数理解析研究所講究録 1470 巻 2006 年 43-49

On Sakaguchi type functions

Shigeyoshi Owa
Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail : owa@math.kindai.ac.jp

Tadayuki Sekine
Office of Mathematics, College of Pharmacy, Nihon University
Narashinodai, Funabashi, Chiba 274-8555, Japan
E-mail : tsekine@pha.nihon-u.ac.jp

and

Rikuo Yamakawa
Office of Mathematics, Shibaura Institute of Technology
Minuma, Saitama, Saitama 337-8570, Japan
E-mail : yamakawa@sic.shibaura-it.ac.jp

Abstract

Two subclasses $\mathcal{S}(\alpha,t)$ and $\mathcal{T}(\alpha,t)$ are introduced concerning with Sakaguchi functions in the open unit disk $\mathbb{U}$. Further, by using the coefficient inequalities for the classes $\mathcal{S}(\alpha,t)$ and $\mathcal{T}(\alpha,t)$, two subclasses $\mathcal{S}_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$ are defined. The object of the present paper is to discuss some properties of functions belonging to the classes $\mathcal{S}_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:\ |z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha,t)$ if it satisfies

$$\mathrm{Re}\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\}>\alpha,\quad |t|\leqq 1,\ t\neq 1 \tag{1.2}$$

2004 *Mathematics Subject Classification*: Primary 30C45.
Kew Words and Phrases: Sakaguchi function, coefficient inequality, distortion inequality.

for some $\alpha(0 \leqq \alpha < 1)$ and for all $z \in \mathbb{U}$. The class $\mathcal{S}(0,-1)$ was introduced by Sakaguchi [5]. Therefore, a function $f(z) \in \mathcal{S}(\alpha,-1)$ is called Sakaguchi function of order α. Incidentally the class of uniformly starlike functions introduced by Goodman [1] is following.

$$\mathcal{UST} = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > 0 \right\}, \quad (z,\zeta) \in \mathbb{U} \times \mathbb{U}.$$

As for $\mathcal{S}(\alpha,t)$ and $\mathcal{UST}$, Rønning [4] showed the following important result.

Remark 1.1 $f(z) \in UST$ if and only if for every $z \in \mathbb{U}$, $|t| = 1$

$$\operatorname{Re} \frac{(1-t)zf'(z)}{f(z)-f(tz)} > 0.$$

We also denote by $\mathcal{T}(\alpha,t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha,t)$. Recently Cho, Kwon and Owa [2], and , recently, Owa, Sekine and Yamakawa [3] have discussed some properties for functions $f(z)$ in $\mathcal{S}(\alpha,-1)$, $\mathcal{T}(\alpha,-1)$. Now we show some results for functions belonging to the classes $\mathcal{S}(\alpha,t)$ and $\mathcal{T}(\alpha,t)$.

2 $\mathcal{S}_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$

We first prove the following two theorems which are similar to the results of Cho, Kwon and Owa [2].

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{|n-u_n| + (1-\alpha)|u_n|\}|a_n| \leqq 1-\alpha, \quad u_n = 1+t+t^2+\cdots+t^{n-1} \tag{2.1}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{S}(\alpha,t)$.

Proof For Theorem 1, we show that if $f(z)$ satisfies (2.1) then

$$|\frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1| < 1-\alpha.$$

Evidently, since

$$\frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1 = \frac{z+\sum_{n=2}^{\infty} na_n z^{n-1}}{z+\sum_{n=2}^{\infty} u_n a_n z^n} - 1 = \frac{\sum_{n=2}^{\infty}(n-u_n)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} u_n a_n z^{n-1}},$$

we see that

$$\left| \frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1 \right| \leqq \frac{\sum_{n=2}^{\infty} |n-u_n||a_n|}{1-\sum_{n=2}^{\infty} |u_n||a_n|}.$$

Therefore, if $f(z)$ satisfies (2.1), then we have

$$\left| \frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1 \right| < 1-\alpha.$$

This completes the proof of Theorem 2.1.

Theorem 2.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n\{|n-u_n|+(1-\alpha)|u_n|\}|a_n| \leqq 1-\alpha \tag{2.2}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{T}(\alpha,t)$.

