数理解析研究所講究録 1473 巻 2006 年 143-163

HIGHER ORDER PAINLEVÉ EQUATIONS OF TYPE $D_l^{(1)}$

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ABSTRACT. A series of systems of nonlinear equations with affine Weyl group of type $D_l^{(1)}$ is studied. This series gives a generalization of Painlevé equations P_{VI} and P_V to higher orders.

0. INTRODUCTION

In this paper we propose a series of systems of nonlinear differential equations which have symmetry under the affine Weyl group of type $D_l^{(1)}$ ($l = 4, 5, 6, ..$). These systems are considered as higher order analogues of the Painlevé equations P_{VI} and P_V. For each $n = 1, 2, ...$, we find an algebraic ordinary differential system with symmetry under the affine Weyl group of type $D_{2n+2}^{(1)}$ for $2n$ unknown functions $q_1, p_1, q_2, p_2, ..., q_n, p_n$, containing complex parameters $(\alpha_1^*), (\alpha_2^*), ..., (\alpha_n^*)$. Here the symbol (α_i^*) denotes the set $(\alpha_i^*) = (\alpha_i^{(0)}, \alpha_i^{(1)}, ..., \alpha_i^{(4)})$. Our differential system is a Hamiltonian system, whose Hamiltonian is given as follows:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, .., n),$$

$$H = \sum_{i=1}^{n} H_{VI}(q_i, p_i, t; \alpha_i^{(0)}, \alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}, \alpha_i^{(4)}) + \sum_{1 \le l < m \le n} \frac{R(q_l, p_l, q_m, p_m, t; \alpha_m^{(2)})}{t(t-1)},$$

where

$$R(q_l, p_l, q_m, p_m, t; \alpha_m^{(2)}) := 2(q_l - t)p_l q_m((q_m - 1)p_m + \alpha_m^{(2)}),$$

and the parameters satisfy the following relations:

$$\begin{cases} \alpha_j^{(0)} + \alpha_j^{(1)} + 2\alpha_j^{(2)} + \alpha_j^{(3)} + \alpha_j^{(4)} = 1 \ (j = 1, 2, .., n), \\ \alpha_j^{(1)} + 2\alpha_j^{(2)} + \alpha_j^{(4)} - \alpha_{j+1}^{(1)} - \alpha_{j+1}^{(4)} = 0 \ (j = 1, 2, .., n-1), \\ \alpha_j^{(3)} - \alpha_j^{(4)} - 2\alpha_{j+1}^{(2)} - \alpha_{j+1}^{(3)} + \alpha_{j+1}^{(4)} = 0 \ (j = 1, 2, .., n-1), \end{cases}$$

and $H_{VI}(q, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denotes the Hamiltonian of the second-order Painlevé VI equations; (see Section 1).

Moreover, for each $n = 1, 2, 3, ...$, we find a $(2n+3)$-parameter family of coupled Painlevé V systems for $2n$ unknown functions $q_1, p_1, q_2, p_2, ..., q_n, p_n$, containing complex parameters $(\alpha_1{}^*), (\alpha_2^*), ..., (\alpha_n^*)$. Here the symbol (α_i^*) denotes the set $(\alpha_i^*) = (\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)})$. Our differential system is a Hamiltonian system, whose Hamiltonian is given as follows:

$$H=\sum_{i=1}^{n} H_V(q_i,p_i,t;\alpha_i^{(1)},\alpha_i^{(2)},\alpha_i^{(3)})+\sum_{1\leq l<m\leq n}\frac{R(q_l,p_l,q_m,p_m,t;\alpha_m^{(2)})}{t},$$

where

$$R(q_l,p_l,q_m,p_m,t;\alpha_m^{(2)}):=2p_lq_m((q_m-1)p_m+\alpha_m^{(2)}),$$

and the parameters satisfy the following relations:

$$\alpha_j^{(1)}-\alpha_j^{(3)}-\alpha_{j+1}^{(1)}+2\alpha_{j+1}^{(2)}+\alpha_{j+1}^{(3)}=0 \ (j=1,2,..,n-1),$$

and $H_V(q,p,t;\alpha_1,\alpha_2,\alpha_3)$ denotes the Hamiltonian of the second-order Painlevé V equations; (see Section 5).

In this paper, we will study the case of dimension 4, that is to say, the systems of type $D_6^{(1)}$ and $D_5^{(1)}$, respectively.

1. Motivation and main results

In the works [10],[11],[12], the author studied higher order Painlevé equations from the viewpoint of algebraic and Hamiltonian vector fields. In the case of the second-order Painlevé vector fields, it is well-known that each of Painlevé vector fields can be expressed as an algebraic vector field satisfying the following conditions:

$$(A) \quad \tilde{v}\in H^0(\mathbb{P}^2,\Theta_{\mathbb{P}^2}(-\log\mathcal{H})(n\mathcal{H})) \ (n=1,2,3).$$

Here, $\Theta_{\mathbb{P}^2}(-\log\mathcal{H})$ is the subsheaf of $\Theta_{\mathbb{P}^2}$ whose local section v satisfies $v(f)\in(f)$ for any local equation f of the boundary divisor $\mathcal{H}$ of $\mathbb{P}^2$. Moreover, each Painlevé vector field has the symmetry under the affine Weyl group (except for the first Painlevé vector field, which does not have the required symmetry). Here, let us summarize the following important properties of the Painlevé vector fields; (see [8],[19]).

Notation.
- $H\in\mathbb{C}(t)[x,y]$,
- $deg(H)$: degree with respect to x,y.

Painlevé equations	P_{VI}	P_V	P_{IV}	P_{III}	P_{II}
symmetry	$W(D_4^{(1)})$	$W(A_3^{(1)})$	$W(A_2^{(1)})$	$W(C_2^{(1)})$	$W(A_1^{(1)})$
degree of Hamiltonian H	5	4	3	4	3
$v\in H^0(\mathbb{P}^2,\Theta_{\mathbb{P}^2}(-\log\mathcal{H})(n\mathcal{H}))$	$n=3$	$n=2$	$n=1$	$n=2$	$n=1$

We are interested in the condition (A) and symmetry under the affine Weyl group, and wish to search for higher order Painlevé vector fields in algebraic vector fields with these favorable properties. As examples of higher order Painlevé vector fields, in 1998, Noumi and Yamada proposed a system of autonomous ordinary differential equations for $l+1$ unknown functions $f_0,f_1,..,f_l$ involving complex parameters $\alpha_0,\alpha_1,..,\alpha_l$ satisfying $\alpha_0+\alpha_1+\cdots+\alpha_l=1$; (see [6]). This system's salient feature is that it has the symmetry under the affine Weyl group of type $A_l^{(1)}$, where $\alpha_0,\alpha_1,..,\alpha_l$ are considered as simple roots of the affine root system of type $A_l^{(1)}$. When $l=2$ (resp. 3), this system of type $A_2^{(1)}$ (resp. $A_3^{(1)}$) is equivalent to the fourth (resp. fifth) Painlevé equation P_{IV} (resp. P_V). When $l>3$, the higher order Painlevé equations corresponding to these systems are not known to satisfy the Painlevé property, but it is widely believed that this is the case. They are considered to be higher order versions of P_V (resp. P_{IV}) when l is odd (resp. even). These two examples by Noumi and Yamada motivated the author to find the examples of higher order versions other than P_V and P_{IV} in this paper. Let us summarize important properties of these two systems as follows:

Notation.
- $H \in \mathbb{C}(t)[x,y,z,w]$,
- $deg(H)$: degree with respect to x,y,z,w.

symmetry	$W(A_5^{(1)})$	$W(A_4^{(1)})$
Hamiltonian H	$H_V(x,y,t)+H_V(z,w,t)$ $-2yzw+\frac{2xyzw}{t}$	$H_{IV}(x,y,t)+H_{IV}(z,w,t)$ $+2yzw$
form of equations	coupled Painlevé V	coupled Painlevé IV
degree of Hamiltonian H	4	3
$\tilde{v} \in H^0(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(-\log \mathcal{H})(n\mathcal{H}))$	$n=2$	$n=1$

These properties suggest the possibility that there exists a procedure for searching for such higher order versions with symmetry under the affine Weyl group of type $D_6^{(1)}$. Here, let us consider the following problem 1.

Problem 1.
Can we show existence of a vector field v associated with coupled Painlevé VI systems in dimension four satisfying the following conditions $(A1)$,$(A2)$? If yes, can we find it explicitly and is it unique?

Condition.
$(A1)$ $deg(H)=5$ with respect to x,y,z,w.
$(A2)$ The vector field v has symmetry under the affine Weyl group of type $D_6^{(1)}$.

