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Duck solutions in a four-dimensional dynamic economic model

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Abstract

We consider the existence of a duck solution in two economic models; one is the Goodwin's nonlinear business cycle model, and the other is a two-region business cycle model where each region is described as the Goodwin model and they are coupled by interregional trade. We show that there exist duck solutions in the two-region business cycle model, while the Goodwin model itself doesn't have a duck solution.

1 Introduction

A duck solution is a segment of a solution to a slow-fast system which first follows the attracting part of the slow manifold and then the repelling one. It was first found for the one parameter family of van der Pol equations and analyzed by making use of techniques from non-standard analysis. In [1], Benoît showed the existence of a duck solution to a slow-fast system in $\mathbb{R}^3$ when it has a pseudo singular saddle point. Tchizawa [5] considered a slow-fast system in $\mathbb{R}^4$ with a two-dimensional slow manifold and obtained the condition for the existence of a duck solution by reducing it to the system in $\mathbb{R}^2$.

In this paper, we consider the existence of a duck solution in two economic models. First we consider the business cycle model of Goodwin [3]. The equation in the Goodwin model can be transformed into a slow-fast system in $\mathbb{R}^2$ when we take the constant concerning a lag to be considerably small, but we can show that a duck solution doesn't exist in it. Tchizawa *et al* [6] considered the Goodwin-like business cycle model where the induced investment function is a cubic polynomial and provided an economic condition for the existence of a duck solution. Second, we present a two-region business cycle model where each region is described as the Goodwin model and they are coupled by interregional trade. Finally we show that there exist duck solutions in the two-region model without using cubic polynomials as the induced invest function.

2 Duck solution

2.1 Duck in $\mathbb{R}^2$

We review some results of a duck solution in $\mathbb{R}^2$ by following Zvonkin and Shubin [7]. Consider the following system of differential equations:

$$\begin{cases} \varepsilon\dot{x} = y - f(x), \\ \dot{y} = a - x, \end{cases} \tag{2.1}$$

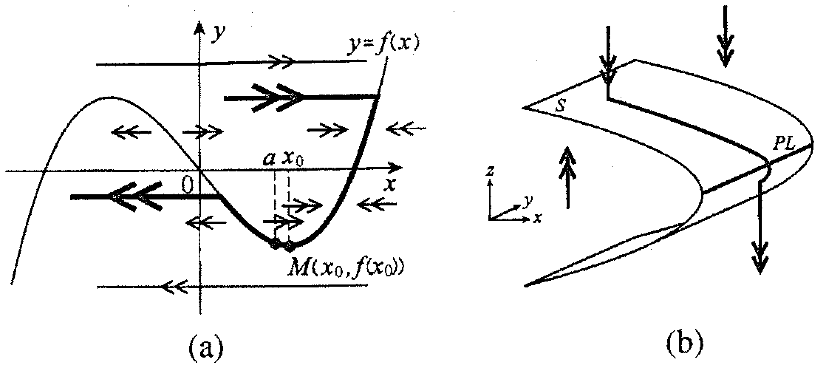

Figure 1 (a) The slow curve of (2.1). The bold line is an example of the solution to (2.1) in the case of $a \approx x_0$. The double arrows indicate rapid motion. The parts of the slow curve where $f(x)$ is increasing are attracting and those where $f(x)$ is decreasing are repelling. (b) The slow manifold S, the pli set PL, and an example solution of (2.2).

where f is defined in $\mathbb{R}^1$, a is a parameter, and ε is infinitesimally small. For system (2.1), the graph $y = f(x)$ is called the *slow curve*. A duck solution is a segment of a solution which first follows the attracting part of the slow manifold and then the repelling one (for the definition, see e.g. [7]). We consider the extremum point x_0 of the slow curve, which separates the attracting and repelling parts. We give a necessary condition for the existence of a duck solution close to the extremum point x_0.

Proposition 2.1 *If there exists a duck solution of system (2.1) close to the extremum point x_0, then $a \approx x_0$.*

The following theorem is due to Diener [2].

