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The partition property of $\mathcal{P}_\kappa\lambda$

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Abstract

We study a relationship between the partition property of $\mathcal{P}_\kappa\lambda$ and the Shelah property.

1 Introduction

The partition property of $\mathcal{P}_\kappa\lambda$ was introduced by Jech [5] as a generalization of the classical partition property of cardinal. In this paper we study a relation between the partition property and the Shelah property of $\mathcal{P}_\kappa\lambda$, the Shalah property is defined by Carr [2] as a generatizatation of weakly compactness. It is well-known that there is a essential connection between the partition property of a cardinal and weakly compactness: for a cardinal κ, κ is weakly compact iff $\kappa \longrightarrow (\kappa)^2_2$. In Carr [4] observed such connection for various partition property and large cardinal property of $\mathcal{P}_\kappa\lambda$, including the Shelah property. We will try more deep analysis. Let $\mathrm{NSh}_{\kappa\lambda}$ is the set of all $X \subseteq \mathcal{P}_\kappa\lambda$ such that X is not Shelah.

Main Theorem 1 *Let* $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \overset{\leq}{\not\longrightarrow} (\mathrm{I}^+_{\kappa\lambda})^2_2\}$. *Assume* $\lambda \geq \kappa$ *is regular,* $\lambda^{<\lambda} = \lambda$ *but not weakly compact. Then there exists a club* C *of* $\mathcal{P}_\kappa\lambda$ *such that* $I|C = \mathrm{NSh}_{\kappa\lambda}$.

Main Theorem 2 *Assume* $\lambda \geq \kappa$ *is regular and* $\lambda^{<\lambda} = \lambda$. *Then the following are equivalent:*

(1) κ *is* λ*-Shelah,*

(2) $C \overset{\leq}{\longrightarrow} (\mathrm{I}^+_{\kappa\lambda})^2_2$ *for every club* C *of* $\mathcal{P}_\kappa\lambda$.

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Theorem 2 shows the Shelah property of $\mathcal{P}_\kappa\lambda$ is *right* analogue of weakly compactness. In this sense, Theorem 2 is not surprising. However Theorem 1 is interesting, since if $\lambda = \kappa$ then it must false; in fact if $\lambda = \kappa$ the partition ideal I in Theorem 1 is just unbounded ideal over $\mathcal{P}_\kappa\lambda$, and $\mathrm{NSh}_{\kappa\kappa}$ is just the weakly compact ideal. Further Theorem 2 shows that the partition ideal I can be locally normal, but I itself cannot be normal. These results indicate that the partition ideal I over $\mathcal{P}_\kappa\lambda$ has a strange structure under GCH. Note that, if GCH fails, the partition ideal can have a simple form, unbounded ideal (see Shioya [9]).

we will give a partial answer of a question of 5.5 in Carr [4] with a method which will be used to prove theorems.

2 Preliminaries

We refer to Kanamori [7] for general background and basic notation. Throughout this paper, κ denotes an inaccessible cardinal and λ a cardinal $\geq \kappa$.

An ideal over $\mathcal{P}_\kappa\lambda$ means that κ-complete fine ideal over $\mathcal{P}_\kappa\lambda$ in this paper. For an ideal I over $\mathcal{P}_\kappa\lambda$, I^* denotes the dual filter of I and $I^+ = \mathcal{P}(\mathcal{P}_\kappa\lambda) \setminus I$. An element of I^+ is called I-positive set. $\mathrm{NS}_{\kappa\lambda}$ ($\mathrm{I}_{\kappa\lambda}$) is the set of all $X \subseteq \mathcal{P}_\kappa\lambda$ such that X is non-stationary (not unbounded) in $\mathcal{P}_\kappa\lambda$.

Definition 2.1 For x, $y \in \mathcal{P}_\kappa\lambda$, we define $x < y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$. For an ideal I on $\mathcal{P}_\kappa\lambda$, I is *strongly normal* if for all $X \in I^+$ and $<$-regressive $f : X \to \mathcal{P}_\kappa\lambda$, that is, $f(x) < x$ for all $x \in X$ with $|x \cap \kappa| > 0$, there exists $y \in \mathcal{P}_\kappa\lambda$ such that $\{x \in X : f(x) = y\} \in I^+$. □

For $x \in \mathcal{P}_\kappa\lambda$, we denote $\mathcal{P}_x = \{y \in \mathcal{P}_\kappa\lambda : y < x\}$. If $x \cap \kappa$ is a regular cardinal, then properties of $\mathcal{P}_\kappa\lambda$ can be translated into $\mathcal{P}_x$ naturally. For example, $X \subseteq \mathcal{P}_x$ is stationary if for all $f : x \times x \to \mathcal{P}_x$ there exists $y \in X$ such that $\bigcup f``(y \times y) \subseteq y$.

Definition 2.2 For $X \subseteq \mathcal{P}_\kappa\lambda$, X is *Shelah* if for all $\langle f_x : x \in X\rangle$ with $f_x : x \to x$, there exists $f : \lambda \to \lambda$ such that the set $\{x \in X : f|y = f_x|y\}$ is unbounded for all $y \in \mathcal{P}_\kappa\lambda$. We say that κ is λ-*Shelah* if $\mathcal{P}_\kappa\lambda$ is Shelah.

$\mathrm{NSh}_{\kappa\lambda}$ is the set of all $X \subseteq \mathcal{P}_\kappa\lambda$ such that X is not Shelah. □

Fact 2.3 (Carr [2, 3]) (1) $\mathrm{NSh}_{\kappa\lambda}$ *is a normal ideal over* $\mathcal{P}_\kappa\lambda$. *Moreover it is strongly normal if* $\mathrm{cf}(\lambda) \geq \kappa$,

(2) *if* κ *is* $2^{\lambda^{<\kappa}}$-*Shelah then* κ *is* λ-*supercompact,*

(3) *if* κ *is* λ-*supercompact then* κ *is* λ-*Shelah.* □

(2) of the above fact shows that the Shelah property of $\mathcal{P}_\kappa\lambda$ is a very strong property.

Fact 2.4 (Abe [1]) $\{x \in \mathcal{P}_\kappa\lambda : \forall\alpha \in x\,(|x \cap \alpha| < |x|)\} \in \mathrm{NSh}^*_{\kappa\lambda}$. $\square$

Now we define the partition property of $\mathcal{P}_\kappa\lambda$.

Definition 2.5 Let n be a natural number > 0. For $X \subseteq \mathcal{P}_\kappa\lambda$,

$$[X]^n_< = \{\{x_1, \dots, x_n\} \subseteq X : x_1 < \dots < x_n\}.$$

For a function f on $[X]^n_<$, H is *homogeneous set for f* if $H \subseteq X$ and $|f``[H]^n_<| = 1$, and $H$ is called *$x$-homogeneous* if $f``[H]^n_< = \{x\}$ for some x. $\square$

When an element of $[X]^n_<$ is written as $\{x_1, \dots, x_n\}$, it is assumed that $x_1 < \dots < x_n$. For $\{x_1, \dots, x_n\} \in [X]^n_<$ and a function f on $[X]^n_<$, we shall write $f(x_1, \dots, x_n)$ instead of $f(\{x_1, \dots, x_n\})$.