To answer this, in this paper, we present an explicit 6-parameter family of fourth-order algebraic ordinary differential equations that can be considered as coupled Painlevé VI systems in dimension four with symmetry under the extended affine Weyl group of type $D_6^{(1)}$, and which is given as follows:

(1)
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{t(t-1)}\{2y(x-t)(x-1)x-(\alpha_0-1)(x-1)x-\alpha_3(x-t)x \\ \qquad -\alpha_4(x-t)(x-1)+2(x-t)z((z-1)w+\beta_2)\}, \\ \dfrac{dy}{dt} = \dfrac{1}{t(t-1)}[-\{(x-t)(x-1)+(x-t)x+(x-1)x\}y^2+\{(\alpha_0-1)(2x-1) \\ \qquad +\alpha_3(2x-t)+\alpha_4(2x-t-1)\}y-\alpha_2(\alpha_1+\alpha_2)-2yz((z-1)w+\beta_2)], \\ \dfrac{dz}{dt} = \dfrac{1}{t(t-1)}\{2w(z-t)(z-1)z-(\beta_0-1)(z-1)z-\beta_3(z-t)z \\ \qquad -\beta_4(z-t)(z-1)+2(x-t)yz(z-1)\}, \\ \dfrac{dw}{dt} = \dfrac{1}{t(t-1)}[-\{(z-t)(z-1)+(z-t)z+(z-1)z\}w^2+\{(\beta_0-1)(2z-1) \\ \qquad +\beta_3(2z-t)+\beta_4(2z-t-1)\}w-\beta_2(\beta_1+\beta_2)-2(x-t)y((2z-1)w+\beta_2)]. \end{cases}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, .., \alpha_4, \beta_0, \beta_1, .., \beta_4$ are complex parameters satisfying the following relations:

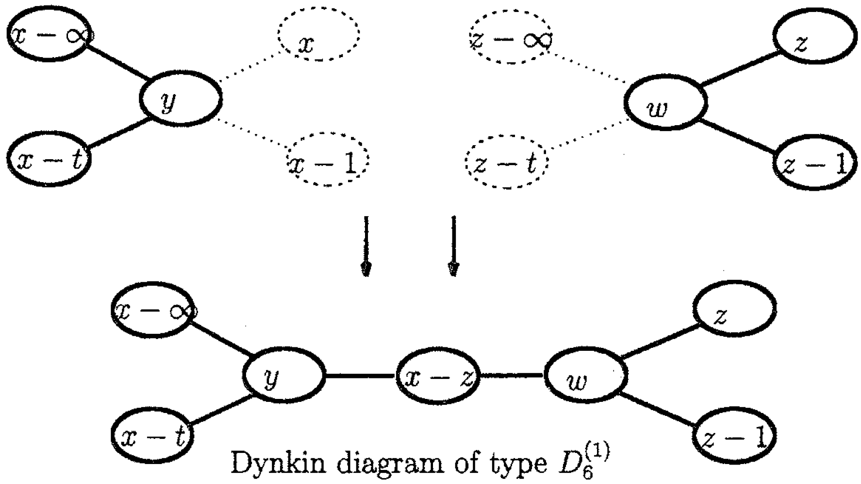
$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \ \beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = 1,$$

$$\alpha_1 + 2\alpha_2 + \alpha_4 - \beta_1 - \beta_4 = 0, \ \alpha_3 - \alpha_4 - 2\beta_2 - \beta_3 + \beta_4 = 0.$$

From the above relations, it is easy to see that the parameters $\alpha_3, \alpha_4, \beta_0, \beta_1$ also satisfy the following relations:

$$\alpha_3 = \frac{1 - \alpha_0 - \alpha_1 - 2\alpha_2 + 2\beta_2 + \beta_3 - \beta_4}{2}, \ \alpha_4 = \frac{1 - \alpha_0 - \alpha_1 - 2\alpha_2 - 2\beta_2 - \beta_3 + \beta_4}{2}$$

$$\beta_0 = \frac{1 + \alpha_0 - \alpha_1 - 2\alpha_2 - 2\beta_2 - \beta_3 - \beta_4}{2}, \ \beta_1 = \frac{1 - \alpha_0 + \alpha_1 + 2\alpha_2 - 2\beta_2 - \beta_3 - \beta_4}{2}.$$



Dynkin diagram of type $D_6^{(1)}$

Our differential system is equivalent to a Hamiltonian system, whose Hamiltonian H is given as follows:

$$\begin{aligned} H &= H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + H_{VI}(z, w, t; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \\ &\quad + \frac{2(x-t)yz\{(z-1)w + \beta_2\}}{t(t-1)}. \end{aligned} \tag{2}$$

The symbol $H_{VI}(q, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denotes the Hamiltonian of the second-order Painlevé VI equations, which is given as follows:

$H_{VI}(q, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{t(t-1)}(p^2(q-t)(q-1)q - \{(\alpha_0 - 1)(q-1)q + \alpha_3(q-t)q + \alpha_4(q-t)(q-1)\}p + \alpha_2(\alpha_1 + \alpha_2)(q-t)) \ (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1)$.

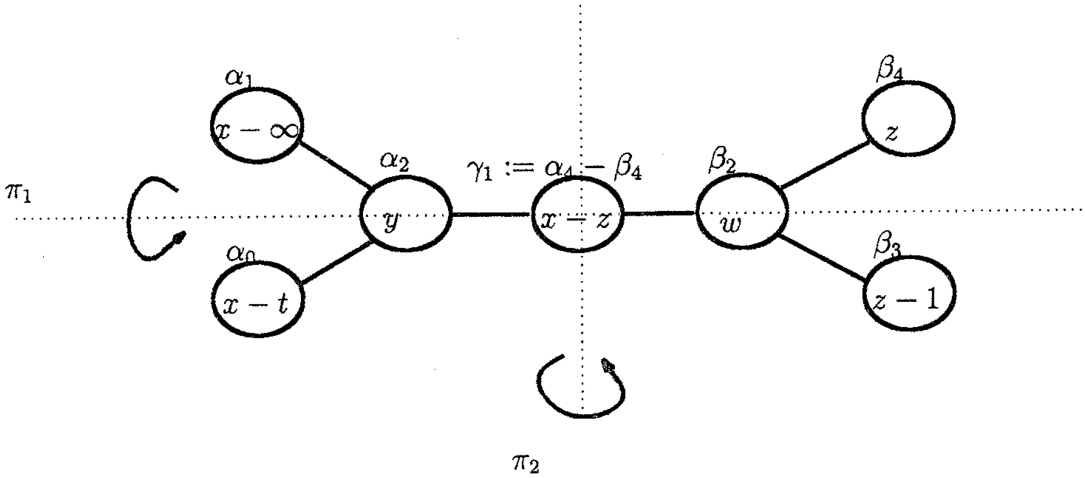
Remark 1.1. *Taking the holomorphic boundary coordinate system* $(X, Y, Z, W) = (x, y, 1/z, -z(zw + \beta_2))$ *of the system* (1), *the interaction term of the Hamiltonian* (2) *changes as follows*:

$$(3)\quad \begin{aligned} H &= H_{VI}(X,Y,t) + H'_{VI}(Z,W,t) + \frac{2(X-t)Y(Z-1)W}{t(t-1)} \\ &= H_{VI}(x,y,t) + H'_{VI}(Z,W,t) + \frac{2(x-t)y(Z-1)W}{t(t-1)}. \end{aligned}$$

Here, $H'_{VI}(Z,W,t)$ is the Hamiltonian in the holomorphic boundary coordinate system $(Z,W) = (1/z, -z(zw+\beta_2))$, which satisfies the following condition:

$$dz \wedge dw - dH_{VI}(z,w,t;\beta_0,\beta_1,\beta_2,\beta_3,\beta_4) \wedge dt = dZ \wedge dW - dH'_{VI}(Z,W,t) \wedge dt.$$

Theorem 1.1. *The system (1) is invariant under the transformations $s_0, s_1, .., s_6, \pi_1, \pi_2, \pi_3$ and π_4 defined as follows: with the notations $\gamma_1 := \alpha_4 - \beta_4$ and $(*) := (x,y,z,w,t;\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)$,*



$$s_0 : (*) \to (x, y - \frac{\alpha_0}{x-t}, z, w, t; -\alpha_0, \alpha_1, \alpha_2+\alpha_0, \gamma_1, \beta_2, \beta_3, \beta_4),$$

$$s_1 : (*) \to (x, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2+\alpha_1, \gamma_1, \beta_2, \beta_3, \beta_4),$$

$$s_2 : (*) \to (x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_0+\alpha_2, \alpha_1+\alpha_2, -\alpha_2, \gamma_1+\alpha_2, \beta_2, \beta_3, \beta_4),$$

$$s_3 : (*) \to (x, y - \frac{\gamma_1}{x-z}, z, w + \frac{\gamma_1}{x-z}, t; \alpha_0, \alpha_1, \alpha_2+\gamma_1, -\gamma_1, \beta_2+\gamma_1, \beta_3, \beta_4),$$

$$s_4 : (*) \to (x, y, z + \frac{\beta_2}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1+\beta_2, -\beta_2, \beta_3+\beta_2, \beta_4+\beta_2),$$

$$s_5 : (*) \to (x, y, z, w - \frac{\beta_3}{z-1}, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2+\beta_3, -\beta_3, \beta_4),$$

$$s_6 : (*) \to (x, y, z, w - \frac{\beta_4}{z}, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2 + \beta_4, \beta_3, -\beta_4),$$

$$\pi_1 : (*) \to (\frac{t(t-1)+t(x-t)}{x-t}, -\frac{(x-t)((x-t)y+\alpha_2)}{t(t-1)}, \frac{t(t-1)+t(z-t)}{z-t},$$

$$-\frac{(z-t)((z-t)w+\beta_2)}{t(t-1)}, t; \alpha_1, \alpha_0, \alpha_2, \gamma_1, \beta_2, \beta_4, \beta_3),$$

$$\pi_2 : (*) \to (\frac{t}{z}, -\frac{z(zw+\beta_2)}{t}, \frac{t}{x}, -\frac{x(xy+\alpha_2)}{t}, t; \beta_3, \beta_4, \beta_2, \gamma_1, \alpha_2, \alpha_0, \alpha_1),$$

$$\pi_3 : (*) \to (1-x, -y, 1-z, -w, 1-t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2, \beta_4, \beta_3),$$

$$\pi_4 : (*) \to (\tfrac{(t-1)x}{t-x}, \tfrac{(t-x)(ty-xy-\alpha_2)}{t(t-1)}, \tfrac{(t-1)z}{t-z}, \tfrac{(t-z)(tw-zw-\beta_2)}{t(t-1)}, 1-t; \alpha_1, \alpha_0, \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4).$$