Theorem 2.2 *Suppose that f has a nondegenerate extremum point x_0, that is, $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Then there are the corresponding values of the parameter a satisfying Proposition 2.1 for which there exist duck solutions in system (2.1).*

2.2 Duck in $\mathbb{R}^3$

We describe some results of Benoît [1] by following Kakiuchi and Tchizawa [4]. Consider the following system of differential equations:

$$\begin{cases} \dot{x} = f(x,y,z), \\ \dot{y} = g(x,y,z), \\ \varepsilon\dot{z} = h(x,y,z), \end{cases} \tag{2.2}$$

where f, g, and h are defined in $\mathbb{R}^3$ and ε is infinitesimally small. We assume that system (2.2) satisfies the following conditions.

(A1) f and g are of class $\mathbb{C}^1$, and h is of class $\mathbb{C}^2$.

(A2) The slow manifold $S = \{(x,y,z) \in \mathbb{R}^3 | h(x,y,z) = 0\}$ is a two-dimensional differentiable manifold and the pli set $PL = \{(x,y,z) \in S | \partial h(x,y,z)/\partial z = 0\}$ is a one-dimensional differentiable manifold.

(A3) Either the value of f or that of g is nonzero at any point $p \in PL$.

We consider the following reduced system (see [4] for details):

$$\begin{cases} \dot{x} = -h_z(x,y,z)f(x,y,z), \\ \dot{y} = -h_z(x,y,z)g(x,y,z), \\ \dot{z} = h_x(x,y,z)f(x,y,z) + h_y(x,y,z)g(x,y,z), \end{cases} \tag{2.3}$$

where $h_\alpha(x,y,z) = \partial h(x,y,z)/\partial\alpha$ $(\alpha = x,y,z)$.

Definition 2.3 A singular point of (2.3) which lies on PL is called a *pseudo singular point.*

(A4) For any $(x,y,z) \in S$, either $h_x(x,y,z) \neq 0$ or $h_y(x,y,z) \neq 0$ holds.

Then the slow manifold S can be expressed like as $y = \varphi(x,z)$ in the neighborhood of PL and we obtain the following system, which restricts system (2.3) on S:

$$\begin{cases} \dot{x} = -h_z(x,\varphi(x,z),z)f(x,\varphi(x,z),z), \\ \dot{z} = h_x(x,\varphi(x,z),z)f(x,\varphi(x,z),z) + h_y(x,\varphi(x,z),z)g(x,\varphi(x,z),z). \end{cases} \tag{2.4}$$

(A5) All singular points of (2.4) are nondegenerate, that is, the linearization of (2.4) at a singular point has two nonzero eigenvalues. Note that all pseudo singular points are the singular points of (2.4).

Definition 2.4 Let λ_1, λ_2 be two eigenvalues of the linearization of (2.4) at a pseudo singular point. The pseudo singular point with real eigenvalues is called a *pseudo singular saddle point* if $\lambda_1\lambda_2 < 0$.

By means of methods in non-standard analysis, Benoît proved the following theorem.

Theorem 2.5 *If system (2.2) has a pseudo singular saddle point, then there exists a duck solution in system (2.2).*

2.3 Duck in $\mathbb{R}^4$

We consider a slow-fast system in $\mathbb{R}^4$ and reduce it to the system in $\mathbb{R}^2$ by following the method in Tchizawa [5], and provide the condition for the existence of a duck solution. Consider the following system of differential equations:

$$\begin{cases} \varepsilon\dot{x}_1 = h_1(x_1,x_2,y_1,y_2), \\ \varepsilon\dot{x}_2 = h_2(x_1,x_2,y_1,y_2), \\ \dot{y}_1 = f_1(x_1,x_2,y_1,y_2), \\ \dot{y}_2 = f_2(x_1,x_2,y_1,y_2), \end{cases} \tag{2.5}$$

where f_1, f_2, h_1, and h_2 are defined in $\mathbb{R}^4$ and ε is infinitesimally small. We put $x = (x_1,x_2)^\top$, $y = (y_1,y_2)^\top$, $f = (f_1,f_2)^\top$, and $h = (h_1,h_2)^\top$. We assume that system (2.5) satisfies the following conditions.

