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THE RELATIVE TRACE FORMULA AND ITS APPLICATIONS

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1. INTRODUCTION

The relative trace formula is a tool introduced by Jacquet to study periods integrals of the form

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\, dh.$$

Here H is subgroup of a reductive G defined over a number field F, and φ is a cusp form on $G(F)\backslash G(\mathbb{A})$. We say that a cuspidal automorphic representation π of $G(\mathbb{A})$ is distinguished by H if this period integral is non-zero for some φ in the space of π. This notion is useful for instance if H is obtained as the fixed points of an involution θ of G defined over F. These periods appear in several contexts – both analytic and geometric. Sometimes they are related to special values of L-functions through a Ranking-Selberg type integral. Moreover, these special values characterize the property that π is obtained as a functorial transfer from an automorphic representation on a third group G'. However, interestingly enough, there are cases where the relation between the period integral and L-values is much more subtle.

The relative trace formula is a variant of the Kuznetsov trace formula, which is in turn the outgrowth of the work of Petterson, and earlier, Kloosterman. The analysis of the Kuznetsov trace from a representation theoretic point of view is carried out in detail in [CPS90]. Since the Kuznetsov trace formula is treated extensively in the article of Michel in this volume, we shall say no further here.

Just like the usual trace formula is a tool for studying harmonic analysis on a group, the relative trace formula does so for the symmetric space G/H. Sometime, two symmetric spaces are related geometrically, and "therefore" they should also compare spectrally. These remarks are equally applicable in the local and global cases. This, however, is easier said than done. The road to obtaining spectral comparison is full of ridges and requires a lot of technical skill and insight. Moreover, the experience of Jacquet-Lai-Rallis suggests that certain automorphic weight factors have to be incorporated into the expressions before a comparison can be carried out.

The goal here is to single out certain cases where the relative trace formula has been worked out, including some exciting new developments. We refer the reader to the excellent expository papers of Jacquet for a more laid back discussion and background about the relative trace formula ([Jac97] ,[Jac04a]). Needless to say that the influence of Jacquet on the author in this subject is immense and the author would like to express his appreciation and gratitude to Jacquet for sharing his deep insight and fantasies.

2. Examples pertaining to residues of Eisenstein series

We already mentioned that periods are often related to special values of L-functions, and that these, in turn, are related to functoriality. However, there is more than one functoriality involved. Namely, the non-vanishing of the special value is also responsible for a pole of an Eisenstein series built from π on a group for which G is a (maximal) Levi subgroup. This double role is extremely important. It is the point of departure for the descent method of Ginzburg-Rallis-Soudry for constructing the inverse lifting from general linear groups to classical groups using Fourier coefficients of residues of Eisenstein series on a larger classical group. Mao and Rallis formulated a sequence of relative trace formula identities which go side by side with the steps ascribed by the descent method. Their work so far deals with the geometric side of the relative trace formula. Once completed, it will give a trace identity for the lifting from (generic cuspidal representations on) classical groups to general linear groups. As a by-product, it will circumvent the current use of the converse theorem for this functorial transfer.

Let then $P = MU$ be a maximal parabolic (over F) of G (quasi-split) with its Levi decomposition and let π be a generic cuspidal automorphic representation of $M(\mathbb{A})$. Suppose, as is often the case, that the existence of an appropriate pole for the Eisenstein series is detected by a period integral over a certain subgroup H_M.

In all known cases there is an "outer period" on G over the period subgroup H which is computed in terms of the "inner" period over $H_M = H \cap M$. Typical examples include:

(1) $G = GL_n$ ([JR92]). The only maximal parabolic which contributes to the discrete spectrum occurs for $n = 2m$ and it is $M = GL_m \times GL_m$; the cuspidal data is $\pi \otimes \pi$ where π is a cuspidal representation of GL_m. The relevant L-function is the

Rankin-Selberg convolution of π with its contragredient. The outer period subgroup is $H = Sp_n$, and the period over $H \cap M$ is essentially the inner product on π.

(2) A twisted analogue of the above is the case $G = U_n$ quasi-split, with respect to a quadratic extension E/F and $M = GL_n(E)$. The pertaining L-function is the Asai L-function and the period subgroup is $GL_n(F)$. This case is considered in the forth-coming thesis of Tanai.

(3) $G = Sp_{4n}$ and P is the Siegel parabolic ([GRS99]). Here, $H = Sp_{2n} \times Sp_{2n}$ and $M \cap H = GL_n \times GL_n$. The relevant period is related to the Bump-Friedberg integral representation of $L(s_1, \pi)L(s_2, \pi, \wedge^2)$ ([BF90]).