Remark 1.2. *It is easy to see that the parameters* $\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_2, \beta_3, \beta_4$ *satisfy the relation:*

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2(\alpha_4 - \beta_4) + 2\beta_2 + \beta_3 + \beta_4 = 1,$$

and the generators π_2, π_3, π_4 *satisfy the relation:*

$$\pi_4 = \pi_2 \pi_3 \pi_2.$$

Remark 1.3. *Taking the holomorphic boundary coordinate system* $(X, Y, Z, W) = (1/x, -x(xy + \alpha_2), z, w)$, *it is easy to see that the transformation* s_1 *can be explicitly written as follows:*

$$s_1 : (X, Y, Z, W, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4)$$

$$\to (X, Y - \alpha_1/X, Z, W, t; \alpha_0, -\alpha_1, \alpha_1 + \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4).$$

Proposition 1.1. *The transformations described in Theorem* 1.1 *define a representation of the affine Weyl group of type* $D_6^{(1)}$, *that is, they satisfy the following relations:*

$s_0{}^2 = s_1{}^2 = s_2{}^2 = s_3{}^2 = s_4{}^2 = s_5{}^2 = s_6{}^2 = (\pi_1{}^2) = (\pi_2{}^2) = (\pi_3{}^2) = (\pi_4{}^2) = (s_0 s_1)^2 = (s_0 s_3)^2 = (s_0 s_4)^2 = (s_0 s_5)^2 = (s_0 s_6)^2 = (s_1 s_3)^2 = (s_1 s_4)^2 = (s_1 s_5)^2 = (s_1 s_6)^2 = (s_2 s_4)^2 = (s_2 s_5)^2 = (s_2 s_6)^2 = (s_3 s_5)^2 = (s_3 s_6)^2 = (s_5 s_6)^2 = 1, (s_0 s_2)^3 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^3 = (s_4 s_5)^3 = (s_4 s_6)^3 = 1,$

$\pi_1(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_0, s_2, s_3, s_4, s_6, s_5)\pi_1,$ $\pi_2(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (s_5, s_6, s_4, s_3, s_2, s_0, s_1)\pi_2,$ $\pi_3(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (s_0, s_1, s_2, s_3, s_4, s_6, s_5)\pi_3,$
$\pi_4(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_0, s_2, s_3, s_4, s_5, s_6)\pi_4.$

Proposition 1.1 is proved by straightforward computations.

Remark 1.4. *The following algebraic and Hamiltonian differential system*

(4)
$$\left\{\begin{aligned}
\frac{dx}{dt} &= \frac{1}{t(t-1)}\{2x^3y-2(t+1)x^2y+(1-\alpha_0-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6)x^2+2txy\\
&\quad +(-1+\alpha_0+\alpha_3+2t\alpha_3+2t\alpha_4+t\alpha_5+\alpha_6+t\alpha_6)x-t(\alpha_3+\alpha_6)\\
&\quad -2(t-x)z(-w+zw+\alpha_4)\},\\
\frac{dy}{dt} &= \frac{1}{t(t-1)}\{-3x^2y^2+2(t+1)xy^2-ty^2-2(1-\alpha_0-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6)xy\\
&\quad -(-1+\alpha_0+\alpha_3+2t\alpha_3+2t\alpha_4+t\alpha_5+\alpha_6+t\alpha_6)y\\
&\quad +\alpha_2(-1+\alpha_0+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6)-2yz(-w+zw+\alpha_4)\},\\
\frac{dz}{dt} &= \frac{1}{t(t-1)}\{2z^3w-2(t+1)z^2w+(1-\alpha_0-\alpha_3-\alpha_5-\alpha_6)z^2+2tzw\\
&\quad +(-1+\alpha_0+\alpha_3+t\alpha_5+\alpha_6+t\alpha_6)z-t\alpha_6-2(t-x)y(-1+z)z\},\\
\frac{dw}{dt} &= \frac{1}{t(t-1)}\{-3z^2w^2+2(t+1)zw^2-tw^2-2(1-\alpha_0-\alpha_3-\alpha_5-\alpha_6)zw\\
&\quad -(-1+\alpha_0+\alpha_3+t\alpha_5+\alpha_6+t\alpha_6)w+\alpha_4(-1+\alpha_0+\alpha_3+\alpha_4+\alpha_5+\alpha_6)\\
&\quad +2(t-x)y(-w+2zw+\alpha_4)\}
\end{aligned}\right.$$

coincides with the system (1) *when* $(\alpha_3,\alpha_4,\alpha_5,\alpha_6)$ *is rewritten as* $(\alpha_4-\beta_4,\beta_2,\beta_3,\beta_4)$, *and this system is invariant under the affine Weyl group* $<w_0,w_1,..,w_6>$ *of type* $D_6^{(1)}$, *whose generators* w_i *are explicitly written as follows:*

$w_0:(*)\rightarrow(x,y-\alpha_0/(x-t),z,w,t;-\alpha_0,\alpha_1,\alpha_2+\alpha_0,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$,

$w_1:(*)\rightarrow(x,y,z,w,t;\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$,

$w_2:(*)\rightarrow(x+\alpha_2/y,y,z,w,t;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4,\alpha_5,\alpha_6)$,

$w_3:(*)\rightarrow(x,y-\alpha_3/(x-z),z,w+\alpha_3/(x-z),t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5,\alpha_6)$,

$w_4:(*)\rightarrow(x,y,z+\alpha_4/w,w,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5+\alpha_4,\alpha_6+\alpha_4)$,

$w_5:(*)\rightarrow(x,y,z,w-\alpha_5/(z-1),t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4+\alpha_5,-\alpha_5,\alpha_6)$,

$w_6:(*)\rightarrow(x,y,z,w-\alpha_6/z,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4+\alpha_6,\alpha_5,-\alpha_6)$.

Here the parameters satisfy the relation $\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6=1$. *We give this alternate formulation* (4) *to the system* (1), *because the system* (4) *will be useful in the proof of Theorem 1.2.*

In addition to Theorem 1.1, we give an explicit description of a confluence to the system of type $A_5^{(1)}$:

Theorem 1.2. *For the system* (4) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

$$\alpha_0=\varepsilon^{-1},\ \alpha_1=A_3,\ \alpha_2=A_2,\ \alpha_3=A_1-B_1,\ \alpha_4=B_2,\ \alpha_5=B_0-B_2-\varepsilon^{-1},\ \alpha_6=B_1,$$

$$B_0=1-2A_1-2A_2-A_3+B_1-B_2,\ t=1+\varepsilon T,\ (x-1)(X-1)=1,\ (z-1)(Z-1)=1,$$

$$(x-1)y+(X-1)Y=-A_2,\ (z-1)w+(Z-1)W=-B_2,$$

from $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,t,x,y,z,w$ *to* $A_1,A_2,A_3,B_1,B_2,\varepsilon,T,X,Y,Z,W$. *Then the system* (4) *can also be written in the new variables* T,X,Y,Z,W *and parameters* $A_1,A_2,A_3,B_1,B_2,\varepsilon$ *as a Hamiltonian system. This new system tends to the system of type* $A_5^{(1)}$ *as* $\varepsilon \to 0$.

2. Review of the systems of type $A_4^{(1)}$ and type $A_5^{(1)}$

Let us recall the system of type $A_5^{(1)}$, which is explicitly written as follows:

(5)
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y+2xzw}{t} - \dfrac{x^2}{t} - 2xy - 2zw + (1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t})x + \alpha_2 + \alpha_4, \\ \dfrac{dy}{dt} = \dfrac{-2xy^2-2yzw}{t} + y^2 + \dfrac{2xy}{t} - (1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t})y + \dfrac{\alpha_1}{t}, \\ \dfrac{dz}{dt} = \dfrac{2z^2w+2xyz}{t} - \dfrac{z^2}{t} - 2zw - 2yz + (1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t})z + \alpha_4, \\ \dfrac{dw}{dt} = \dfrac{-2zw^2-2xyw}{t} + w^2 + \dfrac{2zw}{t} + 2yw - (1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t})w + \dfrac{\alpha_3}{t}. \end{cases}$$

Here, x,y,z and w denote unknown complex variables, and $\alpha_0,\alpha_1,..,\alpha_5$ are complex parameters with $\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=1$. The above differential system (5) is a Hamiltonian system, whose Hamiltonian $H_{A_5^{(1)}}$ is explicitly written as follows:

$$H_{A_5^{(1)}}(x,y,z,w,t;\alpha_0,..,\alpha_5) = \frac{x^2y^2-x^2y}{t} - xy^2 + (1+\frac{\alpha_1+\alpha_3+\alpha_5}{t})xy + (\alpha_2+\alpha_4)y - \frac{\alpha_1 x}{t}$$
$$+\frac{z^2w^2-z^2w}{t} - zw^2 + (1+\frac{\alpha_1+\alpha_3+\alpha_5}{t})zw + \alpha_4 w - \frac{\alpha_3 z}{t} - 2yzw + \frac{2xyzw}{t}.$$

The system (5) admits action of the affine Weyl group $< s_0,s_1,s_2,s_3,s_4,s_5 >$ of type $A_5^{(1)}$ as group of the Bäcklund transformations. By using the notation $(*) := (x,y,z,w,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$, the generators $s_0,s_1,..,s_5$ are explicitly written as follows:

$s_0 : (*) \to (x, y-\alpha_0/(x-t), z, w, t; -\alpha_0, \alpha_1+\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5+\alpha_0)$,
$s_1 : (*) \to (x+\frac{\alpha_1}{y}, y, z, w, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5)$,
$s_2 : (*) \to (x, y-\frac{\alpha_2}{x-z}, z, w+\frac{\alpha_2}{x-z}, t; \alpha_0, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5)$,
$s_3 : (*) \to (x, y, z+\frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5)$,
$s_4 : (*) \to (x, y, z, w-\frac{\alpha_4}{z}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_5+\alpha_4)$,
$s_5 : (*) \to (x+\frac{\alpha_5}{y+w-1}, y, z+\frac{\alpha_5}{y+w-1}, w, t; \alpha_0+\alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4+\alpha_5, -\alpha_5)$.