(B1) f is of class $\mathbb{C}^1$ and h is of class $\mathbb{C}^2$.

(B2) The slow manifold $S = \{(x, y) \in \mathbb{R}^4 | h(x, y) = 0\}$ is a two-dimensional differentiable manifold and the generalized pli set $GPL = \{(x, y) \in S | \det(\frac{\partial h}{\partial x}(x, y)) = 0\}$ is a one-dimensional differentiable manifold.

(B3) Either the value of f_1 or that of f_2 is nonzero at any point $p \in GPL$.

(B4) $\mathrm{rank}(\frac{\partial h}{\partial y}(x, y)) = 2$.

The implicit function theorem ensures that y is uniquely described like as $y = \varphi(x)$. On the set S, differentiating both sides of $h(x, \varphi(x)) = 0$ with respect to x,

$$\frac{\partial h}{\partial x}(x, \varphi(x)) + \frac{\partial h}{\partial y}(x, \varphi(x)) D\varphi(x) = 0, \tag{2.6}$$

where $D\varphi(x)$ is the Jacobian of $\varphi(x)$. On the other hand, $\dot{y} = D\varphi(x)\dot{x}$ holds, where $\dot{x} = (\dot{x}_1, \dot{x}_2)^{\mathrm{T}}$ and $\dot{y} = (\dot{y}_1, \dot{y}_2)^{\mathrm{T}}$. Then we can reduce the system to the following system:

$$D\varphi(x)\dot{x} = f(x, \varphi(x)). \tag{2.7}$$

Using (2.6), (2.7) is described by

$$\dot{x} = -\left[\frac{\partial h}{\partial x}(x, \varphi(x))\right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x)) f(x, \varphi(x)).$$

Moreover we consider the following time scaled reduced system.

$$\dot{x} = -\det\left(\frac{\partial h}{\partial x}(x, \varphi(x))\right)\left[\frac{\partial h}{\partial x}(x, \varphi(x))\right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x)) f(x, \varphi(x)). \tag{2.8}$$

Definition 2.6 A singular point of (2.8) is called a *generalized pseudo singular point.*

(B5) All singular points of (2.8) are nondegenerate.

Definition 2.7 Let λ_1, λ_2 be two eigenvalues of the linearization of (2.8) at a generalized pseudo singular point. The pseudo singular point with real eigenvalues is called a *pseudo singular saddle point* if $\lambda_1\lambda_2 < 0$.

By applying Benoît's criterion, Tchizawa [5] finally obtained the following proposition.

Proposition 2.8 *If system (2.5) has a pseudo singular saddle point, then there exists a duck solution in system (2.5).*

3 Economic models

3.1 Goodwin's business cycle model

The Goodwin model consists of a national income identity $y(t)$, a consumption function $c(t)$, and an investment function $\dot{k}(t)$:

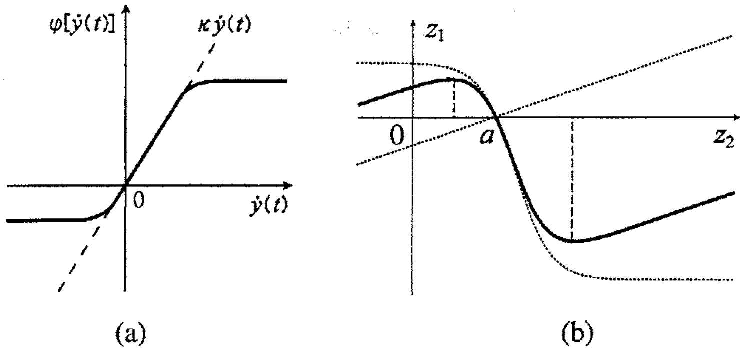

Figure 2 (a) The induced investment function. κ is the acceleration coefficient. (b) The slow curve of (3.3). The dotted line is the first term on the right hand side of (3.4) and the dotted curve is the second one.

