Correspondences between Eisenstein series of Jacobi forms and modular forms with quadratic primitive Dirichlet character

Yoshiki Hayashi

Abstract: Let p be an odd prime and χ_p the quadratic primitive Dirichtlet character modulo p. In this paper, we construct Jacobi-Eisenstein series of weight k and index m on $\Gamma_0(p)\ltimes\mathbb{Z}^2$ with χ_p, and Eisenstein series of elliptic modular forms of weight $2k-2$ on $\Gamma_0(mp)$ with character χ_p^2. Moreover, considering Fourier coefficients of Eisenstein series we construct the correspondence between both Eisenstein series.

0. Introduction

By Shimura, Shintani, Niwa and Kohnen we know the relations between the spaces of modular forms of half-integral weight and of integral weight. Skoruppa considered the relation between modular forms of half-integral weight and Jacobi forms. Skoruppa and Zagier constructed in their paper [SZ] a correspondence between the spaces of Jacobi forms of weight k and index m and modular forms of weight $2k-2$ and level m.

Arakawa([Ara]) and Horie([Hor1], [Hor2]) defined the Jacobi forms on the Jacobi group $\Gamma_0(N)\ltimes\mathbb{Z}^2$ with character.

Recently, Manickam and Ramakrishnan ([MR2]) constructed a new Jacobi-Eisenstein series of square level N with trivial character, and considered the correspondence between Jacobi-Eisenstein series of weight k and index 1 and Eisenstein series of weight $2k-2$ and level N. Moreover, using the theory of new forms of Jacobi forms, they obtained the correspondence between the spaces of these Eisenstein series for arbitrary level with trivial character.

In this paper we consider correspondences between both Eisenstein series with quadratic primitive character.

1. Preparations.

a) Fourier coefficients of Jacobi forms: Let N be an odd positive integer, p an odd prime, χ_N a primitive Dirichlet character modulo N, χ_p a quadratic primitive Dirichlet character modulo p. We recall Jacobi form $\phi_{k,m}$ of weight k and index m on $\Gamma_0(N)\ltimes\mathbb{Z}^2$ with χ_N.

Definition. *We denote $\Gamma = \Gamma_0(N)$. For a holomorphic function $\phi : \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$ we define the actions for fixed positive integers k and m*

(1)

a) $$(\phi|_{k,m}M)(\tau,z) := \frac{1}{(c\tau+d)^k} e^m\left(\frac{-cz^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$$

$$\text{(with } M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \text{ and } e^m(x) := e^{2\pi i m x}),$$

b) $$(\phi|_m X)(\tau,z) := e^m(\lambda^2\tau + 2\lambda z)\phi(\tau, z+\lambda\tau+\mu) \quad (X = [\lambda,\mu] \in \mathbb{Z}^2)$$

and define an action of the semi-direct product $\Gamma \ltimes \mathbb{Z}^2$ with group law $(M,X)(M',X') = (MM', XM'+X')$. This group $\Gamma \ltimes \mathbb{Z}^2$ is called Jacobi group on Γ. A Jacobi form of weight k and index m $(k,m \in \mathbb{N})$ on the group Γ with the character χ_N is a holomorphic function $\phi : \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$ satisfying

i) $\phi|_{k,m}M = \chi_N(d)\phi \quad (M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma)$,

ii) $\phi|_m X = \phi \quad (X \in \mathbb{Z}^2)$,

iii) *ϕ is holomorphic at any cusp of $\Gamma_0(N)$, namely, for each $M \in SL_2(\mathbb{Z})$, $\phi|_{k,m}M$ has a Fourier expansion of the form*

$\sum_{\substack{n\in\mathbb{N}_0, r\in\mathbb{Z} \\ 4mn - r^2 n_M \geq 0}} c_M(n,r) q^{n/n_M} \zeta^r \qquad (q := e(\tau),\ \zeta := e(z))$

with a natural number n_M depending on M and with $c(n,r) = 0$ unless $n \geq r^2 n_M/4m$.