There is the following relation between the generators of type $A_5^{(1)}$ and holomorphic boundary coordinate systems of the system (5):

$$s : (x,y,z,w) \longrightarrow (x+\alpha/y, y, z, w) \Longleftrightarrow (X,Y,Z,W) = (1/x, -x(yx+\alpha), z, w).$$

Let us describe the above relation between all generators of type $A_5^{(1)}$ and holomorphic boundary coordinate systems as follows:

Holomorphic boundary coordinate systems with regard to the transformations s_i

$s_0 : x_0 = -((x-t)y-\alpha_0)y,\ y_0 = 1/y,\ z_0 = z,\ w_0 = w,$
$s_1 : x_1 = 1/x,\ y_1 = -(xy+\alpha_1)x,\ z_1 = z,\ w_1 = w,$

$s_2 : x_2 = -((x-z)y - \alpha_2)y,\ y_2 = 1/y,\ z_2 = z,\ w_2 = w + y,$
$s_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -(zw + \alpha_3)z,$
$s_4 : x_4 = x,\ y_4 = y,\ z_4 = -(zw - \alpha_4)w,\ w_4 = 1/w,$
$s_5 : x_5 = 1/x,\ y_5 = -((y + w - 1)y + \alpha_5)x,\ z_5 = z - x,\ w_5 = w.$

Remark 2.1. *Considering the relation between the generator s_2 and the boundary coordinate system (x_2, y_2, z_2, w_2), we take the linear symplectic transformation $m : (x, y, z, w) \to (x - z, y, z, w + y)$. Then it is easy to see that*

$$m^{-1}s_2m : (x, y, z, w) \to (x, y - \alpha_2/x, z, w).$$

Each coordinate system is a holomorphic coordinate system with a three-parameter family of meromorphic solutions of the system of type $A_5^{(1)}$ as the initial conditions. These coordinate systems can be obtained by blowing up accessible singular points in the boundary divisor $H \cong \mathbb{P}^3$ of $\mathbb{P}^4$.

By using the above relations, we can show the following proposition.

Proposition 2.1. *Let us consider an algebraic and Hamiltonian differential system with Hamiltonian $H \in C(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 4$ with respect to x, y, z, w.

(A2) This system has holomorphic boundary coordinate systems (x_i, y_i, z_i, w_i) $(i = 0, 1, .., 5)$.

Then such a system coincides with the system (5).

By Proposition 2.1, we will now see that — rather than assuming the condition that algebraic and Hamiltonian differential systems have symmetry under the affine Weyl group of type $A_5^{(1)}$ — we can research the algebraic ordinary differential systems here under the assumption that algebraic and Hamiltonian differential system has holomorphic boundary coordinate systems associated with the generators of the affine Weyl group of type $A_5^{(1)}$.

Next, let us recall the system of type $A_4^{(1)}$, which is explicitly written as follows:

$$(6) \quad \begin{cases} \dfrac{dx}{dt} = x^2 + 2xy + 2zw - tx - \alpha_2 - \alpha_4 \\ \dfrac{dy}{dt} = -y^2 - 2xy + ty - \alpha_1 \\ \dfrac{dz}{dt} = z^2 + 2zw + 2yz - tz - \alpha_4 \\ \dfrac{dw}{dt} = -w^2 - 2zw - 2yw + tw - \alpha_3. \end{cases}$$

Here, x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, .., \alpha_4$ are complex parameters with $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -1$. The above differential system (6) is a Hamiltonian system, whose Hamiltonian $H_{A_4^{(1)}}$ is explicitly written as follows:

$$\begin{aligned} H_{A_4^{(1)}}(x, y, z, w, t; \alpha_0, .., \alpha_4) &= x^2y + xy^2 - txy + \alpha_1 x - (\alpha_2 + \alpha_4)y \\ &+ z^2w + zw^2 - tzw + \alpha_3 z - \alpha_4 w + 2yzw. \end{aligned}$$

The system (6) admits action of the affine Weyl group $< s_0, s_1, s_2, s_3, s_4 >$ of type $A_4^{(1)}$ as group of the Bäcklund transformations. By using the notation $(*) :=$

$(x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, the generators $s_0, s_1, .., s_4$ are explicitly written as follows:

$s_0 : (*) \to (x + \frac{\alpha_0}{x+y+w-t}, y - \frac{\alpha_0}{x+y+w-t}, z + \frac{\alpha_0}{x+y+w-t}, w, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3, \alpha_4 + \alpha_0)$,
$s_1 : (*) \to (x+\frac{\alpha_1}{y}, y, z, w, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4)$,
$s_2 : (*) \to (x, y-\frac{\alpha_2}{x-z}, z, w+\frac{\alpha_2}{x-z}, t; \alpha_0, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4)$,
$s_3 : (*) \to (x, y, z+\frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3)$,
$s_4 : (*) \to (x, y, z, w-\frac{\alpha_4}{z}, t; \alpha_0+\alpha_4, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4)$.

There is the following relation between the generators of type $A_4^{(1)}$ and holomorphic boundary coordinate systems of the system (6):

$$s : (x, y, z, w) \longrightarrow (x + \alpha/y, y, z, w) \Longleftrightarrow (X, Y, Z, W) = (1/x, -x(yx + \alpha), z, w).$$

Let us describe the above relation between all generators of type $A_4^{(1)}$ and holomorphic boundary coordinate systems as follows:

Holomorphic boundary coordinate systems with regard to the transformations s_i

$s_0 : x_0 = -((x + y + w - t)y - \alpha_0)y,\ y_0 = 1/y,\ z_0 = z + y,\ w_0 = w,$
$s_1 : x_1 = 1/x,\ y_1 = -(xy + \alpha_1)x,\ z_1 = z,\ w_1 = w,$
$s_2 : x_2 = -((x - z)y - \alpha_2)y,\ y_2 = 1/y,\ z_2 = z,\ w_2 = w + y,$
$s_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -(zw + \alpha_3)z,$
$s_4 : x_4 = x,\ y_4 = y,\ z_4 = -(zw - \alpha_4)w,\ w_4 = 1/w.$

Remark 2.2. *Considering the relation between the generator s_2 and the boundary coordinate system (x_2, y_2, z_2, w_2), we take the linear symplectic transformation $m : (x, y, z, w) \to (x - z, y, z, w + y)$. Then it is easy to see that*

$$m^{-1} s_2 m : (x, y, z, w) \to (x, y - \alpha_2/x, z, w).$$

Each coordinate system is a holomorphic coordinate system with a three-parameter family of meromorphic solutions of the system (6) as the initial conditions. These coordinate systems can be obtained by blowing up accessible singular points in the boundary divisor $H \cong \mathbb{P}^3$ of $\mathbb{P}^4$.

By using the above relations, we can show the following proposition.

Proposition 2.2. *Let us consider an algebraic and Hamiltonian differential system with Hamiltonian $H \in C(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 3$ with respect to x, y, z, w.

(A2) This system has holomorphic boundary coordinate systems (x_i, y_i, z_i, w_i) $(i = 0, 1, .., 4)$.

Then such a system coincides with the system (6).

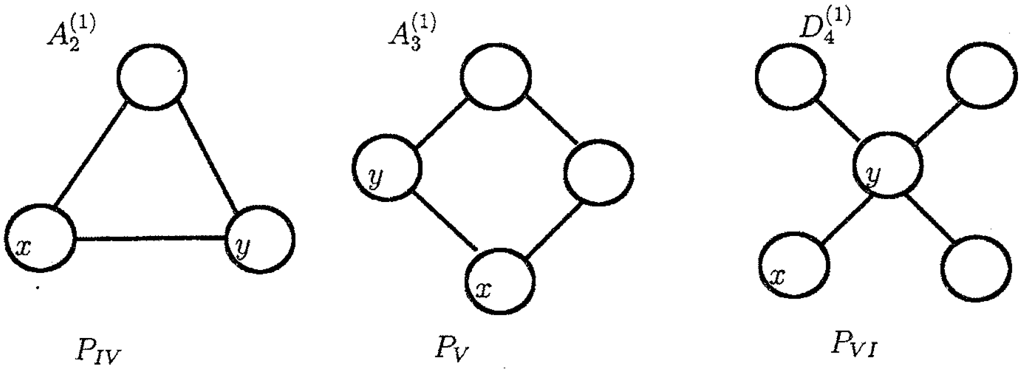
By Proposition 2.2, we will now see that — rather than assuming the condition that algebraic and Hamiltonian differential systems have symmetry under the affine Weyl group of type $A_4^{(1)}$ — we can research the algebraic ordinary differential systems here under the assumption that algebraic and Hamiltonian differential systems have holomorphic boundary coordinate systems associated with the generators of the affine Weyl group of type $A_4^{(1)}$.