$$\begin{aligned} y(t) &= c(t) + \dot{k}(t) - \varepsilon\dot{y}(t), \\ c(t) &= \alpha y(t) + \beta(t), \\ \dot{k}(t+\theta) &= \varphi[\dot{y}(t)] + l(t+\theta), \end{aligned} \tag{3.1}$$

where $k(t)$ denotes capital stock, ε (> 0) a constant expressing a lag in the multiplier process, α ($0 < \alpha < 1$) the marginal propensity to consume, $\beta(t)$ an autonomous consumption, $\varphi[\dot{y}(t)]$ the induced investment function as shown in Figure 2(a), $l(t)$ is the autonomous investment, and θ the lag between the decision to invest and the corresponding outlays, respectively. Goodwin finally obtained the following second-order differential equation (see [3] for details):

$$\varepsilon\theta\ddot{z} + [\varepsilon + (1-\alpha)\theta]\dot{z} - \varphi(\dot{z}) + (1-\alpha)z = 0, \tag{3.2}$$

where z is the deviations from the equilibrium income. Using graphical integration method, Goodwin showed that (3.2) has a unique limit cycle.

By setting new variables, $z_1 = z$, $z_2 = -\dot{z}_1 + a$, (3.2) becomes the following system:

$$\begin{cases} \dot{z}_1 = -z_2 + a, \\ \varepsilon\dot{z}_2 = \dfrac{1-\alpha}{\theta}z_1 - \left(\dfrac{\varepsilon}{\theta} + 1 - \alpha\right)z_2 - \dfrac{1}{\theta}\varphi(-z_2 + a) + a\left(\dfrac{\varepsilon}{\theta} + 1 - \alpha\right), \end{cases} \tag{3.3}$$

which is the type of the slow-fast system (2.1). The slow curve of (3.3) is

$$z_1 = \left(\frac{\varepsilon}{1-\alpha} + \theta\right)(z_2 - a) + \frac{1}{1-\alpha}\varphi(-z_2 + a) \tag{3.4}$$

and is drawn in Figure 2(b). As the figure shows, there is a distance between the value of the parameter a and each extremum point of the slow curve, so that the condition of Proposition 2.1 is not satisfied as far as the induced investment function is the type of the function shown in Figure 2(a). Therefore there does not exist a duck solution in the Goodwin model. Tchizawa *et al* [6] considered the Goodwin-like business cycle model and showed that there exists the condition between the economic parameters under which a duck solution occurs when we use a cubic polynomial as the induced investment function.

3.2 Two-region business cycle model

Now we present a two-region business cycle model which is a natural extension of the Goodwin model obtained by introducing interregional trade. More precisely, the model consists of the following equations:

$$
\begin{aligned}
&y_i(t) = c_i(t) + \dot{k}_i(t) - \varepsilon_i \dot{y}_i(t) + e_i(t) - m_i(t), \\
&c_i(t) = \alpha_i y_i(t) + \beta_i(t), \\
&\dot{k}_i(t+\theta_i) = \varphi_i[\dot{y}_i(t)] + l_i(t+\theta_i),
\end{aligned}
\tag{3.5}
$$

where the subscript i $(i = 1, 2)$ denotes the region i, $e_i(t)$ the export of the region i, and $m_i(t)$ the import of the region i, respectively. For simplicity, we put $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\theta_1 = \theta_2 = \theta$. As to the export and import terms, we put

$$
e_i(t+\theta) = m_j(t+\theta) = a_i \psi_i(\dot{y}_1(t), \dot{y}_2(t)),
$$

where the subscript j $(j = 1, 2)$ denotes the region different from the region i, $a_i \geq 0$ is a constant, and ψ_i is a sufficiently smooth function.

