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A counterpart of strong normality

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Abstract

For non-inaccessible κ we try to define an ideal with the property between normality and strong normality, which is expected to be a natural one.

1 Introduction

Throughout κ is regular uncountable and λ a cardinal $> \kappa$. Let $\mathcal{P}_\kappa\lambda$ denote the set of the subsets of λ with the cardinality less than κ, that is, $\mathcal{P}_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$. All the proofs are easily given by the reader.

Definition 1.1. let $X \subset \mathcal{P}_\kappa\lambda$.
We say X is *unbounded* if for evry $x \in \mathcal{P}_\kappa\lambda$ there exists $y \in X$ such that $x \subset y$.
X is said to be *closed* if it is closed under $\subset$ -increasing sequence of length $< \kappa$.
X is a *club* if it is closed and unbounded.
X is *stationary* if $X \cap C \neq \emptyset$ for any club C.
Let $I_{\kappa,\lambda} = \{X \subset \mathcal{P}_\kappa\lambda : X \text{ is not unbounded}\}$ and $NS_{\kappa,\lambda} = \{X \subset \mathcal{P}_\kappa\lambda : X \text{ is not stationary}\}$.

Usually a large cardinal propertie is characterized by a normal ideal whose members are the sets without the property (or its dual filter):

supercompactness	$\longleftrightarrow$	normal measure
partition property	$\longleftrightarrow$	$NP_{\kappa,\lambda}$
ineffability	$\longleftrightarrow$	$NIn_{\kappa,\lambda}$
Shelah property	$\longleftrightarrow$	$NSh_{\kappa,\lambda}$
subtlety	$\longleftrightarrow$	nonsubtle ideal

Definition 1.2. We say I is an *ideal* if the following hold:

(1) $I \subset \mathcal{P}(\mathcal{P}_\kappa\lambda)$,

(2) $\emptyset \in I$ and $\mathcal{P}_\kappa\lambda \notin I$,

(3) if $X \subset Y \in I$, then $X \in I$,

(4) I is closed under the union of less than κ many mebmers (we say I is κ complete),

(5) $I_{\kappa,\lambda} \subset I$ (we say I is fine).

Let $I^+ = \mathcal{P}(\mathcal{P}_\kappa\lambda) \setminus I$ and $I^* = \{X \subset \mathcal{P}_\kappa\lambda : \mathcal{P}_\kappa\lambda \setminus X \in I\}$.
A function $f : \mathcal{P}_\kappa\lambda \to \lambda$ is *regressive* if $f(x) \in x$ for any $x \in \mathcal{P}_\kappa\lambda$.
An ideal I on $\mathcal{P}_\kappa\lambda$ is *normal* if for any $X \in I^+$ and a regressive function f on X there exists $Y \in \mathcal{P}(X) \cap I^+$ such that $f \restriction Y$ is constant.

Note tha $I_{\kappa,\lambda}$ is the minimal, and $NS_{\kappa,\lambda}$ is the minimal normal ideal on $\mathcal{P}_\kappa\lambda$.
Forementioned ideals have a stronger property:

Definition 1.3. For $x, y \in \mathcal{P}_\kappa\lambda$, $y \prec x$ denotes $y \in \mathcal{P}_{x\cap\kappa}x = \{s \subset x : |s| < |x \cap \kappa|\}$.
We say a function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ is *set-regressive* if $f(x) \prec x$ for any $x \in \mathcal{P}_\kappa\lambda$.
An ideal I on $\mathcal{P}_\kappa\lambda$ is *strongly normal* if for any $X \in I^+$ and set-regressive function f on X there exists $Y \in \mathcal{P}(X) \cap I^+$ such that $f \restriction Y$ is constant.
Let $WNS_{\kappa,\lambda}$ denote the minimal strongly normal ideal on $\mathcal{P}_\kappa\lambda$.

Fact 1.4. *$\mathcal{P}_\kappa\lambda \notin WNS_{\kappa,\lambda}$ if and only if κ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$* [6].

The following figure is known:

$$I_{\kappa,\lambda} \subsetneq NS_{\kappa,\lambda} \subsetneq WNS_{\kappa,\lambda}$$

nonsubtle ideal

$NSh_{\kappa,\lambda}$

2 Motivation

As is shown strong normality gives some limitation to κ. It seems natural to ask:

> Can we define a natural strengthening of normality without assuming inaccessibility?

We consider several aspects of this question.

(1) Reflection.

Usual type of reflection is as follows:

> if κ has property P, we can find $\alpha < \kappa$ which has property P.

The stationary reflection of $\mathcal{P}_{\omega_1}\lambda$ is:

> if $S \subset \mathcal{P}_{\omega_1}\lambda$ is stationary, then we can find A of cardinality ω_1 such that $\omega_1 \subset A \subset \lambda$ and $S \cap \mathcal{P}_{\omega_1}A$ is stationary in $\mathcal{P}_{\omega_1}A$.