Definition 2.6 Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$. For a natural number n, an ordinal α and $X \subseteq \mathcal{P}_\kappa\lambda$, we say that $X \xrightarrow{<} (\mathcal{A})^n_\alpha$ *holds* if for all $f : [X]^n_< \to \alpha$ there exists a homogeneous set $Y \in \mathcal{A}$ for f.

For $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$, $\mathcal{B} \xrightarrow{<} (\mathcal{A})^n_\alpha$ holds if $X \xrightarrow{<} (\mathcal{A})^n_\alpha$ holds for all $X \in \mathcal{B}$.

We say that $\mathrm{Part}(\kappa, \lambda)^n_<$ *holds* if $\mathcal{P}_\kappa\lambda \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^n_2$ holds, and $\mathrm{Part}^*(\kappa, \lambda)^n_<$ *holds* if $\mathcal{P}_\kappa\lambda \xrightarrow{<} (\mathrm{NS}^+_{\kappa\lambda})^n_2$ holds.

As usual, $\not\xrightarrow{<}$ means the negation of the corresponding partition property.

Remark that Jech's partition property was defined with the order $\subsetneq$, not $<$. The partition property with $\subsetneq$ is stronger than with $<$, but the author does not know that there is a essential difference between those properties.

Fact 2.7 (Carr [4], Jech [5], Magidor [8]) (1) *If* $\mathrm{Part}(\kappa, \lambda)^2_<$ *holds for some* λ *then* κ *is weakly compact,*

(2) *if* $\mathrm{Part}(\kappa, \lambda)^3_<$ *holds for all* λ *then* κ *is strongly compact,*

(3) κ *is supercompact iff* $\mathrm{Part}^*(\kappa, \lambda)^2_<$ *holds for all* λ. $\square$

Fix n a natural number > 0 and put $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \not\xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^n_2\}$. Then it is easy to check that I forms an ideal over $\mathcal{P}_\kappa\lambda$. I is often called *the partition ideal* over $\mathcal{P}_\kappa\lambda$.

3 The Shelah property and the partition property

We start proofs of Theorem 1 and 2. First we prove that the Shelah property of $\mathcal{P}_\kappa\lambda$ implies the partition property.

Lemma 3.1 *Assume λ is regular and $\lambda^{<\lambda} = \lambda$. For $X \subseteq \mathcal{P}_\kappa\lambda$, if X is Shelah then X satisfies the following property: for any $\langle f_x : x \in X\rangle$ with $f_x : x \to x$ there exists $f : \lambda \to \lambda$ such that for all $\alpha < \lambda$ $\{x \in X : f|x \cap \alpha = f_x|x \cap \alpha\} \in \mathrm{NSh}^+_{\kappa\lambda}$.*

Proof: Fix $\langle f_\xi : \xi < \lambda\rangle$ an enumeration of $\bigcup\{{}^\alpha\lambda : \alpha < \lambda\}$. Let $Z = \{x \in \mathcal{P}_\kappa\lambda : \forall\alpha \in x \forall f : x\cap\alpha \to x \exists\xi \in x\, (f = f_\xi|(x\cap\alpha))\}$. First we claim $Z \in \mathrm{NSh}^*_{\kappa\lambda}$. Assume not. By the normality of $\mathrm{NSh}_{\kappa\lambda}$ there exists $\alpha < \lambda$ such that $Y = \{x \in \mathcal{P}_\kappa\lambda : \exists f_x : x\cap\alpha \to x \forall\xi \in x\, (f_x \neq f_\xi|(x\cap\alpha))\} \in \mathrm{NSh}^+_{\kappa\lambda}$. For each $x \in Y$, let $g_x : x \to x\cap\alpha$ satisfying $f_x(g_x(\xi)) \neq f_\xi(g_x(\xi))$. Then by the Shelah property of Y, there exists $f : \alpha \to \lambda$ and $g : \lambda \to \alpha$ such that $\{x \in Y : f_x|y = f|y,\ g_x|y = g|y\}$ is unbounded for any $y \in \mathcal{P}_\kappa\lambda$. Then $f = f_\xi$ for some $\xi < \lambda$. Take $y \in \mathcal{P}_\kappa\lambda$ such that y is closed under g and $\xi \in y$. Then we can take $x \in Y$ such that $y \subseteq x$, $f_x|y = f_\xi|y$ and $g_x|y = g|y$. Then $g(\xi) = g_x(\xi) \in y$, hence $f_x(g(\xi)) = f_\xi(g(\xi))$ holds. But this contradict to the definition of g_x, namely $f_x(g_x(\xi)) \neq f_\xi(g_x(\xi))$.

Now let $X \in \mathrm{NSh}^+_{\kappa\lambda}$. We may assume that $X \subseteq Z$. For given $\langle f_x : x \in X\rangle$, define $\langle g_x : x \in X\rangle$ with $g_x : x \to x$ by $f_x|x \cap \xi = f_{g_x(\xi)}|x \cap \xi$. By a theorem of Johnson [6], there exists $g : \lambda \to \lambda$ such that for any $y \in \mathcal{P}_\kappa\lambda$ $\{x \in X : g_x|y = g|y\} \in \mathrm{NSh}^+_{\kappa\lambda}$. Now define $f : \lambda \to \lambda$ by $f(\xi) = f_{g(\eta)}(\xi)$ for some $\eta > \xi$. It is easy to see that f is well-defined. We see that f has the desired property. Let $\alpha < \lambda$. Take $y \in \mathcal{P}_\kappa\lambda$ such that $\alpha \in y$, $\sup(y) > \alpha$ and closed under g. Then $W = \{x \in X : y \subseteq x,\ g|y = g_x|y\} \in \mathrm{NSh}^+_{\kappa\lambda}$. Let $x \in W$. Then $f_x|x\cap\alpha = f_{g_x(\alpha)}|x\cap\alpha = f_{g(\alpha)}|x\cap\alpha$. Hence by the definition of f, $f(\xi) = f_x(\xi)$ holds for any $\xi \in x \cap \alpha$. □

Assume $\lambda^{<\lambda} = \lambda$. Let $\langle x_\xi : \xi < \lambda\rangle$ be an enumeration of $\mathcal{P}_\kappa\lambda$. Then by the strong normality of $\mathrm{NSh}_{\kappa\lambda}$, we have $\{x \in \mathcal{P}_\kappa\lambda : \mathcal{P}_x = \{x_\xi : \xi \in x\}\} \in \mathrm{NSh}^*_{\kappa\lambda}$. Hence we have the following:

Cor. 3.2 *Assume λ is regular and $\lambda^{<\lambda} = \lambda$. Let $X \in \mathrm{NSh}^+_{\kappa\lambda}$. Then X has the following property: for any $\langle f_x : x \in X\rangle$ with $f_x : x \to \mathcal{P}_x$ there exists $f : \lambda \to \mathcal{P}_\kappa\lambda$ such that for all $\alpha < \lambda$ $\{x \in X : f_x|x \cap \alpha = f|x \cap \alpha\} \in \mathrm{NSh}^+_{\kappa\lambda}$.* □

Now we shall prove more strong partition property from the Shelah property. For $X \subseteq \mathcal{P}_\kappa\lambda$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$, we say that $X \overset{<}{\longrightarrow} (\mathcal{A}, \mathcal{B})^n$ *holds* if for any $f : [X]^n_< \to 2$, either there exists a 0-homogeneous set H for f with $H \in \mathcal{A}$ or 1-homogeneous set H for f with $H \in \mathcal{B}$.

Lemma 3.3 *Assume λ is regular and $\lambda^{<\lambda} = \lambda$. For $X \subseteq \mathcal{P}_\kappa\lambda$, if X is Shelah then $X \overset{<}{\longrightarrow} (\mathrm{NSh}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$ holds.*

Proof: Fix an enumeration $\langle x_\xi : \xi < \lambda\rangle$ of $\mathcal{P}_\kappa\lambda$. For each $x \in X$, we may assume that $\mathcal{P}_x = \{x_\xi : \xi \in x\}$. Let $f : [X]^2_< \to 2$. For $x \in X$, we define $g_x : x \cap \alpha_x \to X \cap \mathcal{P}_x$ and $\alpha_x \leq \sup(x)$ by the induction on $\xi \in x$. Let $\xi \in x$ and assume $g_x|x \cap \xi$ is defined. If there exists $z \in \mathcal{P}_x \cap X$ such that $x_\xi \subseteq z$,

$\forall \eta \in x \cap \xi\, (z \not\subseteq g_x(\eta))$ and $\forall \eta \in x \cap \xi\, (g_x(\eta) < z \Rightarrow f(g_x(\eta), z) = f(z, x) = 1)$, then set $g_x(\xi) = z$. If there is no such $z \in X \cap \mathcal{P}_x$, then we set $\alpha_x = \xi$. Assume $g_x(\xi)$ is defined for any $\xi \in x$, then we set $\alpha_x = \sup(x)$.

Note that $\{g_x(\xi) : \xi \in x \cap \alpha_x\} \cup \{x\}$ is 1-homogeneous for f and if $\alpha_x = \sup(x)$ then $\{g_x(\xi) : \xi \in x \cap \alpha_x\}$ is unbounded in $\mathcal{P}_x$.

Now we consider the following two cases.

Case 1. $\{x \in X : \alpha_x < \sup(x)\} \in \mathrm{NSh}^+_{\kappa\lambda}$. By the normality of $\mathrm{NSh}_{\kappa\lambda}$, there exists $\alpha < \lambda$ such that $\{x \in X : \alpha_x = \alpha\} \in \mathrm{NSh}^+_{\kappa\lambda}$. Then by Cor. 3.2, there exists $g : \alpha \to X$ such that $Y = \{x \in X : g_x | x \cap \alpha = g | x \cap \alpha\} \in \mathrm{NSh}^+_{\kappa\lambda}$. Let $H = \{x \in Y : x_\alpha < x,\ \forall \xi < \alpha\, (g(\xi) < x \Rightarrow \xi \in x)\}$. Then it is easy to see that $H \in \mathrm{NSh}^+_{\kappa\lambda}$. We claim that H is 0-homogeneous set. Let $x, y \in H$ with $x < y$. Assume $f(x, y) = 1$. If $f(g_y(\eta), x) = 1$ for all $\eta \in y \cap \alpha$ with $g_y(\eta) < x$, then x witness that $\alpha \in \mathrm{dom}(g_y)$. Hence there must exist $\eta \in y \cap \alpha$ such that $g_y(\eta) < x$ and $f(g_y(\eta), x) = 0$. Since $g_y(\eta) = g(\eta) < x$, we have $\eta \in x$. Thus $g_x(\eta) = g_y(\eta) = g(\eta)$ holds. However $f(g_x(\eta), x) = 1$ by the definition of g_x, a contradiction.

Case 2. $\{x \in X : \alpha_x = \sup(x)\} \in \mathrm{NSh}^+_{\kappa\lambda}$. Let $Y = \{x \in X : \alpha_x = \sup(x)\}$. Then for $x \in Y$, $\{g_x(\xi) : \xi \in x\}$ is a 1-homogeneous set for f and unbounded in $\mathcal{P}_x$. By Cor. 3.2, there exists $g : \lambda \to X$ such that $\{x \in Y : g_x | x \cap \alpha = g | x \cap \alpha\} \in \mathrm{NSh}^+_{\kappa\lambda}$ for all $\alpha < \lambda$. Let $H = g``\lambda$. Then it is easy to see that H is an unbounded 1-homogeneous set for f. □

Next we will show that if $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$ then κ is λ-Shelah. To see this, we need some lemmata.

Lemma 3.4 *Let μ be a cardinal with $\kappa \leq \mu \leq \lambda$. Assume $\lambda^{<\mu} = \lambda$. Then there exists a club C of $\mathcal{P}_\kappa\lambda$ such that for every unbounded subset $X \subseteq C$, $\alpha < \mu$ and $f : \alpha \to \mathcal{P}_\kappa\lambda$, $X \setminus \{x \in X : \forall \zeta \in x \cap \alpha\, (f(\zeta) < x)\}$ is not unbounded.*

Proof: Let $\vec{h} = \langle h_\xi : \xi < \lambda \rangle$ be an enumeration of $\bigcup_{\eta<\mu} {}^{\eta}\lambda$ and $\vec{x} = \langle x_\xi : \xi < \lambda \rangle$ an enumeration of $\mathcal{P}_\kappa\lambda$. We can enumerate with λ-length by our cardinal arithmetic assumption. Let θ be a sufficiently large regular cardinal and $M = \langle \mathcal{H}_\theta, \in, \kappa, \lambda, \vec{h}, \vec{x} \rangle$. Let $C = \{N \cap \lambda : N \prec M,\ |N| < \kappa,\ N \cap \kappa \in \kappa\}$. Then C forms a club. Note that if $N \cap \kappa \in C$ and $x \in N \cap \mathcal{P}_\kappa\lambda$ then $x < N \cap \lambda$. We shall check that C satisfies the conclusion of lemma. Fix X an unbounded subset of C. Let $\alpha < \mu$ and $f : \alpha \to \mathcal{P}_\kappa\lambda$. For f, define $h : \alpha \to \lambda$ by $f(\zeta) = x_{h(\zeta)}$. Then there exists $\xi < \lambda$ such that $h = h_\xi$. By the definition of C, for each $x \in X$ if $\xi \in x$ then $h``(x \cap \alpha) \subseteq x$. Further if $N \prec M$, $N \cap \lambda \in X$ and $h(\zeta) \in N$, then $f(\zeta) = x_{g(\zeta)} \in N$, hence $f(\zeta) \in N$. Therefore for each $N \cap \lambda \in X$ if $\xi \in N \cap \lambda$ then $\forall \zeta \in N \cap \alpha\, (f(\zeta) < N \cap \lambda)$. Since $X \setminus \{x \in X : \xi \in X\}$ is not unbounded, we have done. □