By the same transformation as that in the Goodwin model, we have the following second-order equation:

$$
\varepsilon\theta \ddot{z}_i + [\varepsilon + (1-\alpha_i)\theta]\dot{z}_i - \varphi_i(\dot{z}_i) - a_i\psi_i(\dot{z}_1, \dot{z}_2) + a_j\psi_j(\dot{z}_1, \dot{z}_2) + (1-\alpha_i)z_i = 0.
$$

Setting new variables, $x_i = \dot{z}_i$ $(i = 1, 2)$, we obtain the following system:

$$
\begin{cases}
\varepsilon \dot{x}_1 = -\dfrac{1-\alpha_1}{\theta} z_1 - \left(\dfrac{\varepsilon}{\theta} + 1 - \alpha_1\right) x_1 + \dfrac{1}{\theta}\varphi_1(x_1) + \dfrac{a_1}{\theta}\psi_1(x) - \dfrac{a_2}{\theta}\psi_2(x) \equiv h_1, \\
\varepsilon \dot{x}_2 = -\dfrac{1-\alpha_2}{\theta} z_2 - \left(\dfrac{\varepsilon}{\theta} + 1 - \alpha_2\right) x_2 + \dfrac{1}{\theta}\varphi_2(x_2) + \dfrac{a_2}{\theta}\psi_2(x) - \dfrac{a_1}{\theta}\psi_1(x) \equiv h_2, \\
\dot{z}_1 = x_1, \\
\dot{z}_2 = x_2,
\end{cases}
\tag{3.6}
$$

where $x = (x_1, x_2)^{\mathsf{T}}$. System (3.6) is the specific case of system (2.8), so we can apply Tchizawa's result to (3.6) in order to investigate the existence of a duck solution.

4 Duck solutions in the two-region model

In this section, we examine whether (3.6) has a duck solution. Because we have

$$
\begin{aligned}
&\frac{\partial h_1}{\partial x_1} = -\left(\frac{\varepsilon}{\theta} + 1 - \alpha_1\right) + \frac{1}{\theta}\frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1} + \frac{a_1}{\theta}\frac{\partial \psi_1(x)}{\partial x_1} - \frac{a_2}{\theta}\frac{\partial \psi_2(x)}{\partial x_1}, \\
&\frac{\partial h_1}{\partial x_2} = \frac{a_1}{\theta}\frac{\partial \psi_1(x)}{\partial x_2} - \frac{a_2}{\theta}\frac{\partial \psi_2(x)}{\partial x_2}, \quad \frac{\partial h_2}{\partial x_1} = \frac{a_2}{\theta}\frac{\partial \psi_2(x)}{\partial x_1} - \frac{a_1}{\theta}\frac{\partial \psi_1(x)}{\partial x_1}, \\
&\frac{\partial h_2}{\partial x_2} = -\left(\frac{\varepsilon}{\theta} + 1 - \alpha_2\right) + \frac{1}{\theta}\frac{\mathrm{d}\varphi_2(x_2)}{\mathrm{d}x_2} + \frac{a_2}{\theta}\frac{\partial \psi_2(x)}{\partial x_2} - \frac{a_1}{\theta}\frac{\partial \psi_1(x)}{\partial x_2},
\end{aligned}
$$

and

$$
\frac{\partial h}{\partial z}(x, \varphi(x)) = \begin{bmatrix} -\dfrac{1-\alpha_1}{\theta} & 0 \\ 0 & -\dfrac{1-\alpha_2}{\theta} \end{bmatrix},
$$

where $h = (h_1, h_2)^\top$, $z = (z_1, z_2)^\top$, and $z = \varphi(x)$, we obtain the following time scaled reduced system projected onto $\mathbb{R}^2$, which corresponds to (2.8):

$$\begin{cases} \dot{x}_1 = -\dfrac{(1-\alpha_1)\left(\frac{\varepsilon}{\theta}+1-\alpha_2\right)}{\theta}x_1 + \dfrac{1-\alpha_1}{\theta^2}x_1\dfrac{\mathrm{d}\varphi_2(x_2)}{\mathrm{d}x_2} \\ \qquad -[(1-\alpha_1)x_1+(1-\alpha_2)x_2]\left[\dfrac{a_1}{\theta^2}\dfrac{\partial\psi_1(x)}{\partial x_2} - \dfrac{a_2}{\theta^2}\dfrac{\partial\psi_2(x)}{\partial x_2}\right] \equiv f_1, \\ \dot{x}_2 = -\dfrac{\left(\frac{\varepsilon}{\theta}+1-\alpha_1\right)(1-\alpha_2)}{\theta}x_2 + \dfrac{1-\alpha_2}{\theta^2}x_2\dfrac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1} \\ \qquad -[(1-\alpha_1)x_1+(1-\alpha_2)x_2]\left[-\dfrac{a_1}{\theta^2}\dfrac{\partial\psi_1(x)}{\partial x_1} + \dfrac{a_2}{\theta^2}\dfrac{\partial\psi_2(x)}{\partial x_1}\right] \equiv f_2. \end{cases} \tag{4.1}$$

