omega;I)(\sum_{\alpha\in P_K}|<\alpha,\gamma>|^2),\end{aligned}$$

here we used $<\alpha,\gamma_0>=0$. By (3.11) and Lemma 3.2 we have

$$\begin{aligned}g &= \{(\eta,\omega)+\frac{1}{2}|\omega|^2\}f(\eta+\omega;\omega)-\sum_{\gamma>0}\sum_{I\in\Pi(\omega;\gamma)}(-1)^{\sharp I}a_\omega(I)\\ &\qquad \times S(\eta+\omega;I)\{\sum_{\alpha\in P_K}|<\alpha,\gamma>|^2+\frac{1}{2}|\gamma|^2+(\rho_K,\gamma)\}\\ &= \{(\eta,\omega)-\sum_{\alpha\in P_K}|<\alpha,\omega>|^2-(\rho_K,\omega)\}f(\eta+\omega;\omega).\end{aligned}$$

4 Solution of $(A(\eta) - B(\eta))\mathbf{x} = \mathbf{b}(\eta)$

We now assume $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ with $\dim \mathfrak{k}^* \leq 3$. Since $\mathfrak{k}^*$ is reductive, $\dim \mathfrak{k}^* = 1$ or $\dim \mathfrak{k}^* = 3$. When $\dim \mathfrak{k}^* = 1$ $(\mathfrak{g}, \mathfrak{k})$ is a hermitian symmetric pair. In this case, since $B(\eta) = 0$, the solution of $(A(\eta) - B(\eta))\mathbf{x} = \mathbf{b}(\eta)$ is given by Theorem 3.1. Assume that $\dim \mathfrak{k}^* = 3$, and let K^* the analytic subgroup of K corresponding to $\mathfrak{k}^*$. We denote the root system of $(\mathfrak{k}^*_{\mathbb{C}}, (\mathfrak{k}^* \cap \mathfrak{b})_{\mathbb{C}})$ by $\Sigma_{K^*} = \{\alpha^*\}$. where $\alpha^* \in P_K$. We have $-\gamma + \alpha^* \in P_n$ for all $\gamma \in P_n$.

4.1 Lemma *Let $\gamma, \omega \in \Sigma_n$. Assume that $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbb{C}} \otimes V_\mu) \neq 0$. Then*

$$|P_{\mu+\omega}(X_{-\gamma} \otimes v(\mu))|^2 = \frac{|\alpha^*|^2}{2(\lambda, \alpha^*)}|P_{\mu+\omega}(X_{-\gamma+\alpha^*} \otimes v(\mu))|^2.$$

Since the Casimir operator Ω_{K^*} on K^* belongs to the center of $U(\mathfrak{k}_{\mathbb{C}})$, we can prove this lemma.

Let $\gamma \in P_n$, and put $\gamma^* = -\gamma + \alpha^*$. Then the map $\gamma \to \gamma^*$ is an involutive automorphism of P_n. This implies N is even. We put $N = 2p$ and $J = (\delta_{i,2p-j+1})$. Then by Lemma 4.1 we have $A(\eta) - B(\eta) = (E - \frac{|\alpha^*|^2}{2(\eta,\alpha^*)}J)A(\eta)$. By using Lemma 4.1 and Theorem 3.1 we can prove the following lemma.

4.2 Lemma *Assume $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ with $\dim \mathfrak{k}^* = 3$. Then the solution $\mathbf{x} = {}^t(x_1, x_2, ..., x_N)$ of $(A(\eta) - B(\eta))\mathbf{x} = \mathbf{b}(\eta)$ is given by*

$$x_i = \frac{2(\eta, \alpha^*)}{2(\eta, \alpha^*) + |\alpha^*|^2}\{(\eta, \gamma_i) - \sum_{\alpha \in P_K} | < \alpha, \gamma_i > |^2 + \frac{1}{2}|\alpha^*|^2\}.$$

Theorem 4.1
Let $\mathfrak{g}$ be an inner type noncompact real simple Lie algebra. Assume that the maximal compact subalgebra $\mathfrak{k}$ has an ideal $\mathfrak{k}^*$ satisfying $\dim \mathfrak{k}^* \leq 3$. Let P be the positive root system which contains exactly one noncompact simple root. Then the solution $\mathbf{x}(\eta) = {}^t(x(\eta;\gamma_1), x(\eta;\gamma_2), ..., x(\eta;\gamma_N))$ of the linear equation $(A(\eta) - B(\eta))\mathbf{x} = \mathbf{b}(\eta)$ is given by the followings.

For the case $\dim \mathfrak{k}^* = 1$

$$x(\eta;\gamma_i) = (\eta - \rho_n, \gamma_i) + \frac{1}{2}|\gamma_i|^2.$$

For the cases $\dim \mathfrak{k}^* = 3$

$$x(\eta;\gamma_i) = \frac{2(\eta, \alpha^*)}{2(\eta, \alpha^*) + |\alpha^*|^2}\{(\eta - \rho_n, \gamma_i) + \frac{1}{2}|\gamma_i|^2\}.$$

Let $\mu \in \Gamma_K$ and $\lambda = \mu + \rho_K$. We define $A(\lambda), B(\lambda)$ by

$$\begin{aligned} A(\lambda) &= \lim_{\eta \to \lambda} A(\eta), \\ B(\lambda) &= \lim_{\eta \to \lambda} B(\eta). \end{aligned}$$

Then $A(\lambda)$ and $B(\lambda)$ are welldefined. Moreover we have the following theorem.

Theorem 4.2
Assume that $\mathfrak{k}$ *has an ideal* $\mathfrak{k}^*$ *with* $\dim \mathfrak{k}^* \leq 3$ *and* (π, V) *a lowest* $(\mathfrak{g}, \mathfrak{k})$*-module with a P-dominant nonzero data* μ*. Then* $\mathbf{x} ={}^t (x(\lambda;\omega_1), x(\lambda;\omega_2), ..., x(\lambda;\omega_k))$ *is the unique solution of* $(A_0(\lambda) - B_0(\lambda))\mathbf{x} = \mathbf{b}(\lambda)$*, where* $x(\lambda;\omega_i)$ *is the same as in Theorem 4.1.*

This theorem and Lemma 2.2 imply Theorem 1.1.

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