Lemma 3.5 *Let μ be a cardinal with $\kappa \leq \mu \leq \lambda$ and assume $\lambda^{<\mu} = \lambda$. Then there exists some club C of $\mathcal{P}_\kappa\lambda$ such that for any $X \subseteq C$ if $X \to (\mathrm{I}^+_{\kappa\lambda})^{n+1}_2$ holds then X has the following property: whenever $\langle a_t : t \in [X]^n_< \rangle$ with $a_t \subseteq \min(t) \cap \mu$ there exists an unbounded subset $H \subseteq X$ and $A \subseteq \mu$ such that*

$$\forall \xi < \mu \exists z_\xi \in \mathcal{P}_\kappa\lambda \forall t \in [H]^n_< (z_\xi < \min(t) \Rightarrow A \cap \min(t) \cap \xi = a_t \cap \xi)).$$

Here $\min(t)$ *is the minimal element of* t *with respect to* $<$.

Proof: Let C be a club shown in Lemma 3.4. Let $X \subseteq C$ be such that $X \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^{n+1}_2$ holds. We will see that X has the desired property. Let $\langle a_t : t \in [X]^n_< \rangle$ with $a_t \subseteq \min(t) \cap \mu$. We define $f : [X]^n_< \to 2$ as: for $\{x_1, \ldots, x_{n+1}\} \in [X]^{n+1}_<$, if $a_{x_1 \cdots x_n} = a_{x_2 \cdots x_{n+1}} \cap x_1$, then let $f(x_1, \ldots, x_{n+1}) = 0$. Assume $a_{x_1 \cdots x_n} \neq a_{x_2 \cdots x_{n+1}} \cap x_1$ and let α be the minimal element of $a_{x_1 \cdots x_n} \triangle (a_{x_2 \cdots x_{n+1}} \cap x_1)$. If $\alpha \in a_{x_2 \cdots x_{n+1}}$, then $f(x_1, \ldots x_{n+1}) = 0$. If $\alpha \in a_{x_1 \cdots x_n}$, then $f(x_1, \ldots x_{n+1}) = 1$.

By $X \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^{n+1}_2$, we can take an unbounded homogeneous set H for f. Now we will construct $A \subseteq \mu$ and $\langle z_\xi : \xi < \mu \rangle$ by the induction on $\xi < \mu$. Assume $A \cap \eta$ and z_η is defined for any $\eta < \xi$ and satisfies the following:

(1) $z_\eta \in \mathcal{P}_\kappa\lambda$,

(2) for any $t \in [H]^n_<$, if $z_\eta < \min(t)$ then $A \cap \eta \cap \min(t) = a_t \cap \eta$.

We define z_ξ and decide whether $\xi \in A$ or not. First assume that H is 0-homogeneous. Let $H' = \{x \in H : \exists \eta \in x \cap \xi \, (z_\eta \not< x)\}$. By Lemma 3.4, H' is not unbounded. Fix $z \in H$ such that $\xi \in z$ and $z \not< x$ for all $x \in H'$. Note that if $x \in H$ and $z < x$ then $\forall \eta \in x \cap \xi \, (z_\eta < x)$.

Case 1. If there exists $\{y_1, \ldots, y_n\} \in [H]^n_<$ such that $z < y_1$ and $\xi \in a_{y_1, \ldots, y_n}$, then set $z_\xi = y_n$ and $\xi \in A$. We check that $A \cap \xi + 1$ and z_ξ satisfies the induction hypotheses. Let $\{x_1, \ldots x_n\} \in [H]^n_<$ such that $z_\xi < x_1$. Then since $z < y_1 < \cdots < y_n = z_\xi < x_1 < \cdots < x_n$, $\forall \eta \in x_i \cap \xi \, (z_\eta < x_i)$ and $\forall \eta \in y_i \cap \xi \, (z_\eta < y_i)$ hold for any $i \leq n$. Hence by the induction hypotheses, for any $\eta \in y_1 \cap \xi$, $A \cap y_1 \cap \eta = a_{y_1 \cdots y_n} \cap \eta$. This means that $A \cap y_1 \cap \xi = a_{y_1 \ldots y_n} \cap \xi$. By the same reason we have $A \cap y_2 \cap \xi = a_{y_2 \cdots y_n x_1} \cap \xi$. In particular $a_{y_1 \ldots y_n} \cap \xi = a_{y_2 \cdots y_n x_1} \cap y_1 \cap \xi$. Further H is 0-homogeneous and $\xi \in a_{y_1 \cdots y_n}$, ξ must be an element of $a_{y_2 \cdots y_n x_1}$. Repeating this argument n-times, we have $\xi \in a_{x_1 \cdots x_n}$ and $A \cap (\xi + 1) = a_{x_1 \cdots x_n} \cap \xi + 1$.

Case 2. If there exists no $\{y_1, \ldots, y_n\} \in [H]^n_<$ such that $z < y_1$ and $\xi \in a_{y_1, \ldots, y_n}$, then set $z_\xi = z$ and $\xi \notin A$. Then it is clear that z_ξ and $A \cap \xi + 1$ satisfies the induction hypotheses.

If H is 1-homogeneous, then we consider the following two cases: there exists $\{y_1, \ldots, y_n\} \in [H]^n_<$ such that $z < y_1$ and $\xi \notin a_{y_1, \ldots, y_n}$, and otherwise. The rest follows from a similar argument. □

Now we will prove Theorem 1 and 2 using the above lemma.