In what follows, we put $\alpha_1 = \alpha_2 = \alpha$ and $\varphi_i(x_i) = \tanh(x_i)$ $(i = 1, 2)$, and assume that

$$\frac{\partial\psi_i(x)}{\partial x_i} = \frac{\partial\psi_i(x)}{\partial x_j} \quad (i, j = 1, 2, j \neq i) \tag{4.2}$$

holds. Note that the hyperbolic tangent is a typical example of the function as shown in Figure 2(a). Then the generalized pseudo singular points of (3.6), that is, the singular points of (4.1) are determined by the following system:

$$\begin{cases} -\dfrac{(1-\alpha)\left(\frac{\varepsilon}{\theta}+1-\alpha\right)}{\theta}x_1 + \dfrac{1-\alpha}{\theta^2}x_1\dfrac{\mathrm{d}\varphi_2(x_2)}{\mathrm{d}x_2} - (1-\alpha)(x_1+x_2)\left[\dfrac{a_1}{\theta^2}\dfrac{\partial\psi_1(x)}{\partial x_2} - \dfrac{a_2}{\theta^2}\dfrac{\partial\psi_2(x)}{\partial x_2}\right] = 0, \\ -\dfrac{\left(\frac{\varepsilon}{\theta}+1-\alpha\right)(1-\alpha)}{\theta}x_2 + \dfrac{1-\alpha}{\theta^2}x_2\dfrac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1} - (1-\alpha)(x_1+x_2)\left[-\dfrac{a_1}{\theta^2}\dfrac{\partial\psi_1(x)}{\partial x_1} + \dfrac{a_2}{\theta^2}\dfrac{\partial\psi_2(x)}{\partial x_1}\right] = 0. \end{cases} \tag{4.3}$$

In case $x_1 = -x_2 (\neq 0)$, because $\mathrm{d}\varphi_1(x_1)/\mathrm{d}x_1 = \mathrm{d}\varphi_2(x_2)/\mathrm{d}x_2$ holds, (4.3) can be reduced to the equation:

$$-\frac{(1-\alpha)\left(\frac{\varepsilon}{\theta}+1-\alpha\right)}{\theta}x_1 + \frac{1-\alpha}{\theta^2}x_1\frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1} = -\frac{1-\alpha}{\theta^2}x_1\left[-\left(\frac{\varepsilon}{\theta}+1-\alpha\right)\theta + \frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1}\right] = 0.$$

Therefore the generalized pseudo singular points are solutions of the equation:

$$\frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1} = \varepsilon + (1-\alpha)\theta. \tag{4.4}$$

In case $x_1 = x_2 (\neq 0)$, because (4.2) holds, (4.3) can be reduced to the following equation:

$$-\frac{2(1-\alpha)}{\theta^2}x_1\left[-\left(\frac{\varepsilon}{\theta}+1-\alpha\right)\theta + \frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1}\right] = 0,$$

which implies that the generalized pseudo singular points satisfy (4.4). Because $\varphi_1(x_1) = \tanh(x_1)$ and $\varepsilon + (1-\alpha)\theta > 0$, we have

$$\exp(x_1) + \exp(-x_1) = \sqrt{\frac{4}{\varepsilon + (1-\alpha)\theta}} \equiv Y.$$

Putting $Z = \exp(x_1) (> 0)$, we obtain

$$Z = \frac{Y \pm \sqrt{Y^2 - 4}}{2}.$$

We will restrict ourselves to case where $Y^2 - 4 > 0$, that is, $\varepsilon + (1-\alpha)\theta < 1$ holds since ε and θ are small and $0 < \alpha < 1$. Then we get the following four generalized pseudo singular points:

$$\mathrm{P}_1 = (X, -X)^{\top},\ \mathrm{P}_2 = (-X, X)^{\top},\ \mathrm{P}_3 = (X, X)^{\top},\ \mathrm{P}_4 = (-X, -X)^{\top},$$

where $X = \log((Y + \sqrt{Y^2 - 4})/2)$.

