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# End-extension と standardizable extension について

東京大学・大学院 数理科学研究科　村上 雅彦 (Masahiko Murakami)
Graduate School of Mathematical Sciences,
The University of Tokyo

**Abstract**

We consider end-extension and standardizable extension in nonstandard universes. We have treelike order of nonstandard universes of radius $\leq \aleph_1$ about end-extension (Lemma 4) and about standardizable extension (Theorem 5).

## 1 Nonstandard Universe

**Definitions 1 (superstructure, base set).** Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a *base set* if $\varnothing \notin X$ and each element of $X$ is disjoint from $V(X)$. We call a *set in* $V(X)$ an element of $V(X) \setminus X$.

**Definition 2 (nonstandard universe).** A *nonstandard universe* is a triple $\langle V(X), V(Y), \star\rangle$ such that:

1. $X$ and $Y$ are infinite base sets.
2. (Transfer Principle) The symbol $\star$ is a bounded elementary embedding from $V(X)$ into $V(Y)$: which is

   $$V(X) \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi({}^{\star}a_1, \ldots, {}^{\star}a_n)$$

   holds for any bounded formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in V(X)$.
3. ${}^{\star}X = Y$.
4. For every infinite subset of $A$ of $X$, $\{{}^{\star}a \mid a \in A\}$ is a proper subset of ${}^{\star}A$.

**Definitions 3 (standard, internal).** For $a \in V({}^\star X)$, we call $a$ *standard* if there is an $x \in V(X)$ such that $a = {}^\star x$.

For $a \in V({}^\star X)$, we call $a$ *internal* if there is an $x \in V(X)$ such that $a \in {}^\star x$. We denote by ${}^\star V(X)$ the set of all internal elements in $V({}^\star X)$.

From now on, we denote a nonstandard universe by single ${}^\star V(X)$.

**Definitions 4 (norm, radius).** The *norm (of standardness)* of an internal element $a$ is a cardinal defined by

$$\mathrm{nos}(a) = \min\{|x| \mid a \in {}^\star x\}.$$

The *radius* of ${}^\star V(X)$ is a cardinal defined by

$$\mathrm{rad}({}^\star V(X)) = \min\{\kappa \mid \forall y \in {}^\star V(X)\ \mathrm{nos}(y) < \kappa\}.$$

For further detail, we refer to [1]. We shall consider bounded an elementary embedding $e$ from ${}^{\star_1}V(X)$ into ${}^{\star_2}V(X)$.

## 2 End-extension and Standardizable extension

Let $\mathbb{N}$ be a structure of standard natural numbers in $V(X)$, which is isomorphic to $\omega$.

**Definition 5 (end-extension).** An elementary embedding $e\colon {}^{\star_1}V(X) \to {}^{\star_2}V(X)$ is an *end-extension* if any initial segment of ${}^{\star_1}\mathbb{N}$ is not extended by $e$ :

$$\forall n \in {}^{\star_1}\mathbb{N}\ \forall m_2 \in {}^{\star_2}\mathbb{N}\ \exists m_1 \in {}^{\star_1}\mathbb{N}\ {}^{\star_2}V(X) \models m_2 \leq e(n) \Rightarrow m_2 = e(m_1).$$

We say a set $A$ in ${}^\star V(X)$ is *finite in* ${}^\star V(X)$ or $\star$*-finite* if there is a bijection from an initial segment of ${}^\star\mathbb{N}$ onto $A$ inside ${}^\star V(X)$.

**Lemma 1.** *Any ($\star_1$-)finite set in ${}^{\star_1}V(X)$ is not extended by an end-extension $e\colon {}^{\star_1}V(X) \to {}^{\star_2}V(X)$.*

*Proof.* Let $\sigma$ be a bijection from an initial segment $I$ of ${}^{\star_1}\mathbb{N}$ onto $A$ in ${}^{\star_1}V(X)$. Since $e$ is an end-extension, for an element $a \in e(A)$, there is $n \in I$ such that $e(n) = \big(e(\sigma)\big)^{-1}(a)$. Then we have $e\big(\sigma(n)\big) = \big(e(\sigma)\big)\big(e(n)\big) = a$. □

**Definitions 6 (standardization, standardizable extension).** A set $A_1$ in ${}^{\star_1}V(X)$ is a *standardization* a set $A_2$ in ${}^{\star_2}V(X)$ by $e\colon {}^{\star_1}V(X) \to {}^{\star_2}V(X)$ if

$$\forall x \in {}^{\star_1}V(X)\big[[{}^{\star_1}V(X) \models x \in A_1] \Leftrightarrow [{}^{\star_2}V(X) \models e(x) \in A_2]\big].$$

We say $e$ is *$\kappa$-standardizable* if every set of power less than ${}^{\star_2}\kappa$ in ${}^{\star_2}V(X)$ has its standardization by $e$. We say $e$ is *standardizable* if $e$ is $|V(X)|$-standardizable.

