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Forking in Generic Structures

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Abstract

Baldwin の問題とは，**superstable** であるが ω**-stable** でない **generic** 構造が存在するか，という問題である ([1])．この問題の部分的結果として，**generic** 構造が **saturated** のときは，理論が **superstable** ならば必ず ω**-stable** となることがわかった (定理 10)．

1 Preliminaries

Many papers [2,3,4,8] have laid out the basics of generic structures. So we do not explain all of those details here.

Let L be a countable relational language and $\mathbf{K}^*$ a class of the finite L-structures. Then $\delta : \mathbf{K}^* \to \mathrm{R}^+$ is said to be a *predimension*, if (i) for all $AB \in \mathbf{K}^*, \delta(A/B) \leq \delta(A/A \cap B)$; (ii) if $A \cong B \in \mathbf{K}^*$, then $\delta(A) = \delta(B)$; (iii) $\delta(\emptyset) = 0$; (iv) If $A \subset A', B \subset B', C \subset C'$ and A', B', C' are pairwise disjoint, then $\delta(B/A) - \delta(B/AC) \leq \delta(B'/A') - \delta(B'/A'C')$, where $\delta(A/B)$ denotes $\delta(AB) - \delta(B)$.

Let $A \subset B \in \mathbf{K}^*$. Suppose that A is finite. Then A is said to be *closed* in B (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any finite $X \subset B - A$. In general, A is said to be closed in B, if $A \cap X \leq B \cap X$. We define the *closure* of A in B by $\mathrm{cl}_B(A) = \bigcap\{C : A \subset C \leq B, |C| < \omega\}$. We define a *dimension* of A in B by $\mathrm{d}_B(A) = \delta(\mathrm{cl}_B(A))$.

Let $\mathbf{K}$ be a subclass of $\mathbf{K}^*$ that is closed under substructures, and M a saturated $\mathbf{K}$-generic structure.

$\mathbf{K}$ is said to have *finite closures*, if there are no chains $A_0 \subset A_1 \subset \cdots$ of elements of $\mathbf{K}$ with $\delta_\alpha(A_{i+1}) < \delta_\alpha(A_i)$ for each $i < \omega$. If $\mathbf{K}$ has finite closures, then we can see that there exists a unique $\mathbf{K}$-generic structure M, and moreover that any finite set of M has finite closures. On the other hand, it can be seen that if a $\mathbf{K}$-generic structure M is saturated then $\mathbf{K}$ has finite closures. We summarize our situation.

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Assumption $\mathbf{K} = (K, \leq)$ is derived from a predimension δ such that $\mathbf{K}$ is closed under substructures. M is a saturated $\mathbf{K}$-generic structure.

2 Smallness of algebraic types

Definition Let AB be finite L-structure. Then
(i) A pair (B, A) is said to be $\mathbf{K}$-*normal*, if $A \leq AB \in \mathbf{K}$ and $A \cap B = \emptyset$.
(ii) A $\mathbf{K}$-normal pair (B, A) is said to be *minimal*, if $\delta(C/A) > \delta(B/A)$ for any non-empty proper subset C of B.
(iii) A $\mathbf{K}$-normal pair (B, A) is said to be *weakly small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$-normal, then $\delta(D/C) \geq \delta(B/C)$.
(iv) A $\mathbf{K}$-normal pair (B, A) is said to be *pseudo-small*, if whenever $A \subset C$ and (B, C) is $\mathbf{K}$-normal, then $\delta(B/C) \geq \delta(B/A)$.
(v) A $\mathbf{K}$-normal pair (B, A) is said to be *small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$-normal, then $\delta(D/C) \geq \delta(B/A)$.

Note 1 A $\mathbf{K}$-normal pair (B, A) is small if and only if it is weakly small and pseudo-small.

Lemma 2 Let (B, A) be $\mathbf{K}$-minimal with $A \leq AB \leq M$. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is weakly $\mathbf{K}$-small.

Proof Suppose by way of contradiction that (B, A) is not weakly $\mathbf{K}$-small. Then there are $C \supset A$ and $D \supset B$ such that (D, C) is $\mathbf{K}$-normal and $\delta(D/C) < \delta(B/C)$.
Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of B with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0...B_i, CB_0...B_{i-1}D \leq CB_0...B_iD \in \mathbf{K}$ for each $i < \omega$;
(iii) $D, B_0, B_1, B_2, ...$ are pairwise disjoint.
Proof of Claim 1: We construct $\{B_i\}_{i<\omega}$ inductively. Suppose that $\{B_i\}_{i\leq n}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_nD \in \mathbf{K}$, and so we have $CB_0...B_n \leq CB_0...B_nB \in \mathbf{K}$. By the amalgamation property, we can take a copy B_{n+1} of B over $CB_0...B_n$ such that
(*) $CB_0...B_nD, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}D \in \mathbf{K}$.