Proof Noting that $f(z) \in \mathcal{T}(\alpha,t)$ if and only if $zf'(z) \in \mathcal{S}(\alpha,t)$, we can prove Theorem 2.

We now define

$$\mathcal{S}_0(\alpha,t) = \{f(z) \in \mathcal{A} : f(z) \text{ satisfies } (2.1)\}$$

and

$$\mathcal{T}_0(\alpha,t) = \{f(z) \in \mathcal{A} : f(z) \text{ satisfies } (2.2)\}.$$

In view of the above theorems, we see

Example 2.1 Let us consider a function $f(z)$ given by

$$f(z) = z + (1-\alpha)\left(\frac{\lambda\delta_2}{2(2-\alpha)}z^2 + \frac{(1-\lambda)\delta_3}{7-3\alpha}z^3\right), \quad 0 \leqq \lambda \leqq 1,\ |\delta_2| = |\delta_3| = 1 \tag{2.3}$$

Then for any t $(|t| \leqq 1, t \neq 1), f(z) \in \mathcal{S}_0(\alpha,t) \subset \mathcal{S}(\alpha,t)$.

Example 2.2 Let us consider a function $f(z)$ given by

$$f(z) = z + (1-\alpha)\left(\frac{\lambda\delta_2}{4(2-\alpha)}z^2 + \frac{(1-\lambda)\delta_3}{3(7-3\alpha)}z^3\right), \quad 0 \leqq \lambda \leqq 1,\ |\delta_2| = |\delta_3| = 1 \tag{2.4}$$

Then for any t $(|t| \leqq 1, t \neq 1), f(z) \in \mathcal{T}_0(\alpha,t) \subset \mathcal{T}(\alpha,t)$.

3 Coefficient inequalities

Next applying Carathéodry function $p(z)$ defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{3.1}$$

in $\mathbb{U}$, we discuss the coefficient inequalities for functions $f(z)$ in $\mathcal{S}(\alpha,t)$ and $\mathcal{T}(\alpha,t)$.

Theorem 3.1 *If $f(z) \in \mathcal{S}(\alpha,t)$, then*

$$|a_n| \leqq \frac{\beta}{|v_n|}\left\{1 + \beta\sum_{j=2}^{n-1}\frac{|u_j|}{|v_j|} + \beta^2\sum_{j_2>j_1}^{n-1}\sum_{j_1=2}^{n-2}\frac{|u_{j_1}u_{j_2}|}{|v_{j_1}v_{j_2}|} + \beta^3\sum_{j_3>j_2}^{n-1}\sum_{j_2>j_1}^{n-2}\sum_{j_1=2}^{n-3}\frac{|u_{j_1}u_{j_2}u_{j_3}|}{|v_{j_1}v_{j_2}v_{j_3}|}\right.$$

$$\left. + \cdots + \beta^{n-2}\prod_{j=2}^{n-1}\frac{|u_j|}{|v_j|}\right\}, \tag{3.1}$$

where

$$\beta = 2(1-\alpha), \quad v_n = n - u_n. \tag{3.2}$$

Proof We define the function $p(z)$ by

$$p(z)=\frac{1}{1-\alpha}\left(\frac{(1-t)zf'(z)}{f(z)-f(tz)}-\alpha\right)=1+\sum_{n=1}^{\infty}p_n z^n \tag{3.3}$$

for $f(z)\in\mathcal{S}(\alpha,t)$. Then $p(z)$ is a Carathéodory function and satisfies

$$|p_n|\leqq 2\ \ (n\geqq 1). \tag{3.4}$$

Since

$$(1-t)zf'(z)\ =\ (f(z)-f(tz))\,(\alpha+(1-\alpha)p(z))\,,$$

we have

$$z+\sum_{n=2}^{\infty}na_nz^n=\left(z+\sum_{n=2}^{\infty}u_na_nz^n\right)\left(1+(1-\alpha)\sum_{n=2}^{\infty}p_nz^n\right)$$

where

$$u_n=1+t+t^2+\cdots+t^{n-1}.$$

So we get

$$a_n\ =\ \frac{1-\alpha}{n-u_n}\left(p_1u_{n-1}a_{n-1}+p_2u_{n-2}a_{n-2}+\cdots+p_{n-2}u_2a_2+p_{n-1}\right). \tag{3.5}$$