The statioary reflection of $\mathcal{P}_\kappa\lambda$ is false for $\kappa > \omega_1$ [11]. While the following holds[5][9] :

> if κ is λ Shelah, then for any stationary $S \subset \mathcal{P}_\kappa\lambda$ we can find $x \in \mathcal{P}_\kappa\lambda$ such that $S \cap \mathcal{P}_{x\cap\kappa}x$ is stationary in $\mathcal{P}_{x\cap\kappa}x$.

(2) Diamond and subtlety.

It is known that $\diamondsuit_\kappa$ holds is κ is subtle. Eliminating inaccessibility, this assumption can be weakend to "κ is ethereal with $2^{<\kappa} = \kappa$.
While we have:

> if κ is subtle, then there exists a sequence $\langle S_x | x \in \mathcal{P}_\kappa\lambda\rangle$ such that
>
> (1) $S_x \subset \mathcal{P}_{x\cap\kappa}x$,
> (2) for any $S \subset \mathcal{P}_\kappa\lambda$ $\{x : S_x = S \cap \mathcal{P}_{x\cap\kappa}x\} \in WNS^+_{\kappa,\lambda}$.

We denote the above sequence $\tilde{\diamondsuit}_{\kappa,\lambda}$.
We review some definitions.

Definition 2.1. For $X \subset \kappa$ let $[X]^2$ denote the set $\{(\alpha, \beta) \in X \times X : \alpha < \beta\}$. We say X is *subtle* if for any sequence $\langle S_\alpha \subset \alpha | \alpha \in X\rangle$ and club $C \subset \kappa$ there exists $(\beta, \gamma) \in [C \cap X]^2$ such that $S_\beta = S_\gamma \cap \beta$.
For $Y \subset \mathcal{P}_\kappa\lambda$ let $[Y]^2_\prec$ denote the set $\{(x, y) \in Y \times Y : x \in \mathcal{P}_{y\cap\kappa}y\}$. We say $Y \subset \mathcal{P}_\kappa\lambda$ is *strongly subtle* if for any sequence $\langle S_z \subset \mathcal{P}_{z\cap\kappa}z | z \in Y\rangle$ and $C \in WNS^*_{\kappa,\lambda}$ there exists $(x, y) \in [C \cap Y]^2_\prec$ such that $S_x = S_y \cap \mathcal{P}_{x\cap\kappa}x$.

Note that κ is subtle if and only if $\mathcal{P}_\kappa\lambda$ is strongly subtle [3].
Compare the above with the following:

Definition 2.2. $X \subset \kappa$ is *ethereal* if for any sequence $\langle S_\alpha \subset \alpha | \alpha \in X\rangle$ with $|S_\alpha| = |\alpha|$ and club $C \subset \kappa$ there exists $(\beta, \gamma) \in [C \cap X]^2$ such that $|S_\beta \cap S_\gamma| = |\beta|$.
We say $Y \subset \mathcal{P}_\kappa\lambda$ is *weakly subtle* if for any sequence $\langle S_z \subset \mathcal{P}_{z\cap\kappa}z | z \in Y\rangle$ with $S_x \in I^+_{x\cap\kappa,x}$ and $C \subset \mathcal{P}_\kappa\lambda$ club there exists $(x, y) \in [C \cap Y]^2_\prec$ such that $S_x \cap S_y \in I^+_{x\cap\kappa,x}$.

Fact 2.3. (1) *If $\mathcal{P}_\kappa\lambda$ is weakly subtle, then the corresponding ideal is normal, $\{x : x \cap \kappa$ is regular$\}$ is in its dual filter, hence κ is weakly Mahlo.*
(2) *If $f : \mathcal{P}_\kappa\lambda \to \lambda$ is a bijection and $A = \{x \in \mathcal{P}_\kappa\lambda : f``\mathcal{P}_{x\cap\kappa}x = x\}$, then strongly subtle ideal $=$ weakly subtle ideal $\restriction A$.*

Note that $WNS_{\kappa,\lambda} = NS_{\kappa,\lambda} \restriction A$ in (2).
We have several questions:

Question 2.4. 1) *Is it consistent that there is a non-inaccessible weakly subtle cardinal?*
2) *Does $\tilde{\diamondsuit}_{\kappa,\lambda}$ hold if κ is weakly subtle and $2^{<\kappa} = \kappa$?*
3) *Is $\mathcal{P}_\kappa\lambda$ weakly subtle if κ is ethereal?*
4) *Is the definition of weak subtlety "a right one"?*

(3) Weak normalities.

We have some $\mathcal{P}_\kappa\lambda$ generalizations of weakly normal ideals on κ defined by Kanamori [8].