Lemma 3.6 *Assume λ is regular, $\lambda^{<\lambda} = \lambda$ and λ is not strong limit. Let $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \not\xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2\}$. Then there exists a club D of $\mathcal{P}_\kappa\lambda$ such that* $\mathrm{NSh}_{\kappa\lambda} = I|D$ *holds.*

Proof: Since $\lambda^{<\lambda} = \lambda$ and λ is not strong limit, there exists $\nu < \lambda$ such that $2^\nu = \lambda$. Fix such a ν. Fix $\langle B_\xi : \xi < \lambda\rangle$ a bijective enumeration of $\mathcal{P}(\nu)$. Fix $\pi : \lambda \times \nu \to \lambda$ a bijection. Now let C be a club in Lemma 3.5 with the case $\mu = \lambda$. Let θ be a sufficiently large regular cardinal and $M = \langle \mathcal{H}_\theta, \in, \kappa, \lambda, \pi, \langle B_\xi : \xi < \lambda\rangle, \ldots\rangle$. Now let $D = \{N \cap \lambda \in C : N \prec M,\ |N| < \kappa,\ N \cap \kappa \in \kappa\}$. Then D is a club subset of $\mathcal{P}_\kappa\lambda$. We will show that D works. Note that for any $x \in D$, $\pi``(x \times (x \cap \nu)) = x$ and for all $\xi, \eta \in x$, if $\xi \neq \eta$ then $B_\xi \cap x \neq B_\eta \cap x$.

Since $\mathrm{NSh}_{\kappa\lambda}$ is normal, if $X \in \mathrm{NSh}^+_{\kappa\lambda}$ then $X \cap D \in \mathrm{NSh}^+_{\kappa\lambda}$. Hence by Lemma 3.3 $X \in (I|D)^+$ holds.

To see converse, let $X \in (I|D)^+$. We may assume that $X \subseteq D$. Let $\langle f_x : x \in X\rangle$ with $f_x : x \to x$. For $x \in X$, let $a_x = \pi``\{\langle \eta, \zeta\rangle : \eta \in x,\ \zeta \in B_{f_x(\eta)} \cap x\} \subseteq x$. Then by Lemma 3.5, there exists an unbounded $H \subseteq X$, $A \subseteq \lambda$ and $\langle z_\xi : \xi < \lambda\rangle$ such that $\forall \xi < \lambda \forall x \in H\, (z_\xi < x \Rightarrow A \cap x \cap \xi = a_x \cap \xi)$. For each $\eta < \lambda$, define $A_\eta \subseteq \nu$ by $\zeta \in A_\eta$ iff $\pi(\langle \eta, \zeta\rangle) \in A$. Then define $f : \lambda \to \lambda$ by $A_\xi = B_{f(\xi)}$. We clam for any $y \in \mathcal{P}_\kappa\lambda$ there exists $x \in H$ such that $y \subseteq x$ and $f|y = f_x|y$, this completes a proof. Let $y \in \mathcal{P}_\kappa\lambda$. If necessary we may assume that y is closed under f. Take a large $\xi < \lambda$ such that $\sup(y) < \xi$ and $\pi``(\xi \times \nu) = \xi$. Then we can take $x \in H$ such that $z_\xi < x$, $y < x$ and $A \cap x \cap \xi = a_x \cap \xi$. We check that $f|y = f_x|y$. Note that $\pi``((x \cap \xi) \times (x \cap \mu)) = x \cap \xi$. Let $\eta \in y$. Since $f(\eta), f_x(\eta) \in x$, it suffices to show that $B_{f(\eta)} \cap x = B_{f_x(\eta)} \cap x$. Let $\zeta \in B_{f_x(\eta)} \cap x$. Then $\pi(\langle \eta, \zeta\rangle) \in a_x$. Since $\eta < \xi$, $\pi(\langle \eta, \zeta\rangle) \in a_x \cap \xi = A \cap x \cap \xi$. Then by the definition of A, we have $\zeta \in B_{f(\eta)}$. The converse can be verified by the same argument. □

Lemma 3.7 *Assume λ is regular, $\lambda^{<\lambda} = \lambda$ and there exists a λ-Aronsjazn tree. Let $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \not\xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2\}$. Then there exists a club D of $\mathcal{P}_\kappa\lambda$ such that* $\mathrm{NSh}_{\kappa\lambda} = I|D$ *holds.*

Proof: Fix $T = \langle T, \leq_T\rangle$ a λ-Aronsjazn tree. We may assume that $T = \lambda$. For $\alpha < \lambda$, T_α denotes the α-th level of T. Fix $\pi : \lambda \times \lambda \to \lambda$ a bijection. Let θ be a sufficiently large regular cardinal. Let $M = \langle \mathcal{H}_\theta, \in, \kappa, \lambda, T, \pi, \ldots\rangle$. Let C be a club in Lemma 3.5 with the case $\mu = \lambda$ and $D = \{N \cap \lambda \in C : N \prec M,\ |N| < \kappa,\ N \cap \kappa \in \kappa\}$. We will show that D works.

$I|D \subseteq \mathrm{NSh}_{\kappa\lambda}$ is Lemma 3.3. Let $X \in (I|D)^+$ be such that $X \subseteq D$. For $\langle f_x : x \in X\rangle$ with $f_x : x \to x$, define a_x for $x \in X$ as follows: for $\eta \in x$, take $\alpha^x_\eta \in T_{f_x(\eta)} \cap x$. Note that such an α^x_η exists since $x = N \cap \lambda$ for some $N \prec M$. Let

$b^x_\eta = \{\beta \in T : \beta \leq_T \alpha^x_\eta\} \cap x$. Hence α^x_η is the max element of b^x_η with respect to the order $\leq_T$. Now let $a_x = \pi``\{\langle \eta, \zeta \rangle : \eta \in x,\ \zeta \in b^x_\eta\} \subseteq x$.

We take an unbounded $H \subseteq X$, $A \subseteq \lambda$ $\langle z_\xi : \xi < \lambda \rangle$ by Lemma 3.5. For each $\eta < \lambda$, define $B_\eta \subseteq \lambda$ by $\zeta \in B_\eta \iff \pi(\langle \eta, \zeta \rangle) \in A$.

Fix $\eta < \lambda$. We check B_η forms a chain of T. Let ζ_1, $\zeta_2 \in B_\eta$. Take $\xi < \lambda$ such that $\pi(\langle \eta, \zeta_1 \rangle)$, $\pi(\langle \eta, \zeta_2 \rangle) < \xi$. Then we can take $x \in H$ such that $\pi(\langle \eta, \zeta_1 \rangle)$, $\pi(\langle \eta, \zeta_2 \rangle) \in x$ and $a_x \cap \xi = A \cap x \cap \xi$. Thus $\pi(\langle \eta, \zeta_1 \rangle)$, $\pi(\langle \eta, \zeta_2 \rangle) \in A \cap x \cap \xi = a_x \cap \xi$. By the definition of a_x, both ζ_1 and ζ_2 belong b^x_η, hence ζ_1, ζ_2 are compatible. As the above argument, we can show that if $\zeta_1 \in B_\eta$ and $\zeta_2 \leq_T \zeta_1$ then $\zeta_2 \in B_\eta$.