(If ϕ satisfies the strong condition "$c(n,r) \neq 0 \Rightarrow n > r^2 n_M/4m$", it is called a cusp form). The vector space of all such functions ϕ is denoted by $J_{k,m}(\Gamma,\chi_N)$; if it is not confused we write simply $J_{k,m,N}$ for $J_{k,m}(\Gamma,\chi_N)$ and $J^{\mathrm{cusp}}_{k,m,N}$ is the space of cusp forms.

Remark. *As above, the Fourier coefficients $c_M(n,r)$ are depending on $M \in SL_2(\mathbb{Z})$, $D_M := r^2 n_M - 4mn$ and $r \pmod{2m}$: $\sum_{\substack{D_M, r\in\mathbb{Z}, D_M \leq 0 \\ D_M \equiv r^2 n_M \pmod{4m}}} c_M(D_M, r) q^{(r^2 n_M - D_M)/(4mn_M)} \zeta^r$. Since the calculations in this paper are analogeous for all n_M, we write down our calculations and results simply with the form*

$$\phi_{k,m}(\tau,z) = \sum_{\substack{D,r\in\mathbb{Z} \\ D\leq 0 \\ D\equiv r^2 \pmod{4m}}} c(D,r) q^{(r^2-D)/4m} \zeta^r.$$

b) Operators V_l, U_l: We now define Operators on $J_{k,m,N}$.

We have operators $V_l : J_{k,m,N} \longrightarrow J_{k,ml,N}$ and $U_l : J_{k,m,N} \longrightarrow J_{k,ml^2,N}$ defined by

$$\text{(2a)} \quad (\phi|V_l)(\tau,z) := l^{k-1} \sum_{\substack{M\in\Gamma_0(N)\backslash M_2^*(N) \\ \det M=l}} \frac{\chi_N(a)}{(c\tau+d)^k} \mathbf{e}\left(\frac{-lcz^2}{c\tau+d}\right) \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{lz}{c\tau+d}\right)$$

where $M_2^*(N) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \det M \neq 0, N|c, (a,N)=1\}$, and

$$\text{(2b)} \qquad (\phi|U_l)(\tau,z) := \phi(\tau, lz).$$

c) Operator T_l: For $\phi_{k,m,N} = \sum_{\substack{D\le 0 \\ r\in\mathbb{Z} \\ D\equiv r^2(4m)}} c(D,r) q^{\frac{r^2-D}{4m}} \zeta^r \in J_{k,m,N}$ and a positive integer l prime to mN, we define the Operator $T_l : J_{k,m,N} \longrightarrow J_{k,m,N}$ as follows:

$$\phi_{k,m}|T_l = \sum_{\substack{D\le 0 \\ r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} c^*(D,r) q^{\frac{r^2-D}{4m}} \zeta^r$$

are related to $c(\cdot,\cdot)$ by

$$\text{(4)} \qquad c^*(D,r) = \sum_{\substack{a,r' \\ a|l^2,\, a^2|l^2D \\ l^2D/a^2\equiv 0,1 \pmod 4 \\ r'^2\equiv l^2D/a^2 \pmod{4m} \\ ar'\equiv lr \pmod{2m}}} \chi_N(a)\varepsilon_D(a) a^{k-2} c(\frac{l^2}{a^2}D, r').$$

Here $\varepsilon_D(\cdot)$ is as [EZ] p.50, i.e. defined by

$$\varepsilon_D(a) = \begin{cases} \varepsilon_{D_0}(a/g^2)g \text{ if } (a,D)=g^2 \text{ with } D/g^2 \equiv 0,1 \pmod 4, \\ 0 \text{ otherwise.} \end{cases}$$