From the equation (3.5), we easily have that

$$|a_2|\ =\ \left|\frac{1-\alpha}{2-u_2}p_1\right|\ \leqq\ \frac{2(1-\alpha)}{|2-u_2|},$$

$$|a_3|\ \leqq\ \frac{2(1-\alpha)}{|3-u_3|}(|u_2a_2|+1)\leqq\frac{2(1-\alpha)}{|3-u_3|}\left(1+2(1-\alpha)\frac{|u_2|}{|2-u_2|}\right),$$

and

$$|a_4|\ \leqq\ \frac{2(1-\alpha)}{|4-u_4|}\left\{1+2(1-\alpha)\left(\frac{|u_2|}{|2-u_2|}+\frac{|u_3|}{|3-u_3|}\right)+2^2(1-\alpha)^2\frac{|u_2u_3|}{|2-u_2||3-u_3|}\right\}.$$

Thus, using the mathematical induction, we obtain the inequality (3.1).

Remark 3.1 Equalities in Theorem 3.1 are attended for $f(z)$ given by

$$\frac{zf'(z)}{f(z)-f(tz)}=\frac{1+(1-2\alpha)z}{1-z}.$$

Remark 3.2 If we put $\alpha=0$, $t=0$ in Theorem 3.2, then we have well known result

$$f(z)\in S^*\Longrightarrow|a_n|\ \leqq\ n$$

where S^* is usual starlike class. And if we put $\alpha=0$, $t=-1$, then we have the result due to Sakaguchi [5]

$$f(z)\in STS\Longrightarrow|a_n|\ \leqq\ 1,$$

where $\mathcal{STS}$ is Sakaguchi function class.

For functions $\mathcal{T}(\alpha,t)$, similarly we have

Theorem 3.2 *If $f(z)\in\mathcal{T}(\alpha,t)$, then*

$$|a_n| \leqq \frac{\beta}{n|v_n|}\left\{1+\beta\sum_{j=2}^{n-1}\frac{|u_j|}{|v_j|}+\beta^2\sum_{j_2>j_1}^{n-1}\sum_{j_1=2}^{n-2}\frac{|u_{j_1}u_{j_2}|}{|v_{j_1}v_{j_2}|}+\beta^3\sum_{j_3>j_2}^{n-1}\sum_{j_2>j_1}^{n-2}\sum_{j_1=2}^{n-3}\frac{|u_{j_1}u_{j_2}u_{j_3}|}{|v_{j_1}v_{j_2}v_{j_3}|}\right.$$
$$\left.+\cdots+\beta^{n-2}\prod_{j=2}^{n-1}\frac{|u_j|}{|v_j|}\right\}, \tag{3.6}$$

where

$$\beta=2(1-\alpha),\quad v_n=n-u_n.$$

4 Distortion inequalities

For functions $f(z)$ in the classes $\mathcal{S}_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$, we derive

Theorem 4.1 *If $f(z)\in\mathcal{S}_0(\alpha,t)$, then*

$$|z|-\sum_{n=2}^{j}|a_n||z|^n-A_j|z|^{j+1}\leqq|f(z)|\leqq|z|+\sum_{n=2}^{j}|a_n||z|^n+A_j|z|^{j+1} \tag{4.1}$$

where

$$A_j=\frac{1-\alpha-\sum_{n=2}^{j}\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|}{j+1-\alpha|u_{j+1}|}\quad(j\geqq 2). \tag{4.2}$$

Proof From the inequality (2.1), we know that

$$\sum_{n=j+1}^{\infty}\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|\leqq 1-\alpha-\sum_{n=2}^{j}\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|.$$

On the other hand

$$|n-u_n|+(1-\alpha)|u_n|\geqq n-\alpha|u_n|,$$

and hence $n-\alpha|u_n|$ is monotonically increasing with respect to n. Thus we deduce

$$(j+1-\alpha|u_{j+1}|)\sum_{n=j+1}^{\infty}|a_n|\leqq 1-\alpha-\sum_{n=2}^{j}\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|,$$

which implies that Therefore

$$\sum_{n=j+1}^{\infty}|a_n|\leqq A_j\,. \tag{4.3}$$

Therefore we have that

$$|f(z)| \leqq |z| + \sum_{n=2}^{j} |a_n||z^n| + A_j|z|^{j+1}$$

and

$$|f(z)| \geqq |z| - \sum_{n=2}^{j} |a_n||z^n| - A_j|z|^{j+1}.$$

This completes the proof of the theorem.