Hence B_{n+1} satisfies (i) and (ii). On the other hand, $B_0, B_1, ..., B_{n+1}$ are pairwise disjoint, since $B_{n+1} \cong_{CB_0...B_n} B$ and $B \subset D$. So, to see that (iii) holds it is enough to show that $B' = B_{n+1} \cap D = \emptyset$. If $B' = B_{n+1}$ would hold, then we have $B_{n+1} \subset D$, and so $CB_{n+1} \not\leq CD$, since $\delta(D/C) < \delta(B/C) = \delta(B_{n+1}/C)$. This contradicts (*), and hence we have $B' \neq B_{n+1}$. By (*) again, we have $CB_0...B_nD \leq CB_0...B_nB_{n+1}D$, and so $AB' \leq AB_{n+1}$. Since (B, A) is a minimal pair, we have $B' = \emptyset$. (End of Proof of Claim 1)
Claim 2: $AB, AB_j \leq AB_0...B_iB (\in \mathbf{K})$ for $j \leq i < \omega$
Proof: We prove by induction on i. By (ii) of claim 1, $AB_0...B_iB \leq AB_0...B_{i+1}B$.

By induction hypothesis, we have $AB, AB_j \leq AB_0...B_iB$ for $j \leq i$. Hence $AB, AB_j \leq AB_0...B_{i+1}B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0...B_{i+1}B$. By induction hypothesis again, we have $AB \leq AB_0...B_iB$. From (i) of claim 1, it follows that $AB_{i+1} \leq AB_0...B_{i+1}$. By (ii) of claim 1, $AB_0...B_{i+1} \leq AB_0...B_{i+1}B$. Hence we have $AB_{i+1} \leq AB_0...B_{i+1}B$. (End of Proof of Claim 2)

We show that $\mathrm{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $AB, AB_j \leq ABB_0...B_i \leq M$ for each i, j with $j \leq i < \omega$. So we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j \leq i$. By (iii) of claim 1, B_j's are pairwise disjoint. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Definition We say that $\mathbf{K}$ is *closed under δ-substructures*, if for any disjoint A, B, C with $ABC \in \mathbf{K}$, there is a copy B^* of B over A with $\delta(B^*/CA) = \delta(B^*/A)$.

Lemma 3 Assume that $\mathbf{K}$ is closed under δ-substructures. Let (B, A) be $\mathbf{K}$-minimal with $A \leq AB \leq M$. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is pseudo-$\mathbf{K}$-small.

Proof Suppose by way of contradiction that (B, A) is not pseudo-$\mathbf{K}$-small. Then there is $C \supset A$ such that (B, C) is $\mathbf{K}$-normal and $\delta(B/C) < \delta(B/A)$.
Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of B over A with the following conditions:
(i) $C \leq CB_j \leq CB_0B_1 \cdots B_i \in \mathbf{K}$ for each $j \leq i < \omega$
(ii) $B_0, B_1, B_2, ...$ are pairwise disjoint.
(iii) $B_i \cap C = \emptyset$ for each $i < \omega$;
(iv) $\delta(B_i/C) = \delta(B_i/A)$ for each $i < \omega$.
Proof: Suppose that $\{B_i\}_{i\leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathbf{K}$, and by (i) we have $C \leq CB_0B_1...B_n \in \mathbf{K}$. So we can take a copy B^* of B over C such that $CB_0...B_n, CB^* \leq CB_0...B_nB^* \in \mathbf{K}$. By (iv), $\delta(B_i/C) = \delta(B_i/A)$ for each $i \leq n$. On the other hand, we have $\delta(B^*/C) < \delta(B^*/A)$. So we have $B_i \neq B^*$ for all $i \leq n$. Since (B, A) is $\mathbf{K}$-minimal, B and B_i's are pairwise disjoint. Since $\mathbf{K}$ is closed under δ-substructures, B_{n+1} with $B_{n+1} \cong_{AB_0B_1...B_n} B^*, CB_0B_1...B_nB_{n+1} \in \mathbf{K}$ and $\delta(B_{n+1}/C) = \delta(B_{n+1}/A)$. Then we can see that (i)–(iv) hold. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0...B_i (\in \mathbf{K})$ for $j \leq i < \omega$. So we can assume that $AB_j \leq AB_0...B_i \leq M$ for each i, j with $j \leq i < \omega$. Thus we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, B_j's are pairwise distinct. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Lemma 4 If (B, A) and (C, BA) are $\mathbf{K}$-small, then so is (BC, A).