For l and l' both prime to mN we have an equation

$$\text{(6)} \qquad T_l \cdot T_{l'} = \sum_{d|(l,l')} d^{2k-3} T_{ll'/d^2}.$$

d) Atkin-Lehner Involution on $J_{k,m,N}$(cf. [EZ]p.60,[MR1]p.2613, [SZ]): For $n\|mN$(i.e. $n|mN$, and n and mN/n are coprime), we have W_n with

$$\text{(7)} \qquad \phi|W_n = \sum_{D,r} c(D,\lambda_n r) q^{\frac{r^2-D}{4m}} \zeta^r$$

where λ_n is the modulo mN uniquely determined integer which satisfies $\lambda_n \equiv -1 \pmod{2n}$ and $\lambda_n \equiv +1 \pmod{2mN/n}$. Thus the W_n form a group of involutions. Finally, note that the W_n and T_l commute, as is easily seen by (4) and (7).

Remark: 1) T_l and W_n are hermitian.

2) U_l, V_l commute with all $T_{l'}$ $((l', lmN) = 1)$, and we have

(7a)
$$U_l \circ W_n = W_{(n,mN)} \circ U_l \quad (n \| mNl^2)$$
$$V_l \circ W_n = W_{(n,mN)} \circ V_l \quad (n \| mNl).$$

2. Construction of Eisenstein series: Let be $k \geq 4$. In this paragraph we construct Jacobi-Eisenstein series, and define their spaces. From now on we consider the case $N = p$.

a) Eisenstein series $E^{\kappa}_{k,m,\chi_p}(\tau, z)$: Using the condition $4nm - r^2 \geq 0$ and the decomposition $m = m_1 m_2^2$ (m_1 : squarefree) we can construct m_2 Jacobi-Eisenstein series $E^{\infty}_{k,m,s,\chi_p}$ $(s = 0, 1, \ldots, m_2 - 1)$ of the cusp ∞ given by

(8)
$$E^{\infty}_{k,m,s,\chi_p}(\tau, z) = \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} q^{m_1 s^2} \zeta^{2m_1 m_2 s} |\gamma \quad (s = 0, 1, \ldots, m_2 - 1)$$

with
$$\Gamma^J = \{(M, [\lambda, \mu]) |\ M \in \Gamma,\ \lambda, \mu \in \mathbb{Z}\}$$

and its subgroup
$$\Gamma^J_\infty = \{\gamma \in \Gamma^J |\ 1|_{k,m}\gamma = 1\} = \{(\pm[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}], [0, \mu]) | n, \mu \in \mathbb{Z}\}$$

where 1 denotes the constant function.

Using (1a) and (1b) this is rewritten explicitly

(8a)
$$E^{\infty}_{k,m,s,\chi_p}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1 \\ c \equiv 0 \pmod p}} \frac{\chi_p(d)}{(c\tau + d)^k} \sum_{\substack{\lambda \in \mathbb{Z} \\ u \equiv 2m_1 m_2 s \\ \pmod{2m}}} e^m\left((\lambda + \frac{u}{2m})^2 M\tau + 2(\lambda + \frac{u}{2m})\frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d}\right)$$
$$= \frac{1}{2} \sum_{\substack{u \in \mathbb{Z} \\ u \equiv 2m_1 m_2 s \\ \pmod{2m}}} q^{u^2/4m} (\zeta^u + (-1)^k \zeta^{-u}) + \underbrace{\ldots\ldots}_{c \neq 0 \text{part}}, \quad (s = 0, 1, \ldots, m_2 - 1).$$

For the cusp 0 we define $E^0_{k,m,s,\chi_p}(\tau,z)$ by

(8b) $$E^0_{k,m,s,\chi_p}(\tau,z) := \tau^{-k} e^m\left(\frac{-z^2}{\tau}\right) E^\infty_{k,m,s,\chi_p}(-\frac{1}{\tau},\frac{z}{\tau}).$$

b) Eisenstein series $E^{\kappa,(\chi)}_{k,m,t,\chi_p}$: For the greatest integer whose square divides mp, i.e. for $Q(mp) = \prod_{p^\lambda \| mp} p^{[\lambda/2]}$, we set

(9) $$E^{\kappa,(\chi)}_{k,m,t,\chi_p}(\tau,z) := \sum_{s \pmod{Q(mp)/t}} \chi(s) E^\kappa_{k,m,ts,\chi_p}(\tau,z)$$

where t are divisors of $Q(mp)$, and χ is a primitive Dirichlet character modulo F with $F|\frac{Q(mp)}{t}$ and $\chi(-1) = (-1)^k$.