Definition 2.5. An ideal I on κ is said to be *weakly normal* if for any $f : \kappa \to \kappa$ such that $f(\alpha) < \alpha$ for every $\alpha < \kappa$ there exists $\gamma < \kappa$ with $\{\alpha : f(\alpha) \leq \gamma\} \in I^*$.
We say I on $\mathcal{P}_\kappa\lambda$ is *Kanamori* if for any regressive $f : \mathcal{P}_\kappa\lambda \to \lambda$ there exists $\gamma < \lambda$ with $\{x : f(x) \leq \gamma\} \in I^*$.

D. Burke[4] and Abe[1] proved:

Fact 2.6. *The singular cardinal hypothesis* (SCH) *holds for* $\lambda^{<\kappa}$ *if* $\mathcal{P}_\kappa\lambda$ *carries a Kanamori ideal and one of the following holds:*

(1) λ *is regular or* $\mathrm{cf}(\lambda) \leq \kappa$

(2) $\kappa^+ \leq \mathrm{cf}(\lambda) < \lambda$ *and there is a measurable cardinal above* λ.

Kanamori ideal may be seen as a weakening of strong compactness and has too strong consequences.

Definition 2.7. We say I is an *AN-ideal* if for any set-regressive function f on $\mathcal{P}_\kappa\lambda$ there exists $a \in \mathcal{P}_\kappa\lambda$ such that $\{x : f(x) \subset a\} \in I^*$. (For AN -ideals κ completeness is not assumed.)

Fact 2.8. *Suppose that* I *is a* κ *complete AN-ideal. Then,* I *is strongly normal,* κ *saturated, and* $\{x : S \cap \mathcal{P}_{x\cap\kappa}x \in NS^+_{x\cap\kappa,x}\} \in I^*$ *whenever* $S \subset \mathcal{P}_\kappa\lambda$ *is stationary* [2].

So AN-ideal may be seen as a weakening of supercompactness and is too strong as well.
While Mignon [10] defined a direct weakeninng of normality:

Definition 2.9. An ideal I on $\mathcal{P}_\kappa\lambda$ is *weakly normal* if for any $X \in I^+$ and regressive $f : X \to \lambda$ there exists $\gamma < \lambda$ with $\{x \in X : f(x) \leq \gamma\} \in I^+$.

3 Definition

We just modify Mignon's version of weak normality to define a weakening of strong normality.

Definition 3.1. Let $(*)$ denote the following statement:

$(*)$ for any $X \in I^+$ and set-regressive $f : X \to \mathcal{P}_\kappa\lambda$ there exists $a \in \mathcal{P}_\kappa\lambda$ such that $\{x \in X : f(x) \subset a\} \in I^+$.

Fact 3.2. (1) *If* κ *is inaccessible, then* $(*)$ *is equivalent to strong normality.*
(2) *If* $\mathcal{P}_\kappa\lambda$ *carries an ideal with* $(*)$, *then* κ *is weakly inaccessible.*
(3) *Every normal* κ *saturated ideal on* $\mathcal{P}_\kappa\lambda$ *has the property* $(*)$.

(4) $(*)$ *is equivalent to that* I *is closed under some type of diagonal unions, that is,*

$$I = \widetilde{\nabla}_{\prec} I = \{\nabla_{\prec}\langle X_s | s \in \mathcal{P}_\kappa\lambda\rangle : X_s \in I,\ \underline{X_s \subset X_t \text{ whenever } s \subset t}\}$$

where $x \in \nabla_{\prec}\langle X_s | s \in \mathcal{P}_\kappa\lambda\rangle$ *if and only if* $x \in X_s$ *for some* $s \prec x$.
(5) *Suppose that* I *satisfies* $(*)$ *in the grand model* V, $\mathbb{P}$ *is a* δ*-c.c. forcing with* $\delta < \kappa$, G $\mathbb{P}$ *generic, and* J *defined in* $V[G]$ *as* $J = \{X \subset \mathcal{P}_\kappa\lambda : X \cap V \subset Y$ *for some* $Y \in I\}$. *Then the following hold:*

(a) J *satissfies* $(*)$,

(b) $I = \{X : \Vdash_{\mathbb{P}} \check{X} \in \dot{J}\}$

(6) *Suppose that* $\mathbb{P}$ *is* κ *-c.c.,* J *defined as above satisfies* $(*)$ *in* $V[G]$, *and* $\mathcal{P}_\kappa\lambda \cap V \notin J$. *Then,* I *satisfies* $(*)$.