Next we investigate the eigenvalues of the linearization of (4.1) at these generalized pseudo singular points. Note that

$$Df(x) = \begin{pmatrix} A+B+C & D+B+E \\ F+I+G & H+I+J \end{pmatrix},$$

where

$$A = -\frac{(1-\alpha)\left(\frac{\varepsilon}{\theta}+1-\alpha\right)}{\theta} + \frac{1-\alpha}{\theta^2}\frac{\mathrm{d}\varphi_2(x_2)}{\mathrm{d}x_2},\quad B = -(1-\alpha)\left[\frac{a_1}{\theta^2}\frac{\partial\psi_1(x)}{\partial x_2} - \frac{a_2}{\theta^2}\frac{\partial\psi_2(x)}{\partial x_2}\right],$$

$$C = -(1-\alpha)(x_1+x_2)\left[\frac{a_1}{\theta^2}\frac{\partial^2\psi_1(x)}{\partial x_2\partial x_1} - \frac{a_2}{\theta^2}\frac{\partial^2\psi_2(x)}{\partial x_2\partial x_1}\right],\quad D = \frac{1-\alpha}{\theta^2}x_1\frac{\mathrm{d}^2\varphi_2(x_2)}{\mathrm{d}x_2^2},$$

$$E = -(1-\alpha)(x_1+x_2)\left[\frac{a_1}{\theta^2}\frac{\partial^2\psi_1(x)}{\partial x_2^2} - \frac{a_2}{\theta^2}\frac{\partial^2\psi_2(x)}{\partial x_2^2}\right],\quad F = \frac{1-\alpha}{\theta^2}x_2\frac{\mathrm{d}^2\varphi_1(x_1)}{\mathrm{d}x_1^2},$$

$$G = -(1-\alpha)(x_1+x_2)\left[-\frac{a_1}{\theta^2}\frac{\partial^2\psi_1(x)}{\partial x_1^2} + \frac{a_2}{\theta^2}\frac{\partial^2\psi_2(x)}{\partial x_1^2}\right],\quad H = -\frac{(1-\alpha)\left(\frac{\varepsilon}{\theta}+1-\alpha\right)}{\theta} + \frac{1-\alpha}{\theta^2}\frac{\mathrm{d}\varphi_1(x_1)}{\mathrm{d}x_1},$$

$$I = -(1-\alpha)\left[-\frac{a_1}{\theta^2}\frac{\partial\psi_1(x)}{\partial x_1} + \frac{a_2}{\theta^2}\frac{\partial\psi_2(x)}{\partial x_1}\right],\quad J = -(1-\alpha)(x_1+x_2)\left[-\frac{a_1}{\theta^2}\frac{\partial^2\psi_1(x)}{\partial x_1\partial x_2} + \frac{a_2}{\theta^2}\frac{\partial^2\psi_2(x)}{\partial x_1\partial x_2}\right].$$

At the generalized pseudo singular points, $A = H = 0$ holds. It follows from the property of the hyperbolic tangent that $D = F$ holds. Moreover $C + J = 0$ and $B + I = 0$ can be easily seen because ψ_i is a sufficiently smooth function and satisfies (4.2).

At P_1 and P_2, we have $C = E = G = J = 0$. Therefore the characteristic equation is as follows:

$$\lambda^2 - D^2 = 0.$$

A similar result can be obtained at P_3 and P_4 assuming that

$$\frac{\partial^2\psi_i(x)}{\partial x_i^2} = \frac{\partial^2\psi_i(x)}{\partial x_j^2} \quad \text{and} \quad \frac{\partial^2\psi_i(x)}{\partial x_j\partial x_i} \approx \frac{\partial^2\psi_i(x)}{\partial x_j^2} \quad (i, j = 1, 2, j \neq i)$$