3. An approach for obtaining system (1)

Much effort has been made to investigate the algebraic ordinary differential systems with symmetry under the affine Weyl group of type $D_6^{(1)}$, but these systems have not yet been found. Taking a hint from the representation of the affine Weyl groups of type $A_4^{(1)}$ and $A_5^{(1)}$; (see [7]), we consider Problem 1. We do not yet have the explicit description of the symmetry under the affine Weyl group of type $D_6^{(1)}$ with respect to x, y, z, w, so we will construct the symmetry under the affine Weyl group of type $D_6^{(1)}$ by using a part of the symmetry under the affine Weyl groups of type $A_4^{(1)}$ and type $A_5^{(1)}$. In the case of the Painlevé systems, the affine Weyl groups $W(A_2^{(1)})$, $W(A_3^{(1)})$ and $W(D_4^{(1)})$ have a common subgroup, which is isomorphic to the classical Weyl group $W(A_2)$. Here, the elements u_i of the subgroup $W(A_2) =< u_1, u_2 >$ are explicitly written as follows:

$$u_1 : (x, y) \to (x + \frac{\gamma_1}{y}, y), \; u_2 : (x, y) \to (x, y - \frac{\gamma_2}{x}).$$

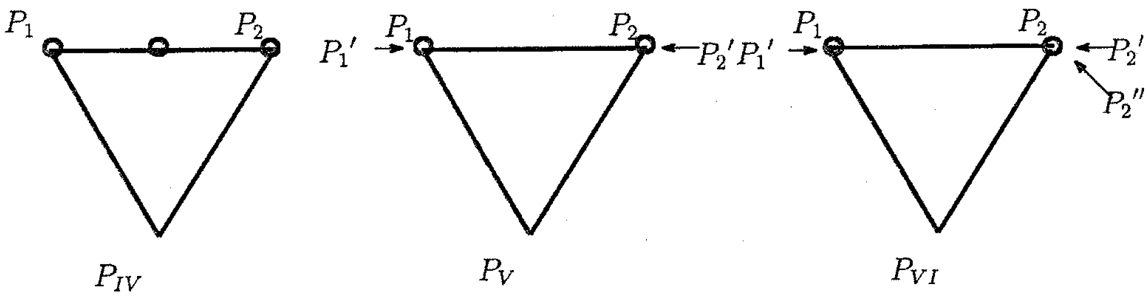
Here, γ_1 and γ_2 are constant parameters.



These transformations u_1, u_2 correspond to holomorphic boundary coordinate systems (x_i, y_i) $(i = 1, 2)$, which are explicitly written as follows:

$$(x_1, y_1) := (1/x, -(xy + \gamma_1)x), \; (x_2, y_2) := (-(xy - \gamma_2)y, 1/y).$$

Moreover, these transformations u_1, u_2 correspond to the accessible singular points P_1, P_2 on the boundary divisor of $\mathbb{P}^2$.



Proposition 3.1. *Let us consider an algebraic and Hamiltonian differential system with Hamiltonian $H \in \mathbb{C}(t)[x, y]$. We assume that*

(A1) $deg(H) = 5$ with respect to x, y.

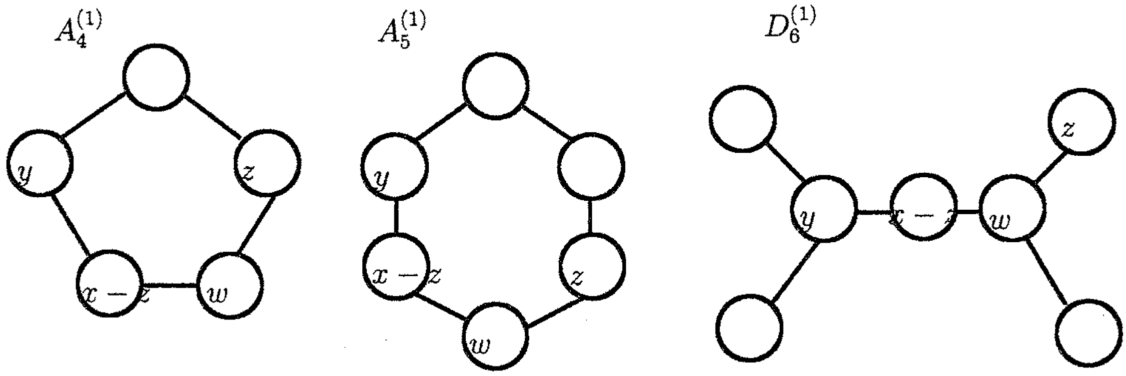
(A2) This system has holomorphic boundary coordinate systems (x_i, y_i) $(i = 1, 2)$ associated with the generators of the Weyl group $W(A_2) =< u_1, u_2 >$, which are explicitly given as follows:

$u_1 : (x_1, y_1) := (1/x, -(xy + \gamma_1)x),$
$u_2 : (x_2, y_2) := (-(xy - \gamma_2)y, 1/y).$
Then such a system is explicitly given as follows:

$$\begin{cases} \dfrac{dx}{dt} = 2a_1x^3y + 3a_2x^2y^2 + 2a_3x^2y + a_4x^2 + 2a_5xy + a_6x - \gamma_2a_5 - \gamma_2^2a_2 \\ \dfrac{dy}{dt} = -3a_1x^2y^2 - 2a_2xy^3 - 2a_3xy^2 - a_5y^2 - 2a_4xy - a_6y - \gamma_1a_4 + {\gamma_1}^2a_1. \end{cases}$$

Here, $a_1, a_2, .., a_6$ are unknown rational functions in t.

By the above proposition, if algebraic and Hamiltonian differential systems in dimension two with the condition (A) (given in Section 1) have symmetry under the group $W(A_2) =< u_1, u_2 >$, then the part of degree 2 with respect to x, y in the right hand side of this differential system is determined by the transformations u_1, u_2. In the case of dimension 4, it is easy to see that the affine Weyl groups $W(A_5^{(1)})$ and $W(A_4^{(1)})$ have a common subgroup W, which is isomorphic to the classical Weyl group $W(A_4)$. Here, the elements g_i of the subgroup $W(A_4) =< g_1, g_2, g_3, g_4 >$ are explicitly written as follows:



$$g_1 : (x, y, z, w) \to (x, y, z + \frac{\gamma_1}{w}, w), \ \ g_2 : (x, y, z, w) \to (x, y, z, w + \frac{\gamma_2}{z}),$$

$$g_3 : (x, y, z, w) \to (x + \frac{\gamma_3}{y}, y, z, w), \ \ g_4 : (x, y, z, w) \to (x, y - \frac{\gamma_4}{x - z}, z, w + \frac{\gamma_4}{x - z}).$$

Here, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are constant parameters.

Proposition 3.2. *Let us consider an algebraic and Hamiltonian differential system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*
$(A1)$ $deg(H) = 5$ with respect to x, y, z, w.
$(A2)$ This system has holomorphic boundary coordinate systems (x_i, y_i, z_i, w_i) $(i = 1, 2, 3, 4)$ associated with the generators of the Weyl group $W(A_4) =< g_1, g_2, g_3, g_4 >$, which are explicitly given as follows:
$g_1 : x_1 = x, \ y_1 = y, \ z_1 = 1/z, \ w_1 = -z(zw + \gamma_1),$
$g_2 : x_2 = x, \ y_2 = y, \ z_2 = -w(zw + \gamma_2), \ w_2 = 1/w,$
$g_3 : x_3 = 1/x, \ y_3 = -x(xy + \gamma_3), \ z_3 = z, \ w_3 = w,$
$g_4 : x_4 = -((x - z)y + \gamma_4)y, \ y_4 = 1/y, \ z_4 = z, \ w_4 = y + w.$
Then such a system is explicitly given as follows:

$$
\begin{cases}
\dfrac{dx}{dt} = (b_1+b_2)x^3y + (b_3+b_4)x^2y + b_5x^2 + 2b_6xy + (b_7-\gamma_1b_3+\gamma_4b_3)x + (\gamma_2+\gamma_4)b_6 \\
\qquad + 2b_6zw + b_4xzw + b_1x^2zw + b_3z^2w + b_2xz^2w + \gamma_1b_3z + \gamma_1b_2xz, \\
\dfrac{dy}{dt} = -\dfrac{3(b_1+b_2)x^2y^2}{2} - (b_3+b_4)xy^2 - b_6y^2 - 2b_5xy - (b_7-\gamma_1b_3+\gamma_4b_3)y \\
\quad + \dfrac{(b_1+b_2)\gamma_3{}^2}{2} - \gamma_5b_5 - b_4yzw - 2b_1xyzw - b_2yz^2w - b_2\gamma_1yz - b_1\gamma_3zw, \\
\dfrac{dz}{dt} = (b_1+b_2)z^3w + (b_3+b_4)z^2w + \dfrac{2b_5+2\gamma_1b_2-2\gamma_3b_1+\gamma_4b_1-\gamma_4b_2}{2}z^2 \\
\quad + 2b_6zw + b_7z + \gamma_2b_6 + 2b_6yz + b_4xyz + b_1x^2yz + b_3yz^2 + b_2xyz^2 + \gamma_3b_1xz, \\
\dfrac{dw}{dt} = -\dfrac{3(b_1+b_2)z^2w^2}{2} - (b_3+b_4)zw^2 - b_6w^2 - (2b_5+2\gamma_1b_2-2\gamma_3b_1+\gamma_4b_1-\gamma_4b_2)zw \\
\quad - b_7w - \dfrac{\gamma_1(2b_5-\gamma_1b_1+\gamma_1b_2-2\gamma_3b_1+\gamma_4b_1-\gamma_4b_2)}{2} - 2b_6yw - b_4xyw - b_1x^2yw \\
\quad - 2b_3yzw - 2b_2xyzw - \gamma_1b_3y - \gamma_1b_2xy - \gamma_3b_1xw.
\end{cases}
$$