Proof Take any $\mathbf{K}$-normal pair (E, D) such that $BC \subset E$ and $A \subset D$. Then note that $(E - B, BD)$ is $\mathbf{K}$-normal. (Proof: Take any $X \subset E - B$. Note that (XB, D) is **K**-normal since (E, D) is so. Since (B, A) is **K**-small, we have $\delta(X/BD) = \delta(XB/D) - \delta(B/D) \geq \delta(XB/D) - \delta(B/A) \geq 0$. Hence $(E - B, BD)$ is **K**-normal.) Since (C, BA) is **K**-small, we have $\delta(E/BD) \geq \delta(C/BA)$. On the other hand, since (B, A) is **K**-small and (B, D) is **K**-normal, we have $\delta(B/D) = \delta(B/A)$. It follows that $\delta(BC/A) = \delta(C/AB) + \delta(B/A) \leq\leq \delta(E/BD) + \delta(B/D) = \delta(E/D)$. Hence (BC, A) is **K**-small.

Thoerem 5 Assume that **K** is closed under δ-substructures. Let (B, A) be **K**-normal with $AB \leq M$. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is **K**-small.

Proof Let $\mathrm{tp}(B/A)$ be algebraic. Take a sequence $A = B_0 \leq B_0B_1 \leq ... \leq B_0B_1...B_n = AB$ with (B_{i+1}, B_i) **K**-minimal for each i. Since each $\mathrm{tp}(B_{i+1}/B_0...B_i)$ is algebraic, it is **K**-small by lemma 2 and 3. So, by lemma 4, $(B, A) = (B_0B_1...B_n, B_0)$ is **K**-small.

3 Forking and dimension

In this section, we assume that **K** is closed under δ-substructures.

Lemma 6 Let $A \subset B$. If B is closed, then so is $B \cup \mathrm{acl}(A)$.

Proof We can assume that A, B are finite. It is enough to show that BA^* is closed for any finite A^* with $A \subset A^* \leq \mathrm{acl}(A)$. Let $A' = A^* \cap B$. Since $\mathrm{tp}(A^*/A')$ is algebraic and $A' \leq A^*$ is closed, $(A^* - A', A')$ is small. Then we can see that BA^* is closed as follows. If not, then there is finite $X \subset N - BA^*$ with $\delta(X/BA^*) < 0$. Then we have $0 \leq \delta(XA^*/B) = \delta(X/BA^*) + \delta(A^*/B) < \delta(A^*/B) \leq \delta(A^*/A')$. This contradict that $(A^* - A', A')$ is small.

Fact([7], [8]) Let B, C be closed and $A = B \cap C$ algebraically closed. Then the following are equivalent.
(i) $\mathrm{d}(B/A) = \mathrm{d}(B/C)$;
(ii) B and C are free over A, and BC is closed;
(iii) $\mathrm{tp}(B/C)$ does not fork over A.

Lemma 7 Assume that **K** is closed under δ-substructures. Let B, C be closed and $A = B \cap C$. If B, C are free over A and BC is closed, then $\mathrm{tp}(B/C)$ does not fork over A.

Proof By lemma 6, $B\mathrm{acl}(A), C\mathrm{acl}(A)$ is closed. Since BC is closed, by lemma 6 again, $BC\mathrm{acl}(A)$ is closed. So, by fact, to show that $\mathrm{tp}(B/C)$ does not fork over A, it is enough to prove that $\mathrm{acl}(A)B, \mathrm{acl}(A)C$ are free over $\mathrm{acl}(A)$. Take finite closed $B_0 \subset B, C_0 \subset C$ such that B_0C_0 is closed. Let $A_0 = B_0 \cap C_0$. Take

finite closed $A^* \subset \mathrm{acl}(A_0)$. Let $D = A^* - B_0C_0$ and $A' = A^* \cap B_0C_0$.
Claim 1: $\delta(B_0C_0/A^*) = \delta(B_0C_0/A')$.
Proof: Since $\mathrm{tp}(D/A')$ is algebraic and (D, A') is normal, (D, A') is small. So we have $\delta(D/B_0C_0) = \delta(D/A')$. Then $\delta(B_0C_0/A^*) - \delta(B_0C_0/A') = \delta(D/B_0C_0) - \delta(D/A') = 0$. (End of Proof of Claim 1)
Claim 2: $\delta(B_0/C_0A^*) = \delta(B_0/A^*)$.
Proof: Since B, C are free over A, we can see that $\delta(C_0/A'B_0) = \delta(C_0/A')$. Then $\delta(B_0/C_0A^*) = \delta(B_0C_0/A^*) - \delta(C_0/A^*) \geq \delta(B_0C_0/A') - \delta(C_0/A') = \delta(B_0C_0/A') - \delta(C_0/A'B_0) = \delta(B_0/A') \geq \delta(B_0/A^*)$. Hence $\delta(B_0/C_0A^*) = \delta(B_0/A^*)$.