(4) Similarly for $G = SO(4n)$ and P – Siegel parabolic. The inner period is the Shalika period (cf. [JS90]). The outer period is a certain Fourier coefficient on the unipotent radical of a Siegel type parabolic, integrated over its stabilizer (which is isomorphic to Sp_{2n}).

(5) There are some interesting exceptional cases ([GJ01], [GL]).

In all these cases, the H-period vanishes on all cuspidal representations (generic or not), cf. [AGR93]. This, however, is not a general feature. It makes a big difference, though, because it then means that the comparison of the relative trace formulas pertaining to M and G will be much simpler. (The functoriality involved is "simply" taking residue of an Eisenstein series.) At any rate, the comparison for the above cases is carried out in [LR04], [JLR04], [GL]. Interestingly enough, the automorphy weight factor, alluded to above, is a certain degenerate Eisensetin series – exactly of the type which appear in the integral representation of the pertinent L-function. Roughly speaking, its role is to cancel out the effect of the residue of the intertwining operator (which is essentially a quotient of "consecutive" values of this L-function. (The residue of the intertwining operator appears in the computation of the inner product of residual Eisenstein series, cf. [Lan90].)

These results should be considered as test cases for the general conjectures of Jacquet-Lai-Rallis (which have to be made precise) – see [JLR93]. They also play a role toward better understanding the descent construction of Ginzburg-Rallis-Soudry alluded to above.

3. THETA CORRESPONDENCE

The previous discussion tried to highlight the fact that explicit constructions in the theory of automorphic forms are handy in obtaining spectral identities. One of the most basic automorphic forms is the theta function used by Jacobi (and in fact, by Riemann in his proof of the analytic continuation of the zeta function which bears his name). The modern theory of Jacobi's theta function and its relatives is the theta correspondence. It gives rise to a way (one of the very few ways known to us) of constructing automorphic forms on a group (from those of another group). I will not review the method here – it is well covered in the literature – but will recall that one of the more famous instances of it is a relation between representations of GL_2 with trivial central character, and representations of the metaplectic cover of SL_2. It was Shimura who first conceived this relation in the context of half-integral weight modular forms. (Shimura used the converse theorem, rather than the Weil reprsentation.) In his seminal work, he related the Dn^2-th Fourier coefficient of a half-integral modular form to the n-th Fourier coefficient of its image F, twisted by χ_D with some proportionality constant depending on D ([Shi73]). (Here D is square-free and χ_D is the corresponding quadratic character.) Subsequently, Waldspurger computed this proportionality constant, namely, he related the square-free Fourier coefficients to $L(\frac{1}{2}, F \otimes \chi_D)$. As is well-known, this fundamental result, combined with other results and conjectures, has a great many far-reaching applications.

Jacquet has used the relative formula to reformulate, reprove and sharpen Waldspurger's results (and their subsequence generalizations) – see [Jac84] [Jac86], [Jac87a], [Jac87b], [Jac91]. One of the advantages of the relative trace formula setup is that it puts the local issues of the problem in their right perspective, and it makes the ultimate formulae (from which special cases may be derived with some effort, cf. [BM03], [BMb], [BMa]) more transparent and conceptual. Following the work of Mao and Rallis, it became clear that the geometric issues of the relative trace formula can in fact be resolved using the machinery of the Weil representation ([MR97], [MR99], [MR04]). Thus, although the relative trace formula voids the use of the theta correspondence in its formulation, the latter can nevertheless be used in its analysis. Recently, we observed that in fact, the spectral identity too can be inferred from the setup of the theta correspondence. This will appear in a joint paper of Baruch, Mao and the author. The result will be a generalization of Waldspurger's formula for generic cuspidal automorphic form f on $SO(2n+1)$ and their theta-lift f' to the metaplectic cover of Sp_{2n}. Namely, if we L^2-normalize the forms then

$$|\frac{c(f')}{c(f)}|^2 \sim L(\frac{1}{2}, f)$$

where $c(f)$ and $c(f')$ are appropriate Fourier coefficients of f and f'. This formula builds on earlier computations of Furusawa, and in fact, it can be thought of as a spectral interpretation thereof ([Fur95]). There is also a local analogue, involving local Bessel distributions, and where the proportionality constant is the root number. This is one in a whole slew of spectral identities which can be obtained using the apparatus of the theta correspondence.