Since T is an Aronsjazn tree, B_η is not cofinal in T. Take $\delta_\eta < \lambda$ such that $B_\eta \subseteq \bigcup_{\beta < \delta_\eta} T_\beta$ but $B_\eta \cap T_{\delta_\eta} = \emptyset$. Now we claim that δ_x is a successor ordinal, hence B_η has the max element. Assume not. Take $\xi < \lambda$ such that $\eta, \delta_x < \xi$, $\bigcup_{\beta \leq \delta_x} T_\beta \subseteq \xi$ and $\pi``(\xi \times \xi) \subseteq \xi$. Take $x \in H$ such that $\delta_\eta \in x$ and $A \cap x \cap \xi = a_x \cap \xi$. By Lemma 3.4 and the fact $H \subseteq C$, we may assume that for each $\beta \in x \cap \delta_x$, the β-th element of B_η (with respect to $\leq_T$) is in x. If $f_x(\eta) \geq \delta_\eta$, there exists $\gamma \in b^x_\eta \cap T_{\delta_\eta} \cap x$. But since $\bigcup_{\beta \leq \delta_\eta} T_\beta \cap x \subseteq x \cap \xi$, we have $\pi(\langle \eta, \gamma \rangle)) \in a_x \cap \xi = A \cap x \cap \xi$. Hence $\gamma \in B_\eta \cap T_{\delta_\eta} \neq \emptyset$, a contradiction. Thus $f_x(\eta) < \delta_\eta$. If $f_x(\eta) + 1 < \delta_\eta$, then $f_x(\eta) + 1 \in x$. Hence we can take $\gamma \in x \cap T_{f_x(\eta)+1} \cap B_\eta$. Then $\gamma < \xi$. Thus $\pi(\langle \eta, \gamma \rangle) \in A \cap x \cap \xi = a_x \cap \xi$. However then $\gamma \in b^x_\eta \cap T_{f_x(\eta)+1}$, a contradiction. Therefore we have $\delta_x = f_x(\eta) + 1$. Further notice that this arguments indicates the max element of b^x_η is equal to of B_η.

For $\eta < \lambda$, let α_η be the max element of B_η. Now define $f : \lambda \to \lambda$ by $f(\eta) =$ the height of α_η. We will see that for any $y \in \mathcal{P}_\kappa\lambda$ there exists $x \in H$ such that $y \subseteq x$ and $f|y = f_x|y$. Let $y \in \mathcal{P}_\kappa\lambda$. If necessary we may assume that y is closed under f. Take a large $\xi < \lambda$ such that $\sup(y) < \xi$ and $f``\xi \subseteq \xi$. Then we can take $x \in H$ such that $y \subseteq x$ and $A \cap x \cap \xi = a_x \cap \xi$. As the above argument, we may assume that for any $\eta \in y$, the max element of b^x_η is equal to of B_η. Then by the definition of b^η_x and f, we have $f_x(\eta) = f(\eta)$ holds for all $\eta \in y$. □

This completes the proof of Main Theorem 1.

Cor. 3.8 *Assume λ is regular, $\lambda^{<\lambda} = \lambda$ but not weakly compact. Then for $X \subseteq \mathcal{P}_\kappa\lambda$, the following are equivalent:*

(1) *X is Shelah,*

(2) $(\mathrm{NS}_{\kappa\lambda}|X)^* \xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2$ *holds,*

(3) $X \xrightarrow{\leq} (\mathrm{NS}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$ *holds,*

(4) $X \xrightarrow{\leq} (\mathrm{NSh}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$ *holds.* □

Proof: (4) ⇒ (3) is trivial. (1) ⇒ (4) is Lemma 3.3. (2) ⇒ (1) follows from Lemma 3.6 and 3.7.

(3) ⇒ (2). Assume $X \cap C \not\xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2$ for some club C of $\mathcal{P}_\kappa\lambda$. Since (3) holds, it must hold that $X \setminus C \xrightarrow{\leq} (\mathrm{NS}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$. However this is impossible; consider the constant function $f : [X \setminus C]^2_< \to \{0\}$. □

In the next section, we will prove that we cannot delete the assumption "λ is not weakly compact" of the above Lemma.

For a proof of Theorem 2, we must prove the case that λ is weakly compact. To see this, we need the following lemma.

Lemma 3.9 *Let ν be a cardinal with $\kappa \leq \nu < \lambda$ and assume $\lambda^\nu = \lambda$. If $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2$ holds then $\mathcal{P}_\kappa\nu$ satisfies the following property: whenever $\langle a_x : x \in \mathcal{P}_\kappa\nu\rangle$ with $a_x \subseteq x$, there exists $A \subseteq \nu$ such that $\{x \in \mathcal{P}_\kappa\nu : a_x = A \cap \nu\}$ is unbounded in $\mathcal{P}_\kappa\nu$.*

Remark that the above property of $\mathcal{P}_\kappa\nu$ is known as *almost ineffability* (see Carr [3]). Almost ineffability of $\mathcal{P}_\kappa\nu$ is stronger than the Shelah property, so the above lemma also shows that if $\lambda^\nu = \lambda$ and $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2$ holds then κ is ν-Shelah.

Proof: Let $\langle a_x : x \in \mathcal{P}_\kappa\nu\rangle$ be a sequence such that $a_x \subseteq x$. For each $x \in \mathcal{P}_\kappa\lambda$, let $b_x = a_{x\cap\nu} \subseteq x \cap \nu$. Then by Lemma 3.5 with the case $\nu^+ = \mu$, there exists unbounded $H \subseteq \mathcal{P}_\kappa\lambda$, $B \subseteq \nu$ and $z \in \mathcal{P}_\kappa\lambda$ such that for any $x \in H$ if $z < x$ then $b_x = B \cap x$. Let $H^* = \{x \cap \nu : x \in H,\ z < x\}$. Then it is easy to see that H^* is unbounded in $\mathcal{P}_\kappa\nu$ and for all $x \in H^*$, $a_x = B \cap x$. □

Lemma 3.10 *Assume λ is weakly compact. Then the followings are equivalent:*

(1) *κ is λ-Shelah,*

(2) *$\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{\leq} (\mathrm{I}^+_{\kappa\lambda})^2_2$ holds.*

Proof: The case that $\lambda = \kappa$ is well-known. Thus we may assume that $\lambda > \kappa$. (1) ⇒ (2) is Lemma 3.3. We see (2) ⇒ (1). By Lemma 3.9, κ is μ-Shelah for any $\mu < \lambda$. Now assume that κ is not λ-Shelah. Let $\vec{f} = \langle f_x : x \in \mathcal{P}_\kappa\lambda\rangle$ be a counterexample of the Shelah property of $\mathcal{P}_\kappa\lambda$. Consider the structure $\langle V_\lambda, \in, \vec{f}, \mathcal{P}_\kappa\lambda\rangle$. The assertion that "$\vec{f}$ is a counterexample of the Shelah property of $\mathcal{P}_\kappa\lambda$" can be describable as Π^1_1-sentence over $\langle V_\lambda, \in, \vec{f}, \mathcal{P}_\kappa\lambda\rangle$. Since weakly compact cardinal is Π^1_1-indescribable, this assertion is reflected to μ for some inaccessible $\mu < \lambda$. However this means that κ is not μ-Shelah, a contradiction. □

Therefore we conclude the following:

Cor. 3.11 *Assume λ is regular and $\lambda^{<\lambda} = \lambda$. Then the following are equivalent:*

(1) κ *is* λ*-Shelah,*

(2) $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$ *holds,*

(3) $\mathcal{P}_\kappa\lambda \xrightarrow{<} (\mathrm{NS}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$ *holds,*

(4) $\mathcal{P}_\kappa\lambda \xrightarrow{<} (\mathrm{NSh}^+_{\kappa\lambda}, \mathrm{I}^+_{\kappa\lambda})^2$ *holds.* □

This and corollary 3.8 are partial answers of a question of 5.5 in Carr [4].