Here, $b_1, b_2, .., b_7$ are unknown rational functions in t. Furthermore, the Hamiltonian H is explicitly written as follows:

$$
\begin{aligned}
H =& \frac{(b_1+b_2)}{2}x^3y^2 + \frac{(b_3+b_4)}{2}x^2y^2 + b_5x^2y + b_6xy^2 + (b_7-\gamma_1b_3+\gamma_4b_3)xy + (\gamma_2+\gamma_4)b_6y \\
&+ (\gamma_5b_5 - \frac{(b_1+b_2)\gamma_3^2}{2})x + \frac{(b_1+b_2)}{2}z^3w^2 + \frac{(b_3+b_4)}{2}z^2w^2 + b_6zw^2 + b_7zw + \gamma_2b_6w \\
&+ \frac{(2b_5+2\gamma_1b_2-2\gamma_3b_1+\gamma_4b_1-\gamma_4b_2)}{2}z^2w + \frac{\gamma_1(2b_5-\gamma_1b_1+\gamma_1b_2-2\gamma_3b_1+\gamma_4b_1-\gamma_4b_2)}{2}z \\
&+ 2b_6yzw + b_4xyzw + b_1x^2yzw + b_3yz^2w + b_2xyz^2w + \gamma_1b_3yz + \gamma_1b_2xyz + b_1\gamma_3xzw.
\end{aligned}
$$

By the above proposition, if algebraic and Hamiltonian differential systems in dimension 4 with the condition $\tilde{v} \in H^0(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(-\log \mathcal{H})(n\mathcal{H}))$ $(n = 1,2,3)$ have symmetry under the group $W(A_4) =< g_1, g_2, g_3, g_4 >$, then the part of degree 2 with respect to x, y, z, w in the right hand side of this differential system is determined by the transformations g_1, g_2, g_3, g_4.

4. Proof of Theorem 1.2

As is well-known, the degeneration from P_{VI} to P_V; (see [16],[17]) is given by

$$\alpha_0 = \varepsilon^{-1},\ \alpha_1 = A_3,\ \alpha_3 = A_0 - A_2 - \varepsilon^{-1},\ \alpha_4 = A_1,$$

$$t = 1 + \varepsilon T,\ (x-1)(X-1) = 1,\ (x-1)y + (X-1)Y = -A_2.$$

Notice that $A_0 + A_1 + A_2 + A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ and the change of variables from (q,p) to (Q,P) is symplectic.

As the fourth-order analogue of the above confluence process, we consider the following coupling confluence process from the system (4). We take the following coupling confluence process $P_{VI} \to P_V$ for each coordinate system (x,y) and (z,w) of the system in (4)

$\alpha_0 = \varepsilon^{-1},\ \alpha_1 = A_3,\ \alpha_2 = A_2,\ \alpha_3 = A_1 - B_1,\ \alpha_4 = B_2,\ \alpha_5 = B_0 - B_2 - \varepsilon^{-1},\ \alpha_6 = B_1,$

$B_0 = 1 - 2A_1 - 2A_2 - A_3 + B_1 - B_2,\ t = 1 + \varepsilon T,\ (x-1)(X-1) = 1,\ (z-1)(Z-1) = 1,$

$$(x-1)y + (X-1)Y = -A_2,\ (z-1)w + (Z-1)W = -B_2,$$

and take the limit $\varepsilon \to 0$. Moreover, by the following transformation φ

$$\varphi : (X, Y, Z, W, T; A_1, A_2, A_3, B_1, B_2) \to (-tx, -y/t, -tz, -w/t, -t; \alpha_2 + \alpha_4, \alpha_1, \alpha_0, \alpha_4, \alpha_3),$$

we obtain the system of type $A_5^{(1)}$, which is explicitly written as follows:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y + 2xzw}{t} - \dfrac{x^2}{t} - 2xy - 2zw + (1 + \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{t})x + \alpha_2 + \alpha_4, \\ \dfrac{dy}{dt} = \dfrac{-2xy^2 - 2yzw}{t} + y^2 + \dfrac{2xy}{t} - (1 + \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{t})y + \dfrac{\alpha_1}{t}, \\ \dfrac{dz}{dt} = \dfrac{2z^2w + 2xyz}{t} - \dfrac{z^2}{t} - 2zw - 2yz + (1 + \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{t})z + \alpha_4, \\ \dfrac{dw}{dt} = \dfrac{-2zw^2 - 2xyw}{t} + w^2 + \dfrac{2zw}{t} + 2yw - (1 + \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{t})w + \dfrac{\alpha_3}{t}. \end{cases}$$

Here, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$.

5. The system of type $D_5^{(1)}$

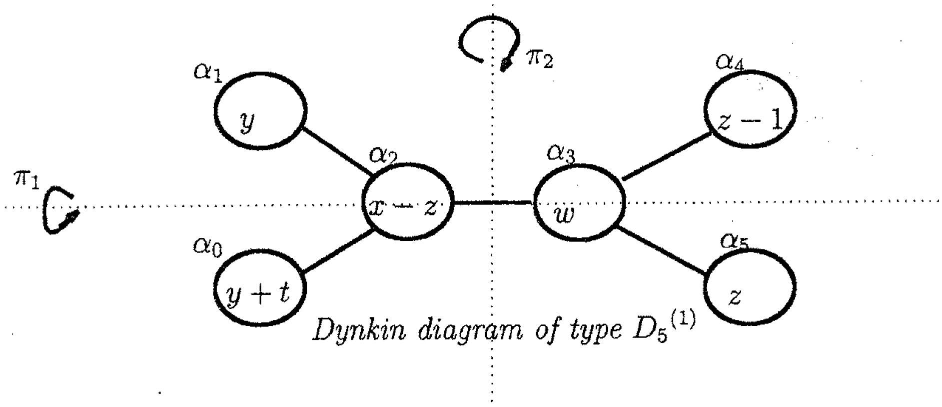
In this section, we present a 5-parameter family of algebraic ordinary differential equations that can be considered as coupled Painlevé V systems in dimension four, and which is given as follows:

(7)
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y}{t} + x^2 - \dfrac{2xy}{t} - (1 + \dfrac{2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4}{t})x + \dfrac{\alpha_2 + \alpha_5}{t} + \dfrac{2z((z-1)w + \alpha_3)}{t}, \\ \dfrac{dy}{dt} = -\dfrac{2xy^2}{t} + \dfrac{y^2}{t} - 2xy + (1 + \dfrac{2\alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_4}{t})y - \alpha_1, \\ \dfrac{dz}{dt} = \dfrac{2z^2w}{t} + z^2 - \dfrac{2zw}{t} - (1 + \dfrac{\alpha_5 + \alpha_4}{t})z + \dfrac{\alpha_5}{t} + \dfrac{2yz(z-1)}{t}, \\ \dfrac{dw}{dt} = -\dfrac{2zw^2}{t} + \dfrac{w^2}{t} - 2zw + (1 + \dfrac{\alpha_5 + \alpha_4}{t})w - \alpha_3 - \dfrac{2y(-w + 2zw + \alpha_3)}{t}. \end{cases}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, .., \alpha_5$ are complex parameters satisfying the following relation:

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1.$$

Theorem 5.1. *The system (7) is invariant under the transformations $s_0, s_1, .., s_5, \pi_1, \pi_2, \pi_3$ and π_4 defined as follows: with the notation $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$,*



Dynkin diagram of type $D_5^{(1)}$

$$s_0 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x+\frac{\alpha_0}{y+t}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2+\alpha_0, \alpha_3, \alpha_4, \alpha_5),$$

$$s_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x+\frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5),$$

$$s_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to$$
$$(x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5),$$

$$s_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x, y, z+\frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5+\alpha_3),$$

$$s_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x, y, z, w-\frac{\alpha_4}{(z-1)}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_5),$$

$$s_5 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x, y, z, w-\frac{\alpha_5}{z}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_5, \alpha_4, -\alpha_5),$$

$$\pi_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (1-x, -y-t, 1-z, -w, t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_5, \alpha_4),$$

$$\pi_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to ((y+w+t)/t, -t(z-1), (y+t)/t, -t(x-z), -t;$$
$$\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0),$$

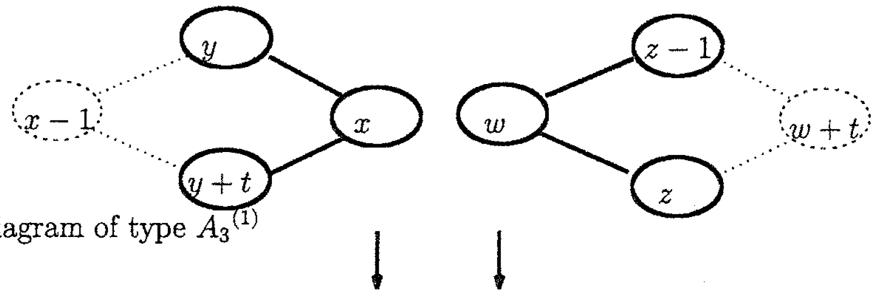
$$\pi_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (1-x, -y, 1-z, -w, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4),$$

$$\pi_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (x, y+t, z, w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5).$$