**Lemma 2.** *If $e$ is $\omega$-standardizable then $e$ is an end-extension.*

*Proof.* Let $n$ be an element of ${}^{*1}\mathbb{N}$ and let $m_2$ be an element of ${}^{*2}\mathbb{N}$ such that ${}^{*2}V(X) \models m_2 \leq e(n)$. Let $A$ be the standardization of the initial segment $\{k \mid k \leq m_2\}$ of ${}^{*2}\mathbb{N}$. Then we have $e(\max A) = m_2$. □

**Corollary 3.** *Any finite set in ${}^{*1}V(X)$ is not extended by an $\omega$-standardizable extension $e\colon {}^{*1}V(X) \to {}^{*2}V(X)$.*

We cannot prove the converse implication of the previous lemma (see [4]).

*Fact 1.* There is an end-extension which is not $\omega$-standardizable, if continuum hypnosis holds.

# 3 Ordering of Nonstandard Universes by Standardization

In this section, we consider relation of two elementary embeddings $e_1\colon {}^{*1}V(X) \to {}^{*}V(X)$ and $e_2\colon {}^{*2}V(X) \to {}^{*}V(X)$.

**Lemma 4.** *Suppose* $\mathrm{rad}({}^{*}V(X)) \leq \aleph_1$. *If $e_1$ and $e_2$ are end-extensions then there is either an end-extension $e\colon {}^{*1}V(X) \to {}^{*2}V(X)$ such that $e_2 \circ e = e_1$ or $e\colon {}^{*2}V(X) \to {}^{*1}V(X)$ such that $e_1 \circ e = e_2$*

*Proof.* Since both $e_1$ and $e_2$ are end-extensions, $e_1``{}^{*1}\mathbb{N} \subseteq e_2``{}^{*2}\mathbb{N}$ or $e_2``{}^{*2}\mathbb{N} \subseteq e_1``{}^{*1}\mathbb{N}$. Without loss of generality, we can assume $e_1``{}^{*1}\mathbb{N} \subseteq e_2``{}^{*2}\mathbb{N}$. Let $f\colon {}^{*1}\mathbb{N} \to {}^{*2}\mathbb{N}$ be a map satisfying $e_2 \circ f = e_1 \restriction {}^{*1}\mathbb{N}$.

Let $a$ in an element of ${}^{*1}V(X)$. Since $\mathrm{rad}({}^{*}V(X)) \leq \aleph_1$, there are a countable set $R_a$ in $V(X)$ such that $a \in {}^{*1}R$ and a bijection $\sigma_a\colon \mathbb{N} \to R_a$. Defining $e$ by $e(a) = {}^{*2}\sigma_a\big(f(({}^{*1}\sigma_a)^{-1}(a))\big)$, we have completed the proof. □

**Theorem 5.** *Suppose* $\mathrm{rad}({}^{*}V(X)) \leq \aleph_1$. *If $e_1$ and $e_2$ are standardizable then there is either a standardizable $e\colon {}^{*1}V(X) \to {}^{*2}V(X)$ such that $e_2 \circ e = e_1$ or $e\colon {}^{*2}V(X) \to {}^{*1}V(X)$ such that $e_1 \circ e = e_2$.*

*Proof.* By the previous lemma, we only check $e$ is standardizable. The standardization of a set $A$ in ${}^{*2}V(X)$ by $e$ is the standardization of the set $e_2(A)$ by $e_1$. □

In the case of ultrapowers, standardizable extension corresponds to Rudin-Frolík order [2, 3, 4] of ultrafilters [5]. On ultrafilters over countable sets, Rudin-Frolík order is treelike: every initial segment are comparably ordered. So the following question rise.

*Question 2.* The assumption $\mathrm{rad}({}^{*}V(X)) \leq \aleph_1$ in the previous theorem is required?

## References

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