By claim 2, $B\mathrm{acl}(A), C\mathrm{acl}(A)$ is free over $\mathrm{acl}(A)$.

Lemma 8 Assume that $\mathbf{K}$ is closed under δ-substructures. Let A, B, C be such that B, C are closed and $A = B \cap C$. Suppose that $\mathrm{tp}(B/C)$ does not fork over A. Then
(i) B, C are free over A (,and moreover $\mathrm{acl}(B), \mathrm{acl}(C)$ are free over $\mathrm{acl}(A)$);
(ii) $B \cup C \cup \mathrm{acl}(A)$ is closed.

Proof Let $A^* = \mathrm{acl}(A), B^* = \mathrm{acl}(B)$ and $C^* = \mathrm{acl}(C)$. Clearly $\mathrm{tp}(B^*/A^*)$ does not fork over A^*, and so $B^* \cap C^* = A^*$.
(i) By fact, B^* and C^* are free over A^*. So we obtain that B and C are free over A^*. Let $B' = B \cap A^*, C' = C \cap A^*$. Note that $\delta(B/B'C) = \delta(B/B'C')$. First we show that B' and C are free over A. If not, then there are finite closed $A_0 \subset A, B_0' \subset B', C_0 \subset C$ such that B_0', C_0 are not free over $A_0 = B_0' \cap C_0$. We can assume that $\mathrm{tp}(B_0'/A_0)$ is algebraic. By theorem 5, $(B_0' - A_0, A_0)$ is small. So we have $\delta(B_0'/A_0) = \delta(B_0'/C_0)$. This contradicts that B_0' and C_0 are not free over A_0. Thus B' and C are free over A. Similarly we see that B and C' are free over A. Then $\delta(B/C) = \delta(BB'/C) = \delta(B/B'C) + \delta(B'/C) = \delta(B/B'C') + \delta(B'/C') = \delta(B/C') = \delta(B/A)$. Hence B and C are free over A.
(ii) By fact, we obtain that B^*C^* is closed. If BCA^* is not closed, then there are finite $X \subset B^*C^* - BCA^*, B_0 \subset B, C_0 \subset C, A_0^* \subset A^*$ such that $\delta(X/B_0C_0A_0^*) < 0$. By lemma, BA^*, CA^* are closed, and hence we can assume that $B_0A_0^*, C_0A_0^*$ are closed. Let $X_B = X \cap B^*$ and $X_C = X \cap C^*$. Then $\delta(X_B/B_0C_0X_CA_0^*) = \delta(X_B/B_0A_0^*)$ and $\delta(X_C/B_0C_0A_0^*) = \delta(X_C/C_0A_0^*)$ since B^*, C^* are free over A^*. Therefore we have $\delta(X/B_0C_0A_0^*) = \delta(X_B/B_0C_0X_CA_0^*) + \delta(X_C/B_0C_0A_0^*) = \delta(X_B/B_0A_0^*) + \delta(X_C/C_0A_0^*) \geq 0 + 0 = 0$. A contradiction.

Lemma 9 Assume that $\mathrm{Th}(M)$ is superstable. Then for any countable model N and $p \in S(N)$ there is finite $A \subset N$ such that $p|A$ is stationary.

Proof Take a realization $\bar{b}$ of p. By superstability, there is finite $X \subset N$ such that p does not fork over X. Let $B = \mathrm{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $\mathrm{tp}(\bar{b}/N)$ does not fork over A. We show that $\mathrm{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such that $\mathrm{tp}(\bar{b}'/A) = \mathrm{tp}(\bar{b}/A)$ and $\mathrm{tp}(\bar{b}'/N)$ does not fork over A. Let $B' = \mathrm{cl}(\bar{b}'A)$. Then we have $B \cong_A B'$. Note that $B' \cap N = A$. By lemma 8, B, N and B', N are free over A respectively, and so we have $B \cong_N B'$. By lemma 8 again, $BN, B'N$ are closed since $\mathrm{acl}(A) \subset N$. It follows that $\mathrm{tp}(BN) = \mathrm{tp}(B'N)$ and hence $\mathrm{tp}(b/N) = \mathrm{tp}(b'/N)$.

By lemma 9, we have the following theorem.

Theorem 10 Let L be a countable relational language. Let $\mathbf{K} = (K, \leq)$ be a class of finte L-structures that is derived from a predimension δ and that is closed under substructures. Let M be a saturated $\mathbf{K}$-generic structure. If $\mathrm{Th}(M)$ is superstable, then it is ω-stable.

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