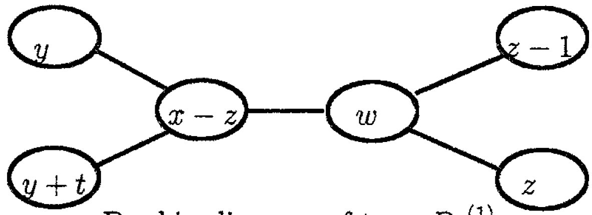
Remark 5.1. *It is easy to see that the generators* π_2, π_3, π_4 *satisfy the following relation:*

$$\pi_4 = \pi_2\pi_3\pi_2.$$

Theorem 5.2. *The transformations described in Theorem* 5.1 *define a representation of the affine Weyl group of type* $D_5^{(1)}$, *that is, they satisfy the following relations*: $s_0{}^2 = s_1{}^2 = s_2{}^2 = s_3{}^2 = s_4{}^2 = s_5{}^2 = (\pi_1{}^2) = (\pi_2{}^2) = 1,\ (s_0s_1)^2 = (s_0s_3)^2 = (s_0s_4)^2 = (s_0s_5)^2 = (s_1s_3)^2 = (s_1s_4)^2 = (s_1s_5)^2 = (s_2s_4)^2 = (s_2s_5)^2 = 1,\ (s_4s_5)^2 = (s_0s_2)^3 = (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_4)^3 = (s_3s_5)^3 = 1,\ \pi_1(s_0,s_1,s_2,s_3,s_4,s_5) = (s_1,s_0,s_2,s_3,s_5,s_4)\pi_1,\ \pi_2(s_0,s_1,s_2,s_3,s_4,s_5) = (s_5,s_4,s_3,s_2,s_1,s_0)\pi_2, \pi_3(s_0,s_1,s_2,s_3,s_4,s_5) = (s_0,s_1,s_2,s_3,s_5,s_4)\pi_3,\ \pi_4(s_0,s_1,s_2,s_3,s_4,s_5) = (s_1,s_0,s_2,s_3,s_4,s_5)\pi_4.$



Dynkin diagram of type $A_3{}^{(1)}$



Dynkin diagram of type $D_5{}^{(1)}$

Our differential system (7) is equivalent to a Hamiltonian system, whose Hamiltonian H is given as follows:

$$
\begin{aligned}
H = H_V(x,y,t;\alpha_2+\alpha_5,\alpha_1,\alpha_2+2\alpha_3+\alpha_4) + H_V(z,w,t;\alpha_5,\alpha_3,\alpha_4) \\
+ \frac{2yz\{(z-1)w+\alpha_3\}}{t}.
\end{aligned} \tag{8}
$$

Here, the symbol $H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3)$ denotes the Hamiltonian of the second-order Painlevé V systems, which is given as follows:

$$
H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3) = \frac{q(q-1)p(p+t) - (\gamma_1+\gamma_3)qp + \gamma_1 p + \gamma_2 tq}{t}.
$$

In addition to Theorems 5.1 and 5.2, we will prove that the system (7) degenerates to the system of type $A_4{}^{(1)}$ by taking the coupling confluence process of $P_V \to P_{IV}$.

Theorem 5.3. *For the system* (7) *of type* $D_5^{(1)}$, *we make the change of parameters and variables*

$$
\alpha_0 = A_0 - A_2 - A_3 + \frac{1}{2}\varepsilon^{-2},\ \alpha_1 = A_1,\ \alpha_2 = A_2,\ \alpha_3 = A_3,\ \alpha_4 = -\frac{1}{2}\varepsilon^{-2},\ \alpha_5 = A_4,
$$

$$
t = \frac{1}{2}\varepsilon^{-2}(1+2\varepsilon T),\ x = -\frac{\varepsilon X}{1-\varepsilon X},\ y = -\varepsilon^{-1}(1-\varepsilon X)[Y - \varepsilon(A_1 + XY)],
$$

$$
z = -\frac{\varepsilon Z}{1-\varepsilon Z},\ w = -\varepsilon^{-1}(1-\varepsilon Z)[W - \varepsilon(A_3 + XY)],
$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, t, x, y, z, w$ to $A_0, A_1, A_2, A_3, A_4, \varepsilon, T, X, Y, Z, W$. Then the system (7) can also be written in the new variables T, X, Y, Z, W and parameters $A_0, A_1, A_2, A_3, A_4, \varepsilon$ as a Hamiltonian system. This new system tends to the system of type $A_4^{(1)}$ as $\varepsilon \to 0$.

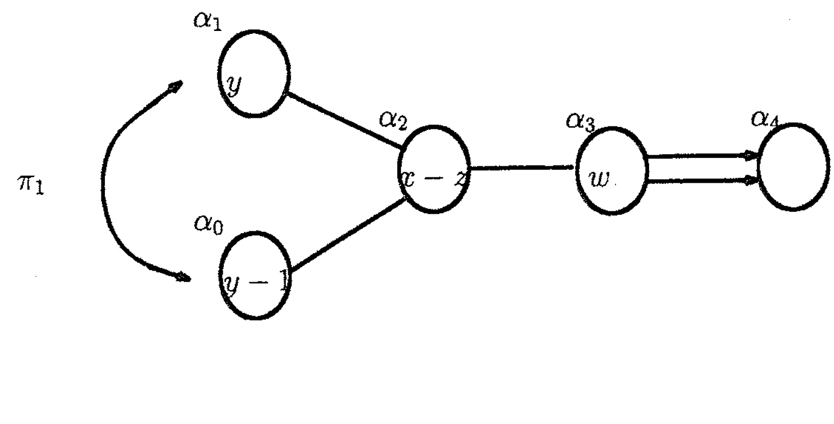
It is well-known that the fifth Painlevé equation P_V has a confluence to the third Painlevé equation P_{III}, where two accessible singularities come together into a single singularity. This suggests the possibility that there exists a procedure for searching for fourth-order versions of Painlevé III, by using Takano's description of the confluence process; (see [16],[17]) from P_V to P_{III} for the coordinate systems (x, y) and (z, w), respectively. In this vein, the goal of this work is to find a fourth-order version of the Painlevé III equation with symmetry under the group which degenerates from the affine Weyl group of type $D_5^{(1)}$ by the coupling confluence process. In this paper, we also present a 4-parameter family of algebraic ordinary differential equations that can be considered as coupled Painlevé III systems in dimension four, and which is given as follows:

$$
(9) \quad \begin{cases} \dfrac{dx}{dt} = \dfrac{2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3 z + 2z^2 w}{t} + 1 \\ \dfrac{dy}{dt} = \dfrac{-2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1}{t} \\ \dfrac{dz}{dt} = \dfrac{2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2}{t} + 1 \\ \dfrac{dw}{dt} = \dfrac{-2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3 y - 4yzw + \alpha_3}{t}. \end{cases}
$$

Here x, y, z and w denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 are complex parameters satisfying the following relation:

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = 1.$$

Theorem 5.4. *The system (7) is invariant under the transformations $s_0, s_1, .., s_4, \pi_1, \pi_2$ defined as follows: with the notation $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*



$$s_0 : (*) \to (x + \frac{\alpha_0}{y - 1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4),$$

$$s_1:(*)\to(x+\frac{\alpha_1}{y},y,z,w,t;\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4),$$

$$s_2:(*)\to(x,y-\frac{\alpha_2}{x-z},z,w+\frac{\alpha_2}{x-z},t;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4),$$

$$s_3:(*)\to(x,y,z+\frac{\alpha_3}{w},w,t;\alpha_0+,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3),$$

$$s_4:(*)\to(x,y,z,w-\frac{2\alpha_4}{z}+\frac{t}{z^2},-t;\alpha_0,\alpha_1,\alpha_2,\alpha_3+2\alpha_4,-\alpha_4),$$

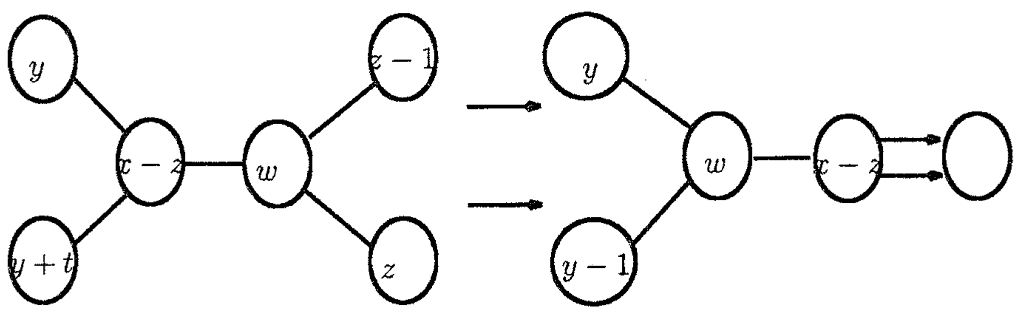
$$\pi_1:(*)\to(-x,1-y,-z,-w,-t;\alpha_1,\alpha_0,\alpha_2,\alpha_3,\alpha_4),$$

$$\pi_2:(*)\to(\frac{t}{z},-\frac{z}{t}(zw+\alpha_3),\frac{t}{x},-\frac{x}{t}(xy+\alpha_1),t;2\alpha_4+\alpha_3,\alpha_3,\alpha_2,(\alpha_0-\alpha_1)/2,\alpha_1).$$

Theorem 5.5. *The transformations described in Theorem* 5.4 *define a representation of the affine Weyl group of type $B_4^{(1)}$, that is, they satisfy the following relations:* $s_0{}^2=s_1{}^2=s_2{}^2=s_3{}^2=s_4{}^2=(\pi_1{}^2)=(\pi_2{}^2)=1,\ (s_0s_1)^2=(s_0s_3)^2=(s_0s_4)^2=(s_1s_3)^2=(s_1s_4)^2=(s_2s_4)^2=1,\ (s_0s_2)^3=(s_1s_2)^3=(s_2s_3)^3=1,\ (s_3s_4)^4=1,\ \pi_1s_0=s_1\pi_1,\ \pi_1s_1=s_0\pi_1,\ \pi_1s_2=s_2\pi_1,\ \pi_1s_3=s_3\pi_1,\ \pi_1s_4=s_4\pi_1.$

