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非線形固有値問題の解の漸近挙動

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1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$-u''(t)+f(u(t)) = \lambda u(t), \quad t\in I:=(0,1), \tag{1.1}$$

$$u(t) > 0, \quad t\in I, \tag{1.2}$$

$$u(0) = u(1)=0, \tag{1.3}$$

where $\lambda>0$ is an eigenvalue parameter. We assume that $f(u)$ satisfies the following conditions (A.1)-(A.3).

(**A.1**) $f(u)$ is a function of C^1 for $u\geq 0$ satisfying $f(0)=f'(0)=0$.

(**A.2**) $g(u):=f(u)/u$ is strictly increasing for $u\geq 0$ $(g(0):=0)$.

(**A.3**) $g(u)\rightarrow\infty$ as $u\rightarrow\infty$.

The typical examples of $f(u)$ which satisfy (A.1)-(A.3) are

$$f(u)=u^{p}, \quad f(u)=u^{p+2}/(1+u^{2}), \quad f(u)=u^{p}e^{u} \quad (p>1).$$

We know from [1] that for each given $\alpha>0$, there exists a unique solution $(\lambda, u)=(\lambda(\alpha), u_{\alpha})\in \mathbf{R}_{+}\times C^{2}(\bar{I})$ with $\|u_{\alpha}\|_{2}=\alpha$. Furthermore, The set $\{(\lambda(\alpha), u_{\alpha}):\alpha>0\}$ gives all solutions and is an unbounded curve of class C^{1} in $\mathbf{R}_{+}\times L^{2}(I)$ emanating from $(\pi^{2},0)$.

The purpose here is to study precisely the global structure of this bifurcation branch in $\mathbf{R}_+ \times L^2(I)$. To do this, we establish several types of precise asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to \infty$ under some additional conditions on f.

We know from [1] that for $t \in \bar{I}$,

$$g^{-1}(\lambda - \pi^2)\sin \pi t \le u_\lambda(t) \le g^{-1}(\lambda). \tag{1.4}$$

In particular, put $t = \frac{1}{2}$. Then as $\lambda \to \infty$

$$g^{-1}(\lambda - \pi^2) \le \|u_\lambda\|_\infty \le g^{-1}(\lambda). \tag{1.5}$$

Therefore, for $\lambda \gg 1$,

$$\lambda = g(\|u_\lambda\|_\infty) + O(1). \tag{1.6}$$

For instance, let $f(u) = u^p$. Then since $g(u) = f(u)/u = u^{p-1}$, for $\lambda \gg 1$

$$\lambda = \|u_\lambda\|_\infty^{p-1} + O(1). \tag{1.7}$$

Furthermore, we know that as $\lambda \to \infty$

$$\frac{u_\lambda(t)}{g^{-1}(\lambda)} \to 1 \tag{1.8}$$

uniformly on any compact set in I. Then we obtain

$$\alpha = \|u_\alpha\|_2 = \left(\int_I g^{-1}(\lambda)^2 dt\right)^{1/2} (1 + o(1)) = g^{-1}(\lambda)(1 + o(1)).$$

This implies that, in many cases,

$$\lambda(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.9}$$

For instance, let $f(u) = u^p$. Then for $\alpha \gg 1$,

$$\lambda(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}). \tag{1.10}$$

This asymptotic formula has been improved as follows.

Theorem 1 [6]. *Let $f(u) = u^p$ $(p > 1)$. Further, let an arbitrary $n \in \mathbf{N}_0$ be fixed. Then as $\alpha \to \infty$*

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2}\alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}),$$

where

$$C_1 = (p+3)\int_I \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}ds$$

and $a_k(p)$ *(deg* $a_k(p) \leq k+1$*) is the polynomial determined inductively by* $a_0, a_1, \cdots, a_{k-1}$.

For instance, we have

$$a_0(p) = 1, \ a_1(p) = \frac{(5-p)(9-p)}{24}, \ a_2(p) = \frac{(3-p)(5-p)(7-p)}{24}.$$

We also obtain the information about the slope of the boundary layer of u_α for $\alpha \gg 1$.