Remark. The condition underlined in (4) is equivalent to the following:

$$\bigcup\{X_s : s \subset x\} \in J \text{ for every } x \in \mathcal{P}_\kappa\lambda.$$

Concernning the consistency of the existence of a non-strongly normal ideal with $(*)$ we have the following:

Fact 3.3. *Let* κ *be Mahlo,* $\mathbb{P}$ *adding* κ *many Cohen real forcing, and* $V[G] \models$ "$J = \{X \subset \mathcal{P}_\kappa\lambda : X \cap V \subset Y$ *for some* $Y \in WNS^V_{\kappa,\lambda}\}$". *Then* J *is the minimal ideal with* $(*)$ *such that* $\mathcal{P}_\kappa\lambda \cap V \in J^*$.

4 Combinatorial characterization of the minimal ideal with $(*)$

$NS_{\kappa,\lambda}$ and $WNS_{\kappa,\lambda}$ are characterized as follows:

Fact 4.1. *Let* $X \subset \mathcal{P}_\kappa\lambda$.
(1) $X \in NS_{\kappa,\lambda}$ *if and only if there exists* $f : \lambda^2 \to \mathcal{P}_\kappa\lambda$ *such that* $C_f \cap X = \emptyset$, *where* $C_f = \{x : f``x^2 \subset \mathcal{P}(x)\}$.
(2) $X \in WNS_{\kappa,\lambda}$ *if and only if there exists* $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ *such that* $C_f \cap X = \emptyset$, *where* $C_f = \{x : f``\mathcal{P}_{x\cap\kappa}x \subset \mathcal{P}(x)\}$.

If κ is inaccessible or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$, then $\cup f``\mathcal{P}_{x\cap\kappa}x \in \mathcal{P}_\kappa\lambda$ for every $f \in {}^{\mathcal{P}_\kappa\lambda}\mathcal{P}_\kappa\lambda$ and $x \in \mathcal{P}_\kappa\lambda$.

Definition 4.2. Let $\mathcal{F} = \{f \in {}^{\mathcal{P}_\kappa\lambda}\mathcal{P}_\kappa\lambda : \cup f``\mathcal{P}_{x\cap\kappa}x \in \mathcal{P}_\kappa\lambda$ for every $x \in \mathcal{P}_\kappa\lambda\}$, and $\tilde{C}_f = \{x : f``\mathcal{P}_{x\cap\kappa}x \subset \mathcal{P}(x)\}$ for $f \in \mathcal{F}$.
Set $I_0 = \{X \subset \mathcal{P}_\kappa\lambda : \tilde{C}_f \cap X = \emptyset$ for some $f \in \mathcal{F}\}$.

Fact 4.3. *Let κ be weakly Mahlo. Then,*
(1) *For any $f \in \mathcal{F}$ $\tilde{C}_f \in I^+_{\kappa,\lambda}$.*
(2) *I_0 satisfies $(*)$.*

Recall that $WNS_{\kappa,\lambda}$ has another characterization:

Fact 4.4. *For any $X \subset \mathcal{P}_\kappa\lambda$, $X \in WNS_{\kappa,\lambda}$ if and only if there exists a set-regressive $f : X \to \mathcal{P}_\kappa\lambda$ such that $f^{-1}(\{a\}) \in I_{\kappa,\lambda}$ for any $a \in \mathcal{P}_\kappa\lambda$.*

We now define another ideal.

Definition 4.5. Define J_0 by:

$X \in J_0$ if $X \subset \mathcal{P}_\kappa\lambda$ and there exists a set regressive $f : X \to \mathcal{P}_\kappa\lambda$ such that for any $a \in \mathcal{P}_\kappa\lambda$ $\{x \in X : f(x) \subset a\} \in I_{\kappa,\lambda}$.

We easily have:

Fact 4.6. $NS_{\kappa,\lambda} \subset J_0 = \tilde{\nabla}_\prec I_{\kappa,\lambda}$.

We know $\nabla\nabla\nabla I = \nabla\nabla I$ and $\nabla_\prec\nabla_\prec I = \nabla_\prec I$ for every ideal I. (If $NS_{\kappa,\lambda} \subset I$, then $\nabla\nabla I = \nabla I$.) The author does not know how about for the operation $\tilde{\nabla}_\prec$.

Question 4.7. (1) *Is J_0 normal?*
(2) *$\tilde{\nabla}_\prec I = \tilde{\nabla}_\prec\tilde{\nabla}_\prec I$ for every ideal I?*

Fact 4.6 suggests a different ideal.

Definition 4.8. Define J_1 by:

$X \in J_1$ if $X \subset \mathcal{P}_\kappa\lambda$ and there exists a set regressive $f : X \to \mathcal{P}_\kappa\lambda$ such that for any $a \in \mathcal{P}_\kappa\lambda$ $\{x \in X : f(x) \subset a\} \in NS_{\kappa,\lambda}$.

Clearly J_1 is normal.

Question 4.9. $J_1 = I_0$?

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