Then we can show

(9a) $$\begin{aligned} E^{\infty,(\chi)}_{k,m,t,\chi_p}|T_l &= \sigma^{(\chi)}_{2k-3,\chi_p^2}(l) E^{\infty,(\chi)}_{k,m,t,\chi_p} \quad \text{if } (l,mp)=1 \\ E^{\infty,(\chi)}_{k,m,t,\chi_p}|W_n &= \chi(\lambda_n) E^{\infty,(\chi)}_{k,m,t,\chi_p} \quad \text{if } n\|mp \end{aligned}$$

where $\sigma^{(\chi)}_{k-1,\chi_p^2}(l) := \sum_{0<d|l} d^{k-1}\chi_p^2(l/d)\overline{\chi(d)}\chi(l/d)$, and λ_n denotes any integer as above.

c) Eisenstein series $E^{\kappa,(\chi)}_{k,\chi_p^2}(\tau)$: (cf. [Miy]). We define Eisenstein series in the space of $M_k(mp,\chi_p^2)$ by

(10)
$$E^{\infty,(\chi)}_{k,\chi_p^2}(\tau) = \sum_{l\geq 0} \sigma^{(\chi)}_{k-1,\chi_p^2}(l) q^l \text{ with } \sigma^{(\chi)}_{k-1,\chi_p^2}(l) := \sum_{0<d|l} d^{k-1}\chi_p^2(l/d)\overline{\chi(d)}\chi(l/d) \quad (l\neq 0)$$

and $$\sigma^{(\chi)}_{k-1,\chi_p^2}(0) := \begin{cases} 0 & (m>1) \\ \frac{1}{2}\left(1-p^{k-1}\right)\cdot\zeta(1-k) = \frac{1}{2}L(1-k,\chi_p^2) & (m=1), \end{cases}$$

and $$E^{0,(\chi)}_{k,\chi_p^2}(\tau) := \tau^{-k} E^{\infty,(\chi)}_{k,\chi_p^2}(\frac{-1}{p\tau}).$$

d) The space of cuspforms, Eisenstein series and new forms: We define $J^{\text{Eis,new}}_{k,m,p} := J^{\text{Eis,new}}_{k,m}(\Gamma_0(p),\chi_p)$ as the span of the functions $E^{\kappa,(\chi)}_{k,m,1,\chi_p}$ $(\kappa = 0,\infty)$ if $mp = F^2$ $(F\in\mathbb{N})$ and 0 otherwise, i.e.

$$J^{\text{Eis,new}}_{k,m,p} = \begin{cases} \langle E^{\kappa,(\chi)}_{k,m,1,\chi_p} \mid \chi \bmod F\rangle \text{ if } mp = F^2, \\ \{0\} \text{ otherwise.} \end{cases}$$

Analogeous to [EZ] we can estimate $E^{\kappa,(\chi)}_{k,m,1,\chi_p}$ from $E^{\kappa,(\chi)}_{k,1,1,\chi_p}$ by U_l and V_l operators:

$$E^{\kappa,(\chi)}_{k,m,1,\chi_p} = m^{-k+1} \prod_{q|m} (1+q^{-k+1})^{-1} \sum_{d^2|m} \mu(d) E^{\kappa,(\chi)}_{k,1,1,\chi_p} |U_d V_{m/d^2}$$

Moreover we can show:

$$J^{\mathrm{Eis}}_{k,m,p} = \bigoplus_{\substack{l,l' \\ l^2 l' | mp \\ l_1 l_2 = l^2 l'}} J^{\mathrm{Eis,new}}_{k,\frac{m}{l_1},\frac{p}{l_2}} |U_l V_{l'}. \tag{11}$$