Theorem 2 [6]. *Let* $f(u) = u^p$ $(p > 1)$*. Further, let an arbitrary* $n \in \mathbf{N}_0$ *be fixed. Then as* $\alpha \to \infty$

$$\begin{aligned} u'_\alpha(0)^2 &= u'_\alpha(1)^2 = \frac{p-1}{p+1}\alpha^{p+1} + C_1\alpha^{(p+3)/2} + \sum_{k=0}^{n} \frac{2A_k(p)}{(p-1)^{k+1}} C_1^{k+2}\alpha^{2+k(1-p)/2} \\ &\quad + o(\alpha^{2+n(1-p)/2}), \end{aligned}$$

where $A_k(p)$ *(deg* $A_k(p) \leq k+1$*) is the polynomial determined by* $a_0, a_1, \cdots, a_{k-1}$.

For instance,

$$A_0(p) = 1, \ A_1(p) = \frac{(9-p)(13-p)}{48}, \ A_2(p) = \frac{(5-p)(7-p)(9-p)}{48}.$$

So it is natural to consider the following problem. Consider $f(u)$ which satisfies (A.1)–(A.3). Then is the following formula valid or not for $\alpha \gg 1$?

$$\lambda(\alpha) = g(\alpha) + B_1 g(\alpha)^{1/2} + \cdots, \tag{1.11}$$

where B_1 is a constant. To treat this problem, we assume additional conditions. Let $f(u) = u^p h(u)$ $(p > 1)$. Assume that $h(u)$ is C^2 function for $u \geq 0$. Besides, $h(u)$ satisfies the following conditions (B.1)–(B.4).

(**B**.1) As $u \to \infty$

$$\frac{uh'(u)}{h(u)} \to 0. \tag{1.12}$$

Furthermore, there exists a constant $C_0 \geq 0$ such that as $u \to \infty$

$$uh'(u) \to C_0. \tag{1.13}$$

(**B.2**) There exist constants $C > 0$ and $\delta > 0$ such that for $u \gg 1$

$$|h'(u) + uh''(u)| \le Cu^{-(1+\delta)}. \tag{1.14}$$

(**B.3**) For $0 \le a \le 1$ and $u \gg 1$

$$\frac{h(au)}{h(u)} \le C. \tag{1.15}$$

Furthermore, for a fixed $0 < a \le 1$, as $u \to \infty$

$$\frac{h(au)}{h(u)} \to 1. \tag{1.16}$$

(**B.4**) (a) $u^{p+1}|h'(u)|$ is non-decreasing for $u \ge 0$ or,

(b) $u^{p+1}|h'(u)|$ is bounded for $u \ge 0$.

The typical examples of h are: (i) $h(u) \equiv 1$, (ii) $h(u) = \log(u+1)$, (iii) $h(u) = u^2/(1+u^2)$.

Theorem 3 [7]. *Let $p > 1$ be fixed. Assume that $f(u) := u^p h(u)$ satisfies (A.1)–(A.3) and (B.1)–(B.4). Then as $\alpha \to \infty$*

$$\lambda(\alpha) = \alpha^{p-1}h(\alpha) + \frac{1}{p+1}C_0\alpha^{p-1} + (p+3)C_1\alpha^{(p-1)/2}\sqrt{h(\alpha)}(1+o(1)).$$

Remark 4. (i) For $\alpha \gg 1$, by (B.2), we see that

$$C_0 = \alpha h'(\alpha)(1+o(1)). \tag{1.17}$$

Therefore, as $\alpha \to \infty$

$$\frac{\alpha^{p-1}C_0}{\alpha^{p-1}h(\alpha)} = \frac{\alpha^p h'(\alpha)(1+o(1))}{\alpha^{p-1}h(\alpha)} \to 0. \tag{1.18}$$

So we find that the leading term of $\lambda(\alpha)$ in Theorem 3 is $\alpha^{p-1}h(\alpha)$.

(ii) If $C_0 \ne 0$, then the second term of $\lambda(\alpha)$ is $C_0\alpha^{p-1}/(p+1)$. Therefore, our conjecture (1.11) is valid if and only if $C_0 = 0$. We note that, if $h(u) = \log(u+1)$, then $C_0 = 1$. Further, if $h(u) = u^2/(1+u^2)$, then $C_0 = 0$.

Now we consider the case where $f(u) = u^p e^u$ $(p > 1)$.