Similarly we have

Theorem 4.2 *If $f(z) \in \mathcal{T}_0(\alpha, t)$, then*

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - B_j|z|^{j+1} \leqq |f(z)| \leqq |z| + \sum_{n=2}^{j} |a_n||z|^n + B_j|z|^{j+1} \tag{4.4}$$

and

$$1 - \sum_{n=2}^{j} n|a_n||z|^{n-1} - C_j|z|^{j-1} \leqq |f'(z)| \leqq 1 + \sum_{n=2}^{j} n|a_n||z|^{n-1} + C_j|z|^{j-1} \tag{4.5}$$

where

$$B_j = \frac{1 - \alpha - \sum_{n=2}^{j} n\{|n - u_n| + (1-\alpha)|u_n|\}|a_n|}{(j+1)\{j+1-\alpha|u_{j+1}|\}} \quad (j \geqq 2). \tag{4.6}$$

and

$$C_j = \frac{1 - \alpha - \sum_{n=2}^{j} n\{|n - u_n| + (1-\alpha)|u_n|\}|a_n|}{j+1-\alpha|u_{j+1}|} \quad (j \geqq 2). \tag{4.7}$$

5 Relation between the classes

By the definitions for the classes $\mathcal{S}_0(\alpha, t)$, and $\mathcal{T}_0(\alpha, t)$, evidently we have

$$\mathcal{S}_0(\alpha, t) \subset \mathcal{S}_0(\beta, t) \quad (0 \leqq \beta \leqq \alpha < 1)$$

and

$$\mathcal{T}_0(\alpha, 1) \subset \mathcal{T}_0(\beta, t) \quad (0 \leqq \beta \leqq \alpha < 1)\ .$$

Let us consider a relation between $\mathcal{S}_0(\beta, t)$ and $\mathcal{T}_0(\alpha, t)$.

Theorem 5.1 *If $f(z) \in \mathcal{T}_0(\alpha, t)$, then $f(z) \in \mathcal{S}_0\left(\frac{1+\alpha}{2}, t\right)$.*

Proof Let $f(z) \in \mathcal{T}_0(\alpha, t)$. Then, if β satisfies

$$\frac{|n - u_n| + (1-\beta)|u_n|}{1-\beta} \leqq n\,\frac{|n - u_n| + (1-\alpha)|u_n|}{1-\alpha} \tag{5.1}$$

for all $n \geqq 2$, then we have that $f(z) \in \mathcal{S}_0(\beta, t)$. From (5.1), we have

$$\beta \leqq 1 - \frac{(1-\alpha)|n-u_n|}{n|n-u_n| + (1-\alpha)(n-1)|u_n|}. \tag{5.2}$$

Furthermore, since for all $n \geqq 2$

$$\frac{|n-u_n|}{n|n-u_n| + (1-\alpha)(n-1)|u_n|} \leqq \frac{1}{n} \leqq \frac{1}{2}, \tag{5.3}$$

we obtain

$$f(z) \in S_0\left(\frac{1+\alpha}{2}, t\right).$$

References

[1] A.W.Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), 364 - 370.

[2] N. E. Cho, O. S. Kwon and S. Owa, *Certain subclasses of Sakaguchi functions*, SEA Bull. Math. **17**(1993), 121 - 126.

[3] S. Owa, T. Sekine and Rikuo Yamakawa, *Notes on Sakaguchi functions*, RIMS. Kokyuroku **1414**(2005), 76 - 82.

[4] F. Rønning, *On Uniform Starlikeness and Related Properties of Univalent Functions*, Complex Variables, **24**(1994), 233 - 239.

[5] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11**(1959), 72 - 75.