We define the space of Eisenstein series of modular forms. $M_k^{\mathrm{Eis,new}}(mp,\chi_p^2)$ is defined to be zero if mp is not a square, while if mp is a square then $M_k^{\mathrm{Eis,new}}(mp,\chi_p^2)$ is defined to be the span of the series $E^{0,(\chi)}_{k,\chi_p^2}(\tau)$ and $E^{\infty,(\chi)}_{k,\chi_p^2}(\tau)$. For $d \geq 1$ we define an operator B_d by

$$(f|B_d)(\tau) := f(d\tau).$$

Using this operator we obtain

$$M_k^{\mathrm{Eis}}(mp,\chi_p^2) = \bigoplus_{rd|mp} M_k^{\mathrm{Eis,new}}(r,\chi_p^2)|B_d.$$

3. Main Theorem: In this paragraph we construct the mapping between Eisenstein series, and gain the main theorem of this paper:

Definition. *$M^-_{2k-2}(mp,\chi_p^2)$ denote the space of all forms $f \in M_{2k-2}(mp,\chi_p^2)$ satisfying the functional equation*

$$f(\frac{-1}{mp\tau}) = (-1)^k (mp)^{k-1} \tau^{2k-2} f(\tau).$$

(Here the minus-sign means that the L-series of such an f satisfies a functional equation under $s \to 2k-2-s$ with root number -1 and, in particular, vanishes at $s = k-1$). Let be

$$M^{\mathrm{Eis},-}_{2k-2}(mp,\chi_p^2) := M^{\mathrm{Eis}}_{2k-2}(mp,\chi_p^2) \cap M^-_{2k-2}(mp,\chi_p^2).$$

Now we have followings:

Theorem. *Let k, m be integers with $k \geq 4$ even and $m > 0$. Let χ, χ_p be as above. For any fixed fundamental discriminant $d_0 < 0$ and any fixed integer r_S with $d_0 \equiv r_S^2 \bmod 4m$ there is a map*

$$\mathcal{S}_{d_0,r_S} : J_{k,m}^{\mathrm{Eis}}(\Gamma_0(p), \chi_p) \xrightarrow{\text{into}} M_{2k-2}^{\mathrm{Eis},-}(mp, \chi_p^2)$$

given by

$$\sum_{\substack{D \leq 0 \\ r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4m}}} e_{k,m}^{\kappa,(\chi)}(D,r) q^{\frac{r^2-D}{4m}} \zeta^r \mapsto \sum_{l \geq 0} \left\{ \sum_{a|l} a^{k-2} \chi_p(a) \varepsilon_{d_0}(a) e_{k,m}^{\kappa,(\chi)}\left(\frac{l^2}{a^2} d_0, \frac{l}{a} r_S\right) \right\} q^l$$

with the convention

$$\sum_{a|0} a^{k-2} \chi_p(a) \varepsilon_{d_0}(a) e_{k,m}^{\kappa,(\chi)}(0,0) := \frac{1}{2} e_{k,m}^{\kappa,(\chi)}(0,0) \cdot L(2-k, \chi_p \cdot \varepsilon_{d_0}).$$

The maps $\mathcal{S}_{d_0,r_S}$ commute with all Hecke operators T_l ($(l, mp) = 1$) and Atkin-Lehner involutions W_n ($n \| mp$), and map Eisenstein series to Eisenstein series.

Remark: 1) For integers $l, n > 0$ with $(l, mp) = 1$ and $n \| mp$ we denote by T_l and W_n the l-th Hecke operator and the n-th Atkin-Lehner involution on $M_k(mp, \chi_p^2)$, respectively. Thus, for any $f \in M_k(mp, \chi_p^2)$ one has

$$f|T_l = \sum_{r \geq 0} \sum_{d|(l,r)} \chi_p^2(d) d^{k-1} a_f\left(\frac{lr}{d^2}\right) q^n$$

$$f|W_n(\tau) = n^{k/2}(cmp\tau + nd)^{-k} f\left(\frac{an\tau + b}{cmp\tau + dn}\right).$$

Here $a_f(\tau)$ denote the r-th Fourier coefficients of f and a, b, c, d are any integers satisfying $adn^2 - bcmp = n$.