4. UNITARY PERIODS

Let E/F be a quadratic extension and consider cuspidal representations of $GL_n(\mathbb{A}_E)$ which are distinguished by a unitary group U_n (not necessarily quasi-split). It has been long conjectured by Jacquet that these representations are characterized as the functorial transfer of automorphic representations of $GL_n(\mathbb{A}_F)$ via base change. Jacquet has proposed a relative trace formula comparison between a Kuznetsov trace formula on $GL_n(\mathbb{A}_F)$ on the one hand and a relative version of it on the other hand. Several low rank cases where extensively studied by Jacquet and Ye ([JY90], [JY92], [Jac92], [Ye97], [Jac95], [JY96], [JY99], [Ye99], [Ye98], [Jac01]). However, it was clear that more ideas are required to deal with the higher rank case. In the last few years Jacquet succeeded in dealing with the issues coming from the geometric side, namely proving the existing of matching functions ([Jac02], [Jac03a], [Jac03b]) and subsequently, proving the fundamental lemma in this setup, initially for the unit element of the Hecke algebra ([Jac04b]) and ultimately, in general ([Jac]). Parallel to these developments, the author obtained the fine spectral expansion of the relative trace formula ([Lap]). Finally, Jacquet's conjecture is proved!

For the geometric part of the trace formula, Jacquet first linearized the problem by considering functions on the spaces of all $n \times n$ matrices and Hermitian matrices, rather than the invertible ones. He then expressed the orbital integral of a Fourier transform of a function by means of the *Jacquet-Klooseterman transform* of the original orbital integral. This builds on an idea of Waldspurger for the usual trace formula (on the Lie algebra) – see [Wal97], [Wal00]. Using this it is "not too difficult" to establish the transfer (at least in the p-adic case). The fundamental Lemma requires, in addition, some "uncertainty principle" for the Jacquet-Kloosterman transform which can be reduced to the usual uncertainty principle (for a function and its Fourier transform). To carry out this reduction Jacquet developed an elaborate combinatorial scheme using certain boxed diagrams. We will not venture into this very aesthetic approach. Strictly speaking, the combinatorics only gives matching of Hecke functions, but it is rather intricate to explicate the ensuing linear map (say β) between the Hecke algebras, and in particular, to compare it with the expected base change map (say b). To that end, Jacquet used global means to show that β is in fact an algebra homomorphism. Therefore, it is enough to show that β agrees with b on a set of generators, and this is much less daunting, and in any case, doable.

To deal with the spectral side, the point of departure, as always, is to describe the kernel using Langlands L^2-theory, namely as sums of integrals of the Eisenstein series, and then integrate in the group variables. The unipotent integration, being compact, poses no analytical difficulties, and handsomely kills all contribution from residual Eisenstein series. However, the period integral of the Eisenstein series does not converge, and *a fortiori* it is not possible to interchange the order of integrations over the spectral parameter and the group variable. The standard approach to overcome these kind of difficulties is to use Arthur's truncation operator, and then analyze the behavior in the truncation parameter. This can be made to work, but we chose a slightly different approach. Namely, we first write the Eisenstein series in terms of its truncated parts (along different parabolics). (This is the inversion formula for truncation, based on a well-known geometric lemma of Langlands. For technical reasons, a slightly different variant called mixed truncation is used.) Each term is indexed by a parabolic subgroup and a Weyl chamber thereof. The obstruction to interchange the order of integration boils down to the non-integrability of $e^{\lambda x}$ (on $x \in \mathbb{R}_{>0}$) for $\mathrm{Re}\lambda = 0$. However, shifting the contour of integration slightly to $\mathrm{Re}\lambda$ in the appropriate chamber, the double integral will converge and it is possible to interchange the order, obtaining an exponential divided by a product of linear factors from the inner integration. Shifting the spectral parameter back to $\mathrm{Re}\lambda = 0$ is again subtle, exactly because of these linear factors in the denominators. However, we can at least do it in "Cauchy's sense", not forgetting to take into account terms coming from iterated residues. All in all, we obtain sums of expressions which miraculously group together to terms which are independent of the truncation parameter (as they should!) and which are actually holomorphic for $\mathrm{Re}\lambda = 0$. The latter terms are the regularized periods defined in [JLR99]. They were studied further in [LR03] (see also [LR02]) and a formula for them was given in terms of intertwining periods – an analogue of the standard intertwining operators in the theory of Eisenstein series. The rearranging of terms and the residue calculus requires some knowledge of analytic properties of Eisenstein, namely polynomial bounds (uniformly, not only on average) in terms of the spectral parameter (as well as the group variable) on, or even near, $\mathrm{Re}\lambda = 0$. This problem can be translated, using Arthur's formalism of (G, M)-families and the manipulation of [Art82], into a problem of lower bounds of L-functions near $\mathrm{Re}s = 1$, together with some knowledge toward the Ramanujan Hypothesis. (A similar analysis is carried out by Müller in his investigations on the Arthur-Selberg trace formula and its analytic properties – cf. [Mül02], [MS04].) For reductive groups these problems are unsolved in general. Fortunately for GL_n, the required lower bounds for Rankin-Selberg L-function are known by the work of Brumley ([Bru]), while uniform bounds toward the Ramanujan Hypothesis were given by Luo-Rudnick-Sarnak ([LRS99]). Ultimately, the relative trace formula is spectrally expanded in terms of intertwining periods, and the sum-integral is absolutely convergent.