hold. Then we have $E + G = 0$ and $C \approx E$, and the characteristic equation becomes as follows:

$$0 = \lambda^2 + (2B + C + E)(E - C) - D^2 \approx \lambda^2 - D^2.$$

Because at each generalized pseudo singular point,

$$|D| = \left|\frac{1-\alpha}{\theta^2}X\frac{\partial^2\varphi_1(X)}{\partial x_1^2}\right| = \frac{8(1-\alpha)X\sqrt{Y^2-4}}{\theta^2 Y^3} \neq 0,$$

system (3.6) has pseudo singular saddle points. Therefore, the following theorem is established by Proposition 2.8.

Theorem 4.1 *If $\alpha_1 = \alpha_2$, $\varphi_i(x_i) = \tanh(x_i)$ $(i = 1, 2)$, and (4.2) hold, then there exist duck solutions in system (3.6).*

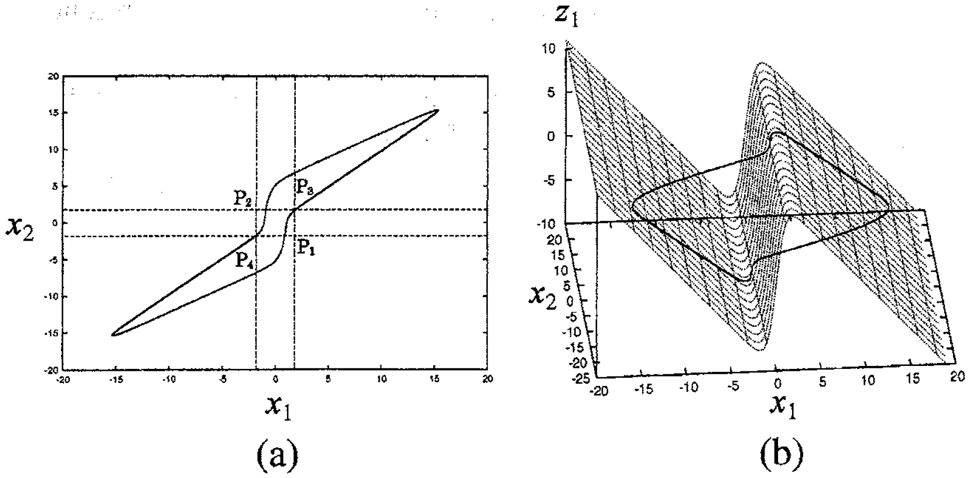

Figure 3 (a) The periodic solution of (3.6) projected onto the (x_1, x_2) plane. The dotted lines are GPL and the four intersection points are the generalized pseudo singular points P_1-P_4. (b) The periodic solution and the slow manifold of (3.6) projected onto the (x_1, x_2, z_1) space.

5 Numerical example

Finally, we present a numerical example. In the experiment, we put

$$\psi_1(x) = \psi_2(x) = \frac{1}{1+\exp(-b(x_1+d))} \times \frac{1}{1+\exp(-b(x_2+d))},$$

which is a monotonically increasing function with upper and lower limits, and satisfies the assumptions mentioned above. The values of the parameters are as follows:

$$\alpha = 0.9,\ \varepsilon = 0.005,\ \theta = 1,\ a_1 = 0.01,\ a_2 = 0.02,\ b = 0.5,\ d = 30.$$

The values of a_1 and a_2 mean that the export of the region 1 is less than that of the region 2. The results shown in Figure 3 are calculated by using the forth-order Runge-Kutta method. Figure 3(a) shows a two-dimensional projection of the solution to (3.6) onto (x_1, x_2) plane. Note that the four intersection points of the dotted lines in Figure 3(a) are the generalized pseudo singular points we calculated. After the solution passes through the generalized pseudo singular points P_3 and P_4, it slides along GPL and then it jumps. Figure 3(b) shows a three-dimensional projections of the solution and the slow manifold of (3.6) onto (x_1, x_2, z_1) space. The solution moves along the attracting part of the slow manifold till it crosses the fold of the slow manifold, and then it slides along the repelling part before it jumps to the other side, which can be interpreted as a duck solution.

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