Let be

$$M_{2k-2}^{\mathrm{new},-}(mp, \chi_p^2) := M_{2k-2}^{\mathrm{new}}(mp, \chi_p^2) \cap M_{2k-2}^{-}(mp, \chi_p^2).$$

Comparing 9a) with the description of Eisenstein series $E_{2k-2,\chi_p^2}^{\infty,(\chi)} \in M_{2k-2}^{\mathrm{Eis,new}}(mp, \chi_p^2)$ with $mp = F^2$, we gain

$$E_{2k-2,\chi_p^2}^{\kappa,(\chi)} | T_l = \sigma_{2k-3,\chi_p^2}^{(\chi)}(l) E_{2k-2,\chi_p^2}^{\kappa,(\chi)}$$

$$E_{2k-2,\chi_p^2}^{\kappa,(\chi)} | W_n = \chi(\lambda_n) E_{2k-2,\chi_p^2}^{\kappa,(\chi)}$$

where $(l, F^2) = 1, n\|F^2$ and $\lambda_n \equiv -1 \pmod{2n}$ and $\lambda_n \equiv +1 \pmod{2mp/n}$.

2) For the proof we need the explicit form of Fourier coefficients of Jacobi-Eisenstein series $E^{\infty}_{k,1,\chi_p}$, which are calculated in [Hay2] analogeous to [EZ] p17, where

$$\begin{aligned} E^{\infty}_{k,m,\chi_p}(\tau, z) &:= \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} (1|_{k,m}\gamma)(\tau, z) \\ &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1 \\ c \equiv 0 \pmod N}} \frac{\chi_p(d)}{(c\tau + d)^k} \sum_{\lambda \in \mathbb{Z}} e^m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right). \end{aligned}$$

For the Fourier coefficients we have

Proposition. *Assume $\chi_p(-1) = (-1)^k$. Then the series E^{∞}_{k,m,χ_p} ($k \geq 4$: even) for the cusp $\kappa = \infty$ converges and defines a non-zero element of $J_{k,m,p}$. For*

$$E^{\infty}_{k,m,\chi_p}(\tau, z) = \sum_{\substack{r, D \in \mathbb{Z} \\ D \leq 0 \\ D \equiv r^2 \pmod{4m}}} e^{\infty}_{k,m}(D, r) q^{\frac{r^2 - D}{4m}} \zeta^r.$$

we have the constant terms $e^{\infty}_{k,m}(0, r)$ equals 1 if $r \equiv 0 \pmod{2m}$ and 0 otherwise. For $D < 0$ we have

$$e^{\infty}_{k,1,\chi_p}(D, r) = \lambda_{k,p,\chi_p} \cdot \frac{L_D(2 - k, \chi_p)}{L(3 - 2k, \chi_p^2)} \cdot \frac{c_p(2k - 3, D)}{1 - p^{2-2k}} \text{ with } \lambda_{k,p,\chi_p} = \frac{\chi_p(-1)^{1/2}}{p^{k-1/2}}$$

and $c_p(2k - 3, D)$ elementary representation with p^{2k-3} and D.