Jacquet is now in the process of completing the geometric aspects of the comparison in the real case. (So far, a technical assumption that all real places of F split in E was put into place in order to circumvent this issue.) Even without this, it is still possible to obtain an interesting result about periods over anisotropic unitary groups. Namely, suppose that F is totally real and E is totally complex (i.e. a CM-field). In principle, all unitary groups appear in the relative trace formula. However, we can single out the anisotropic ones by choosing an appropriate test function, without losing much spectrally. The Kuznetsov counterpart of the formula is reasonably well understood spectrally. The result would be a formula for the anisotropic unitary periods in terms of L-function, roughly speaking

> The norm square of the anisotropic unitary period of the base change Π of π (assumed cuspidal) is "equal" to the quotient
>
> $$\frac{L(1, \pi \times \tilde{\pi} \otimes \eta_{E/F})}{\mathrm{res}_{s=1} L(s, \pi \otimes \tilde{\pi})}$$

where $\eta_{E/F}$ is the quadratic Hecke character of F attached to E by class field theory. To explicate this formula, it will be useful to have a comparison of the local Bessel distributions (defined appropriately) with an explicit proportionality constant. For $n = 3$ and principal series this was done in [LR00] using global methods. Omer Offen is working on generalizing this for $n > 3$, taking into account the aforementioned recent developments. It will require the detailed results of Hironaka on spherical functions on Hermitian matrices ([Hir99]).

This kind of formula, once explicated completely, has an application toward some recent L^∞-norm conjecture of Sarnak ([Sar04]). Namely, thinking of Π as a Maass form φ on a locally symmetric space $\Gamma\backslash G/K$

(where G is a product of $GL_n(\mathbb{C})$'s and K is its maximal compact – a product of compact U_n's), the period becomes a finite sum of point evaluations of φ. On the right hand side, the finite part of the L-function is expected to be sub-exponential in the logarithm of the spectral parameter. (This can be proved sometimes, at least on average.) Therefore the behavior is dictated by the archimedean part of the L-function which by simple properties of the Γ-function is easily seen to be roughly $\lambda^{n(n-1)/4}$, where λ is the eigenvalue of the Casimir (or Laplacian), at least when the parameters of π_∞ are in general position. This would give a lower bound of the order of magnitude of $\lambda^{n(n-1)/8}$ for $\|\varphi\|_\infty$. The upper bound $\lambda^{n(n-1)/4}$ is the "convexity" bound in this setup, and comes from local considerations of the symmetric space (cf. [Sar04]). For $n = 2$ these results had been obtained by Rudnick and Sarnak using the theta correspondence ([RS94]). However, the latter is not applicable for $n > 2$.

5. Concluding remarks

The reader may have been conveyed the impression, which is shared by the author, that at this stage the development of the relative trace formula is more by means of examples (or families of them), rather than by general setup and methodology (even a conjectural one). This is a main lacking feature in the theory, especially compared to other leading themes in automorphic forms, for instance endoscopy (in its various guises). Such formulation, if possible, will have to take into account automorphic forms on metaplectic covers (i.e. non-algebraic groups) for which strictly speaking functoriality does not apply in its current formulation. (They already appear when comparing the pair PGL_2 and the torus, with the metaplectic cover of SL_2.)

For the more impatient reader there are many other important cases, beyond those considered above, where the relative trace formula has a substantial payoff, though hard earned.

One of the more tantalizing ambitions is to find a trace formula interpretation (even a conjectural one) of the Gross-Zagier formula. As a wishful thinking, such a formulation will provide an interpretation of the value of the derivative $L'(\frac{1}{2},\pi)$ of a GL_2 L-function, even in cases where π comes from a Maass form. At the moment, there is no really cogent reason to believe that such an approach is possible. It will certainly require new ideas.

There are also interesting cases in which the H-period does not support generic representations, but nevertheless it is either known or expected that there exist distinguished cuspidal representations.

Those representations are presumably CAP representations attached to a residual Eisenstein series. Our understanding of these representations has developed in recent years due to the fundamental work of Ginzburg, Rallis and Soudry. It would be of great interest to see to what extent these constructions can shed light on the trace formula and vice versa.

It will also be interesting to interpret the recent results of Luo-Sarnak (cf. this volume) in the context of the relative trace formula. This is plausible since Watson's formula for the triple product can be interpreted in terms of the see-saw formalism of the theta correspondence, which in turn can also be interpreted using the relative trace formula.

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