Theorem 5 [8]. *Assume that $f(u) = u^p e^u$ $(p > 1)$ in (1.1). Then as $\alpha \to \infty$*

$$\lambda(\alpha) = \alpha^{p-1}e^\alpha + \frac{\pi}{4}\alpha^{(p+1)/2}e^{\alpha/2}(1+o(1)).$$

2 Sketch of the proof of Theorem 3

We begin with notations and the fundamental properties of $\lambda(\alpha)$ and u_α. Let $F(u) := \int_0^u f(s)ds$. Let $\|\cdot\|_q$ $(1 \le q \le \infty)$ denote the usual L^q-norm. C denotes various positive constants independent of $\alpha \gg 1$. It is known by [1] that (1.1)–(1.3) has a unique solution u_α for a given $\alpha > 0$ and the mapping $\alpha \mapsto u_\alpha \in C^2(\bar{I})$ is C^1 for $\alpha > 0$. By (1.4) and (1.5), for $\alpha \gg 1$

$$\lambda(\alpha) = \alpha^{p-1}h(\alpha) + o(\alpha^{p-1}h(\alpha)), \tag{2.1}$$

$$u_\alpha(t) = \|u_\alpha\|_\infty(1 + o(1)) = \alpha(1 + o(1)), \quad t \in I. \tag{2.2}$$

We put

$$\lambda_1(\alpha) := \lambda(\alpha) - \alpha^{p-1}h(\alpha), \tag{2.3}$$

$$\gamma(\alpha) := \|u'_\alpha\|_2^2 + 2\int_I F(u_\alpha(t))dt. \tag{2.4}$$

To show Theorem 3, we find $\lambda_1(\alpha)$ when $\alpha \gg 1$. To do this, we define the second term $\gamma_1(\alpha)$ of $\gamma(\alpha)$, which plays important roles, as follows.

$$\gamma_1(\alpha) := \gamma(\alpha) - \frac{2}{p+1}\alpha^{p+1}h(\alpha). \tag{2.5}$$

The rough idea of the proof is as follows.

(i) We obtain three estimates in Lemmas 2.1, 2.3 and 2.4.

(ii) We establish the relationship between $\lambda_1(\alpha)$ and $\gamma_1(\alpha)$ in Lemma 2.2.

(iii) We derive the first order differential equation for $\gamma_1(\alpha)$ by using (i) and (ii). Then by solving it, we obtain the asymptotic formula for $\lambda_1(\alpha)$.

Lemma 2.1. $\|u'_\alpha\|_2^2 = 2C_1(1 + o(1))\alpha^{(p+3)/2}\sqrt{h(\alpha)}$ *for* $\alpha \gg 1$.

Lemma 2.2. *For* $\alpha > 0$

$$\frac{d\gamma_1(\alpha)}{d\alpha} = 2\alpha\lambda_1(\alpha) - \frac{2}{p+1}\alpha^{p+1}h'(\alpha). \tag{2.6}$$

Lemma 2.3. *For* $\alpha \gg 1$

$$\int_0^{\|u_\alpha\|_\infty} s^{p+1}h'(s)ds = \int_I \left(\int_0^{u_\alpha(t)} s^{p+1}h'(s)ds\right) dt + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right).$$

Lemma 2.4. *For* $\alpha \gg 1$

$$\int_0^{\|u_\alpha\|_\infty} s^{p+1}h'(s)ds = \int_0^{\alpha} s^{p+1}h'(s)ds + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right). \tag{2.7}$$

Proof of Theorem 3. By simple calculation, we have

$$\frac{2}{p+1}\lambda(\alpha)\alpha^2 - \gamma(\alpha) = -\frac{p-1}{p+1}\|u'_\alpha\|_2^2 + \frac{2}{p+1}\int_I\left(\int_0^{u_\alpha(t)} s^{p+1}h'(s)ds\right)dt.$$

By this, Lemmas 2.1, 2.3 and 2.4,

$$\begin{aligned}\frac{2}{p+1}\lambda_1(\alpha)\alpha^2 - \gamma_1(\alpha) &= -\frac{2(p-1)}{p+1}C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1+o(1)) \\ &\quad + \frac{2}{p+1}\int_0^{\alpha} s^{p+1}h'(s)ds.\end{aligned}$$