Using Lemma 3.9, we have a slit improvement of a Magidor's theorem((3) of Fact 2.7). Notice that $\mathrm{Part}^*(\kappa, \lambda)^2_<$ implies $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$, but the converse does not hold in general.

Cor. 3.12 *The followings are equivalent:*

(1) κ *is supercompact,*

(2) $\mathrm{NS}^*_{\kappa\lambda} \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$ *holds for any* λ*,*

(3) *for any countable language structure* M *with* $\kappa \subseteq M$ *and* $f : [\{N \in \mathcal{P}_\kappa M : N \prec M,\ N \cap \kappa \in \kappa\}]^2_< \to 2$ *there exists an* H *such that* H *is unbounded in* $\mathcal{P}_\kappa M$ *and homogeneous for* f*. Where for* $X \subseteq \mathcal{P}_\kappa M$*,* $[X]^2_< = \{\{N, N'\} \subseteq X : N \subseteq N',\ |N| < |N' \cap \kappa|\}$. □

4 Some related results

In Theorem 1 and Cor. 3.8, it was assumed that λ is not weakly compact. Now we show that this assumption is needed.

Fact 4.1 *Let* θ *be a sufficiently large regular cardinal and* $\mu < \theta$ *a cardinal. Let* Δ *be a well-order of* $\mathcal{H}_\theta$*. Let* $M = \langle \mathcal{H}_\theta, \in, \Delta, \mu, \ldots \rangle$*. For* $N \prec M$ *and* $\alpha < \mu$*, let* $N[\alpha] = \{f(\alpha) : f \in {}^{\mu}N \cap N\}$*. Then* $N \subseteq N[\alpha]$*,* $\alpha \in N[\alpha]$ *and* $N[\alpha] \prec M$*.* □

In fact $N[\alpha]$ is just the Skolem hull of $N \cup \{\alpha\}$ under M.

Lemma 4.2 *Assume* λ *is weakly compact* $> \kappa$ *and* κ *is* λ*-Shelah. Let* $W = \{x \in \mathcal{P}_\kappa\lambda : \exists\alpha \in x\,(|x| = |x \cap \alpha|)\}$*. Then for any club* C *of* $\mathcal{P}_\kappa\lambda$*,* $(C \cap W) \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$ *holds.*

Proof: Let C be an arbitrary club and $g : \lambda \times \lambda \to \lambda$ generating C, that is, if $x \cap \kappa \in \kappa$ and x is closed under g then $x \in C$. Fix a sufficiently large regular cardinal θ and a well-order Δ on $\mathcal{H}_\theta$. Let $M = \langle \mathcal{H}_\theta, \in, \Delta, \kappa, \lambda, g \rangle$. Let $M^* = \mathrm{Skull}^M(\lambda)$.

Then by Carr [2], there exists a λ-complete proper M^*-normal ultra filter F over λ, here M^*-normal ultra mean that for all $A \in M^* \cap \mathcal{P}(\lambda)$, either $A \in F$ or $\lambda \setminus A \in F$, and for any regressive $f \in {}^\lambda\lambda \cap M^*$ there exists $\beta < \lambda$ such that $\{\alpha < \lambda : f(\alpha) = \beta\} \in F$.

By Abe [1], we can take $Y \in \mathrm{NSh}^*_{\kappa\lambda}$ such that

(1) for each $x \in Y$, $x \cap \kappa \in \kappa$ and $\mathrm{Skull}^M(x) \cap \lambda = x$, here $\mathrm{Skull}^M(x)$ is the Skolem hull of x under M,

(2) for $x, y \in Y$, if $x \neq y$ then $\sup(x) \neq \sup(y)$.

For $x \in Y$, let $M_x = \mathrm{Skull}^M(x)$. Note that $|M_x| = |x|$. Now define $\langle s_x : x \in Y\rangle$ by the induction on $\sup(x) < \lambda$. Let $x \in Y$ and assume $s_y < \lambda$ is defined for any $y \in Y$ with $\sup(y) < \sup(x)$. Consider $A = \bigcap\{B \in F : B \in M_x \cap \mathcal{P}(\lambda)\}$. Since F is λ-complete, $A \in F$. Hence we can take $s_x \in A$ such that $s_x > \sup(M_y[s_y] \cap \lambda)$ for any $y \in Y$ with $\sup(y) < \sup(x)$.

Now we claim the following:

Claim 4.3 $\{x \in Y : M_x[s_x] \cap s_x \neq x\}$ *is non-stationary.*

Proof: Assume not. Let $\pi : \lambda \to M^*$ be a bijection. Then $\{x \in \mathcal{P}_\kappa\lambda : M_x \cap \lambda = x, \pi``x = M_x\}$ is club, so $Z = \{x \in Y : \pi''x = M_x,\ M_x[s_x] \cap s_x \neq x\}$ is stationary. Let $x \in Z$. Then by the definition of $M_x[s_x]$, there exists $f_x \in M_x$ such that $f_x(s_x) \in (M[s_x] \cap s_x) \setminus x$. Then we may assume that $f_x \in {}^\lambda\lambda$ and f_x is regressive. By Fodor's lemma, there exists $f \in M^* \cap {}^\lambda\lambda$ such that $\{x \in Z : f_x = f\}$ is stationary. Since $f \in M^*$ and f is regressive, there exists $\beta < \lambda$ such that $\{\alpha < \lambda : f(\alpha) = \beta\} \in F$. Then we can take $x \in Z$ such that $f = f_x$ and $\beta \in x$. Since $f, \beta \in x$, we have $\{\alpha < \lambda : f(\alpha) = \beta\} \in F \cap M_x$. Then $s_x \in \{\alpha < \lambda : f(\alpha) = \beta\}$ thus $f_x(s_x) = \beta \in x$, a contradiction. □