*Using V_l, U_l operators we can determine the Fourier coefficients with $m > 1$ (see;**2**).* The L-functions $L(s, \chi_p^2)$ and $L(s, \varepsilon_{D_0} \cdot \chi_p)$ are given by $L(s, \chi_p^2) = \sum_{n>0} \chi_p^2(n) n^{-s}$ and convolution $L(s, \varepsilon_{D_0} \cdot \chi_p) = \sum \varepsilon_{D_0}(n) \chi_p(n) n^{-s}$, respectively. For $D = D_0 f^2$ with $f \in \mathbb{Z}$ and fundamental discriminant D_0 we define

$$L_D(s, \chi_p) = \begin{cases} 0 \text{ if } D \not\equiv 0, 1 \pmod 4, \\ L(2s - 1, \chi_p) \text{ if } D = 0, \\ L(s, \varepsilon_{D_0} \cdot \chi_p) \cdot \gamma^s_{D_0,\chi_p}(f) \text{ if } D \equiv 0, 1 \pmod 4, D \neq 0 \end{cases}$$

where $\gamma^s_{D_0,\chi_p}(f) = \sum_{d|f} \mu(d) \varepsilon_{D_0}(d) \chi_p(d) d^{-s} \sigma_{1-2s,\chi_p^2}(f/d)$
with $\sigma_{s,\chi_p^2}(d) = \sum_{d'|d} \chi_p^2(d') d'^s$.

Proof: [Hay2].

References:

[Ara] Arakawa, T.: Saito-Kurokawa lifting for odd weights, Comment.Math.Univ.St. Paul. 49(2000), 159-176

[EZ] Eichler, M., Zagier, D.: The Theory of Jacobi Forms, Birkhäuser, Boston-Basel-Stuttgart (1985)

[Evd1] Evdokimov, S.A.: Euler products for Congruence subgroups of the Siegel Group of Genus 2, Math. USSR Sbornik, Vol. **28** (1976) No.4 pp431-458

[Evd2] Evdokimov, S.A.: A characterization of the Maass space of Siegel cusp forms of second degree, Math. USSR Sbornik, Vol.**40** (1981), No.1, 125-133

[Hay1] Hayashi, Y.: Remarks on Spinor Zeta Function, Saito-Kurokawa Lifting and Jacobi Forms on the Jacobi Group $\Gamma_0(N)\ltimes\mathbb{Z}^2$ with Primitive Dirichlet Character, preprint (2000), to be submitted

[Hay2] Hayashi, Y.: Fourier coefficients of Jacobi Eisenstein series with quadratic primitive character, preprint (2001),

[Hor1] Horie, T.: A note on the Rankin-Selberg method for Siegel cusp form of genus 2, Proc. Japan Acad. **75**, Ser.A (1999), p18-21

[Hor2] Horie, T.: A note on the Rankin-Selberg method for Siegel cusp form of genus 2, RIMS-Kokyuroku. **1103**, (2000), p169-181

[Koh] Kohnen, W.: Modular Forms of Half-Integral Weight on $\Gamma_0(4)$, Math.Ann.248. p249-266 (1980)

[Kra] Kramer, J.: Jacobiformen und Thetareihen, manuscr. math.54, 279-322 (1986)

[MR1] Manickam, M., Ramakrishnan, B.: On Shimura, Shintani and Eichler-Zagier correspondences, Trans. AMS Vol **352**, No 6, p2601-2617 (2000)

[MR2] Manickam, M., Ramakrishnan, B.: On Saito-Kurokawa Correspondence of Degree two for Arbitrary Level, J.Ramanujan Math.Soc.17,No.3 pp149-160, (2002)

[MRV] Manickam, M., Ramakrishnan, B., Vasudevan,T.C.: On Saito-Kurokawa Descent for congruence subgroups, manuscripta math. **81**, 161-182 (1993)

[Miy] Miyake,T.: Modular forms, Springer-Verlag, Berlin-Heidelberg, (1989)

[Skog] Skogman, H.; On the Fourier Expansions of Jacobi Forms, preprint

[SZ] Skoruppa N.-P., Zagier D.: Jacobi forms and a certain space of modular forms, Invent.Math. 94. (1988)

Address of Author:

Class Preparation Office (Room No. kyo203-2)

Department for the Promotion of General Education and Liberal Arts

Kyoto University

Yoshida-Nihonmatsu-cho, Sakyo-ku, Kyoto

606-8501 JAPAN

yoshiki@kurims.kyoto-u.ac.jp