By integration by parts,

$$\int_0^{\alpha} s^{p+1}h'(s)ds = \frac{1}{p+1}\alpha^{p+2}h'(\alpha) - \frac{1}{p+1}R(\alpha),$$

where

$$R(\alpha) := \int_0^{\alpha} s^{p+1}(h'(s) + sh''(s))ds. \tag{2.8}$$

By this and Lemma 2.2,

$$\begin{aligned}\frac{1}{p+1}\alpha\gamma_1'(\alpha) - \gamma_1(\alpha) &= -\frac{2(p-1)}{p+1}C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1+o(1)) \\ &\quad - \frac{2}{(p+1)^2}R(\alpha).\end{aligned} \tag{2.9}$$

Now we put $\gamma_1(\alpha) = \eta(\alpha)\alpha^{p+1}$. Then for $\alpha \gg 1$, we obtain

$$\begin{aligned}\eta'(\alpha) &= -2(p-1)C_1\alpha^{-(p+1)/2}\sqrt{h(\alpha)}(1+o(1)) \\ &\quad - \frac{2}{p+1}R(\alpha)\alpha^{-(p+2)} \\ &:= \eta_1'(\alpha) + \eta_2'(\alpha),\end{aligned} \tag{2.10}$$

where

$$\eta_1(\alpha) = (1+o(1))\int_\alpha^\infty 2(p-1)C_1 s^{-(p+1)/2}\sqrt{h(s)}ds, \tag{2.11}$$

$$\eta_2(\alpha) = \frac{2}{p+1}\int_\alpha^\infty R(s)s^{-(p+2)}ds. \tag{2.12}$$

Then it is easy to show that for $\alpha \gg 1$

$$\eta_1(\alpha)\alpha^{p+1} = 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1+o(1)). \tag{2.13}$$

We next calculate $\eta_2(\alpha)$. By (B.2), we have

$$|R(\alpha)| \le \int_0^\alpha s^{p+1}|h'(s)+sh''(s)|ds \le C\int_0^\alpha s^{p-\delta}ds \le C\alpha^{p+1-\delta}. \tag{2.14}$$

By this, we easily see that $\eta_2(\alpha)$ is well defined. Then by integration by parts and simple calculation, we have

$$\begin{aligned} \eta_2(\alpha) &= \frac{2}{p+1}\int_\alpha^\infty R(s)s^{-(p+2)}ds \\ &= \frac{2}{p+1}\left[-\frac{1}{p+1}s^{-(p+1)}R(s)\right]_\alpha^\infty + \frac{2}{(p+1)^2}\int_\alpha^\infty (h'(s)+sh''(s))ds \\ &= \frac{2}{(p+1)^2}R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2}\int_\alpha^\infty (sh'(s))'ds \\ &= \frac{2}{(p+1)^2}R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2}(C_0-\alpha h'(\alpha)). \end{aligned} \tag{2.15}$$

Therefore,

$$\begin{aligned} \gamma_1(\alpha) &= (\eta_1(\alpha)+\eta_2(\alpha))\alpha^{p+1} \\ &= 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1+o(1)) \\ &\quad + \frac{2}{(p+1)^2}(R(\alpha)+C_0\alpha^{p+1}-\alpha^{p+2}h'(\alpha)). \end{aligned} \tag{2.16}$$

By this and Lemma 2.1, we obtain

$$\begin{aligned} \frac{2}{p+1}\lambda_1(\alpha)\alpha^2 &= \gamma_1(\alpha) - \frac{p-1}{p+1}\|u'_\alpha\|_2^2 + \frac{2}{p+1}\int_0^\alpha s^{p+1}h'(s)ds \\ &\quad + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right) \\ &= 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)} - \frac{2C_1(p-1)}{p+1}\alpha^{(p+3)/2}\sqrt{h(\alpha)} \\ &\quad + \frac{2}{(p+1)^2}C_0\alpha^{p+1} + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right). \end{aligned} \tag{2.17}$$

By this, we obtain

$$\begin{aligned} \lambda_1(\alpha) &= \frac{1}{p+1}C_0\alpha^{p-1} + (p+3)C_1\alpha^{(p-1)/2}\sqrt{h(\alpha)} \\ &\quad + o\left(\alpha^{(p-1)/2}\sqrt{h(\alpha)}\right). \end{aligned} \tag{2.18}$$

Thus the proof is complete. ∎

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