Our differential system is equivalent to a Hamiltonian system. The Hamiltonian H is given as follows:

$$H=\frac{x^2y(y-1)+x\{(1-2\alpha_2-2\alpha_3-2\alpha_4)y-\alpha_1\}+ty}{t}+\frac{z^2w(w-1)+z\{(1-2\alpha_4)w-\alpha_3\}+tw}{t}+\frac{2yz(zw+\alpha_3)}{t}. \tag{10}$$



Dynkin diagram of type $D_5{}^{(1)}$ Dynkin diagram of type $B_4{}^{(1)}$

Theorem 5.6. *For the system* (7) *of type $D_5^{(1)}$, we make the change of parameters and variables*

$$\alpha_0=A_0,\ \alpha_1=A_1,\ \alpha_2=A_2,\ \alpha_3=A_3,\ \alpha_4=2A_4-\frac{1}{\varepsilon},\ \alpha_5=\frac{1}{\varepsilon},$$

$$\beta_2 = A_4,\ \beta_3 = 2A_3 - \varepsilon^{-1},\ t = -\varepsilon T,\ x = 1 + \frac{X}{\varepsilon T},\ y = \varepsilon TY,\ z = 1 + \frac{Z}{\varepsilon T},\ w = \varepsilon TW,$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, t, x, y, z, w$ *to* $A_0, A_1, A_2, A_3, A_4, \varepsilon, T, X, Y, Z, W$. *Then the system* (7) *can also be written in the new variables* T, X, Y, Z, W *and parameters* $A_0, A_1, A_2, A_3, A_4, \varepsilon$ *as a Hamiltonian system. This new system tends to the system* (9) *of type* $B_4^{(1)}$ *as* $\varepsilon \to 0$.

By the following theorem, we show how the degeneration process in Theorem 5.6 works on the Bäcklund transformation group $W(D_5^{(1)}) =< s_0, s_1, .., s_5 >$ described in Theorem 5.1.

Theorem 5.7. *For the degeneration process in Theorem* 5.6, *we can choose a subgroup* $W_{D_5^{(1)} \to B_4^{(1)}}$ *of the Bäcklund transformation group* $W(D_5^{(1)})$ *so that* $W_{D_5^{(1)} \to B_4^{(1)}}$ *converges to* $W(B_4^{(1)})$ *as* $\varepsilon \to 0$.

6. Proof of Theorem 5.3

As is well-known, the degeneration from P_V to P_{IV}; (see [16]) is given by

$$\alpha_0 = A_0 + \frac{1}{2}\varepsilon^{-2},\ \alpha_1 = A_1,\ \alpha_2 = A_2,\ \alpha_3 = -\frac{1}{2}\varepsilon^{-2},$$

$$t = \frac{1}{2}\varepsilon^{-2}(1 + 2\varepsilon T),\ x = -\frac{\varepsilon X}{1 - \varepsilon X},\ y = -\varepsilon^{-1}(1 - \varepsilon X)[Y - \varepsilon(A_1 + XY)],$$

As the fourth-order analogue of the above confluence process, we consider the following coupling confluence process from the system (7) by taking the above process for each coordinate system (x, y) and (z, w) in (7), respectively. If we take the following coupling confluence process $P_V \to P_{IV}$ for each coordinate system (x, y) and (z, w) in (7)

$$\alpha_0 = A_0 - A_2 - A_3 + \frac{1}{2}\varepsilon^{-2},\ \alpha_1 = A_1,\ \alpha_2 = A_2,\ \alpha_3 = A_3,\ \alpha_4 = -\frac{1}{2}\varepsilon^{-2},\ \alpha_5 = A_4,$$

$$t = \frac{1}{2}\varepsilon^{-2}(1 + 2\varepsilon T),\ x = -\frac{\varepsilon X}{1 - \varepsilon X},\ y = -\varepsilon^{-1}(1 - \varepsilon X)[Y - \varepsilon(A_1 + XY)],$$

$$z = -\frac{\varepsilon Z}{1 - \varepsilon Z},\ w = -\varepsilon^{-1}(1 - \varepsilon Z)[W - \varepsilon(A_3 + XY)],$$

and take the limit $\varepsilon \to 0$, then we can obtain the system of type $A_4^{(1)}$, which is given as follows:

$$(11)\quad \begin{cases} \dfrac{dx}{dt} = -x^2 + 4xy + 4zw - 2tx - 2A_2 - 2A_4 \\ \dfrac{dy}{dt} = -2y^2 + 2xy + 2ty + A_1 \\ \dfrac{dz}{dt} = -z^2 + 4zw + 4yz - 2tz - 2A_4 \\ \dfrac{dw}{dt} = -2w^2 + 2zw - 4yw + 2tw + A_3. \end{cases}$$

Remark 6.1. *The system* (11) *is invariant under the transformations* $s_0, s_1, .., s_4$ *defined as follows: with the notation* $(*) := (x, y, z, w, t; A_0, A_1, A_2, A_3, A_4)$,

$s_0 : (*) \rightarrow (x - \frac{2A_0}{x-2y-2w+2t}, y - \frac{A_0}{x-2y-2w+2t}, z - \frac{2A_0}{x-2y-2w+2t}, w, t; -A_0, A_1+A_0, A_2, A_3, A_4+A_0)$,
$s_1 : (*) \rightarrow (x+\frac{A_1}{y}, y, z, w, t; A_0+A_1, -A_1, A_2+A_1, A_3, A_4)$,
$s_2 : (*) \rightarrow (x, y-\frac{A_2}{x-z}, z, w+\frac{A_2}{x-z}, t; A_0, A_1+A_2, -A_2, A_3+A_2, A_4)$,
$s_3 : (*) \rightarrow (x, y, z+\frac{A_3}{w}, w, t; A_0, A_1, A_2+A_3, -A_3, A_4+A_3)$,
$s_4 : (*) \rightarrow (x, y, z, w-\frac{A_4}{z}, t; A_0+A_4, A_1, A_2, A_3+A_4, -A_4)$.

These transformations are generators of the affine Weyl group $< s_0, s_1, s_2, s_3, s_4 >$ *of type* $A_4^{(1)}$.

7. Proof of Theorem 5.7

The degeneration process from the system (7) to the system (9) in Theorem 5.6 is given by

$$\alpha_0 = A_0,\ \alpha_1 = A_1,\ \alpha_2 = A_2,\ \alpha_3 = A_3,\ \alpha_4 = 2A_4 - \frac{1}{\varepsilon},\ \alpha_5 = \frac{1}{\varepsilon},$$

$$\beta_2 = A_4,\ \beta_3 = 2A_3 - \varepsilon^{-1},\ t = -\varepsilon T,\ x = 1 + \frac{X}{\varepsilon T},\ y = \varepsilon TY,\ z = 1 + \frac{Z}{\varepsilon T},\ w = \varepsilon TW,$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, t, x, y, z, w$ to $A_0, A_1, A_2, A_3, A_4, \varepsilon, T, X, Y, Z, W$. Notice that $A_0 + A_1 + 2A_2 + 2A_3 + 2A_4 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1$ and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose S_i, $i = 0, 1, 2, 3, 4$ as

$$S_0 := s_0,\ S_1 := s_1,\ S_2 := s_2,\ S_3 := s_3,\ S_4 := s_4 s_5 = s_5 s_4$$

which are reflections of

$$A_0 = \alpha_0,\ A_1 = \alpha_1,\ A_2 = \alpha_2,\ A_3 = \alpha_3,\ A_4 = \frac{\alpha_4 + \alpha_5}{2} \quad \text{respectively.}$$

By using the notation $(*) := (A_0, A_1, A_2, A_3, A_4, \varepsilon)$, we can easily check

$S_0(*) = (-A_0, A_1, A_2 + A_0, A_3, A_4, \varepsilon)$,
$S_1(*) = (A_0, -A_1, A_2 + A_1, A_3, A_4, \varepsilon)$,
$S_2(*) = (A_0 + A_2, A_1 + A_2, -A_2, A_3 + A_2, A_4, \varepsilon)$,
$S_3(*) = (A_0, A_1, A_2 + A_3, -A_3, A_4 + A_3, \frac{\varepsilon}{1+\varepsilon A_3})$,
$S_4(*) = (A_0, A_1, A_2, A_3 + 2A_4, -A_4, -\varepsilon)$.

By the above relation, we will see that the group $< S_0, S_1, S_2, S_3, S_4 >$ can be considered to be an affine Weyl group of the affine Lie algebra of type $B_4^{(1)}$ with respect to simple roots A_0, A_1, A_2, A_3, A_4.

Now we investigate how the generators of $< S_0, S_1, S_2, S_3, S_4 >$ act on T, X, Y, Z and W. By using the notation $(**) := (X, Y, Z, W, T)$, we can verify

$$
\begin{aligned}
S_0(**) &= (X + \tfrac{A_0}{Y-1}, Y, Z, W, T),\\
S_1(**) &= (X + \tfrac{A_1}{Y}, Y, Z, W, T),\\
S_2(**) &= (X, Y - \tfrac{A_2}{X-Z}, Z, W + \tfrac{A_2}{X-Z}, T),\\
S_3(**) &= (X, Y, Z + \tfrac{A_3}{W}, W, T(1 + \varepsilon A_3)),\\
S_4(**) &= (X, Y, Z, \tfrac{T+\varepsilon TZW+Z^2W}{Z(\varepsilon T+Z)} - \tfrac{2A_4}{Z}, -T).
\end{aligned}
$$

The proof of Theorem 5.7 has thus been completed.

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