Let $X = \{x \in Y : M_x[s_x] \cap s_x = x\}$. By the above claim $X \in \mathrm{NSh}^*_{\kappa\lambda}$. Note that for $x \in X$, $M_x[s_x] \cap \kappa = x \cap \kappa \in \kappa$, and $M_x[s_x]$ is closed under g. Thus we have $M_x[s_x] \cap \lambda \in C$. For $x \in X$, $|x| = |M_x[s_x] \cap s_x| = |M_x[s_x] \cap \lambda|$. Therefore $\{M_x[s_x] \cap \lambda : x \in X\} \subseteq C \cap \{x \in \mathcal{P}_\kappa\lambda : \exists\alpha \in x\,(|x| = |x \cap \alpha|)\}$. We will see that $\{M_x[s_x] \cap \lambda : x \in X\} \xrightarrow{<} (\mathrm{I}^+_{\kappa\lambda})^2_2$. To see this, we claim the following: for any $x, y \in X$, if $M_x[s_x] \cap \lambda < M_y[s_y] \cap \lambda$ then $x < y$. Since $|x| = |M_x[s_x] \cap \lambda|$ and $|y| = |M_y[s_y] \cap \lambda|$, we have $|x| < |y \cap \kappa|$. We check that $x \subseteq y$. We consider three cases.

1. If $\sup(x) = \sup(y)$, then $x = y$ by the definition of Y, a contradiction.

2. If $\sup(x) > \sup(y)$. Then $s_x > \sup(M_y[s_y] \cap \lambda)$ by the choice of s_x. Hence $s_x \notin M_y[s_y] \cap \lambda$, but this contradict to $M_x[s_x] \cap \lambda \subset M_y[s_y] \cap \lambda$.

3. If $\sup(x) < \sup(y)$. Note that then $s_x < s_y$. Hence $x = M_x[s_x] \cap s_x \subseteq M_y[s_y] \cap s_y = y$ and we have done.

For given $f : [\{M_x[s_x] \cap \lambda : x \in X\}]^2_< \to 2$, define $f' : [X]^2_< \to 2$ by $f'(x,y) = f(M_x[s_x] \cap \lambda, M_y[s_y] \cap \lambda)$ if $M_x[s_x] \cap \lambda < M_y[s_y] \cap \lambda$. Since $X \in \mathrm{NSh}^*_{\kappa\lambda}$, there exists an unbounded homogeneous set H for f'. Then it is easy to see that $\{M_x[s_x] \cap \lambda : x \in H\}$ is unbounded homogeneous set for f. □

Combing the above lemma and Fact 2.4, we have the following.

Cor. 4.4 *Assume λ is weakly compact $> \kappa$ and κ is λ-Shelah. Then $I|C \subsetneq \mathrm{NSh}_{\kappa\lambda}$ holds for any club C of $\mathcal{P}_\kappa\lambda$, here $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \overset{\leq}{\not\to} (\mathrm{I}^+_{\kappa\lambda})^2_2\}$.* □

Next we argue more possibility of the local normality of the partition ideal. The local normality of the 2-array partition ideal was shown. We see the case $n \geq 2$ with a bit weak assumption.

Lemma 4.5 *Let n be a natural number > 0 and $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \overset{\leq}{\not\to} (\mathrm{I}^+_{\kappa\lambda})^{n+1}_2\}$. Assume $\lambda = 2^\nu$ for some $\nu < \lambda$. Then there exists a club D of $\mathcal{P}_\kappa\lambda$ such that $I|D$ is normal.*

Proof: Fix $\nu < \lambda$ with $2^\nu = \lambda$. Note that $\kappa \leq \nu < \mathrm{cf}(\lambda)$ and $\lambda^\nu = \lambda$ holds. Fix $\vec{A} = \langle A_\xi : \xi < \lambda \rangle$ a bijective enumeration of $\mathcal{P}(\nu)$. Take a club C shown in Lemma 3.5 with the case $\mu = \nu^+$. Fix a sufficiently large regular cardinal θ and let $M = \langle \mathcal{H}_\theta, \in, \kappa, \lambda, \vec{A} \rangle$. Let $D = \{N \cap \lambda \in C : N \prec M,\ |N| < \kappa,\ N \cap \kappa \in \kappa\}$. We will prove D is a desired club.

Let $X \in (I|D)^+$ with $X \subseteq D$. Let $g : X \to \lambda$ be a regressive function. Assume $X_\alpha = \{x \in X : g(x) = \alpha\} \in I$ for all $\alpha < \lambda$. Let $f_\alpha : [X_\alpha]^{n+1}_< \to 2$ be a couterexample of $X_a \overset{\leq}{\longrightarrow} (\mathrm{I}^+_{\kappa\lambda})^n_2$. For $t \in [X]^n_<$, set $a_t = A_{g(\min(t))} \cap \min(t) \subseteq \min(t) \cap \nu$. Now define $f : [X]^{n+1}_< \to 2$ as: for $\{x_1, \ldots, x_{n+1}\} \in [X]^{n+1}_<$, if $g(x_1) = \cdots = g(x_{n+1}) = \alpha$, then $f(x_1, \ldots, x_{n+1}) = f_\alpha(x_1, \ldots, x_{n+1})$. Suppose not. Assume $a_{x_1 \cdots x_n} \neq a_{x_2 \cdots x_{n+1}} \cap x_1$ and let $\xi = \min(a_{x_1 \cdots x_n} \triangle a_{x_2 \cdots x_{n+1}} \cap x_1)$. If $\xi \in a_{x_2 \cdots x_{n+1}} \cap x_1$ then set $f(x_1, \ldots, x_{n+1}) = 0$. If $\xi \in a_{x_1 \cdots x_n}$ then set $f(x_1, \ldots, x_{n+1}) = 1$.

Then, by a similar argument of Lemma 3.5, there exists an unbouned homogeneous set $H \subseteq X$ for f, $A \subseteq \nu$ and $z \in \mathcal{P}_\kappa\lambda$ such that for any $t \in [H]^n_<$ if $z < \min(t)$ then $A \cap \min(t) = a_t$. Take $\alpha < \lambda$ such that $A = A_\alpha$ and put $H^* = \{x \in H : z < x,\ \alpha \in x\}$. It is easy to check that $H^* \subseteq X_\alpha$. Then by the definiton of f, H^* is an unbouned homogeneous set for f_α, a contradiction. □

Note that the above lemma shows that the partition ideal over $\mathcal{P}_\kappa\lambda$ can be locally normal even if λ is singular.

Combing an arguments of Lemma 3.7 with Lemma 4.5, we have the following.

Lemma 4.6 *Let n be a natural number > 0 and $I = \{X \subseteq \mathcal{P}_\kappa\lambda : X \overset{\leq}{\not\to} (\mathrm{I}^+_{\kappa\lambda})^{n+1}_2\}$. Assume λ is inaccessible but not weakly compact. Then there exists a club D of $\mathcal{P}_\kappa\lambda$ such that $I|D$ is normal.* □

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