数理解析研究所講究録 1474 巻 2006 年 55-59



# Functional differential equations of a type similar to $f'(x)=2f(2x+1)-2f(2x-1)$ and its application.

大阪教育大学　米田 剛 (Tsuyoshi Yoneda)

Department of Mathematics

Osaka Kyoiku University

Frederickson [1], [2] (1971) investigated functional-differential equations of advanced type

$$f'(x)=af(\lambda x)+bf(x), \tag{0.1}$$

where $\lambda>1$, and provided several properties of solutions. Kato and McLeod [4] (1971) and Kato [3] (1972) studied asymptotic behavior of solutions of (0.1).

By using another method, the author [6] constructed one of the solutions for the equation

$$\begin{cases} f'(x)=af(2x), & x\in\mathbb{R}=(-\infty,+\infty), \\ f(0)=0, \end{cases} \tag{0.2}$$

where $a$ is a constant with $a\neq 0$. The solution is not unique. If $f$ is a solution, then a constant times $f$ is also a solution. Our solutions are infinitely differentiable and bounded on $\mathbb{R}$. In [6], the author gave the graph of the solution $f(x)$ of (0.2) with $a=4$ (Figure 1).

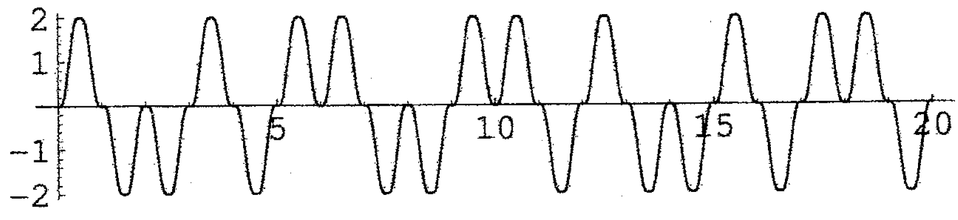



FIGURE 1. $f'(x)=4f(2x)$

In this paper, we construct a solution for the functional differential equation;

$$f^{(n)}(x) = \lambda^{n+1} \sum_j c_j \beta_j^n \sum_{l=0}^{n} (-1)^l \binom{n}{l} f(\lambda x - \frac{l - n/2 + nk_j}{\beta_j}), \tag{0.3}$$

where $\sum_j c_j = 1, \sum_j c_j \beta_j^n < \infty, c_j \geq 0, \inf_j \beta_j > 0$, $\sup_j |k_j| < \infty$ and $\sum_j$ is finite sum or infinite sum. The solution is unique in $L^1(\mathbb{R})$ up to a multiplicative constant and it is in $C^\infty_{comp}(\mathbb{R}) \cap L^1(\mathbb{R})$. (Some special cases were treated in [5].) We also give the method of caluculating numerical data. For example, the following is a special case of (0.3).

$$f'(x) = 4f(2x) - 4f(2x-1). \tag{0.4}$$

The graph of the solution of (0.4) is in Figure 2, which is a component of the graph of Figure 1. Our main result is as follows.

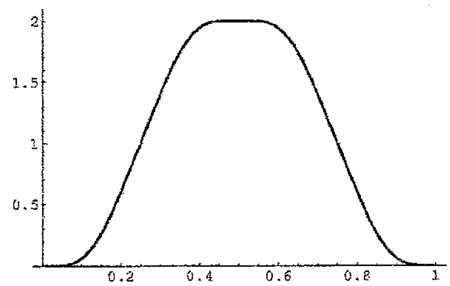



FIGURE 2. The solution of (0.4).

**Theorem 0.1.** *The equation* (0.3) *has a unique solution in* $L^1(\mathbb{R})$ *up to a multiplicative constant and it is in* $C^\infty_{comp}(\mathbb{R}) \cap L^1(\mathbb{R})$.

We give some examples of (0.3).

(E1) $f'(x) = 2f(2x+1) - 2f(2x-1)$,

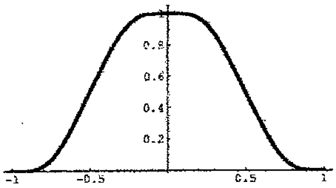
(E2) $f'(x) = 16f(4x) - 16f(4x-1)$,

(E3) $f'(x) = (9/4)f(3x/2) - (9/4)f(3x/2-1)$,

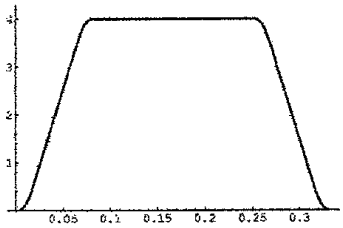
(E4) $f'(x) = 3f(2x) - 2f(2x-1) - f(2x-2)$,

(E5) $f'(x) = 3f(2x) - 3f(2x-1) + f(2x-2) - f(2x-3)$,

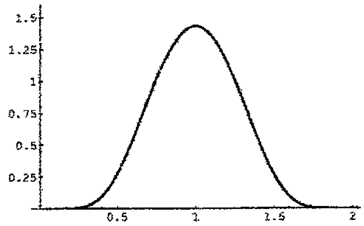
(E6) $f'(x) = 4\sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \frac{1}{2^{k+1}} \sum_{\ell=-2,-1,1,2} f(2^{k+2}x - \frac{4j+2\ell}{2^{k+2}})$, (E7) $f''(x) = 8f(2x+1) - 16f(2x) + 8f(2x-1)$.



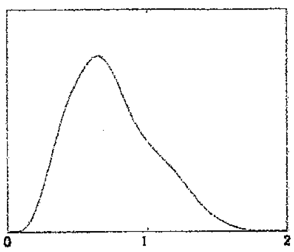

(E1), $\operatorname{supp} f = [0,1]$



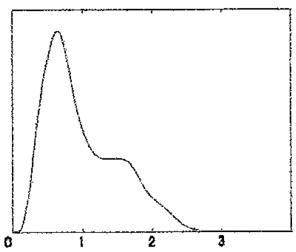

(E2), $\operatorname{supp} f = [0,1/3]$




(E3), $\operatorname{supp} f = [0,2]$

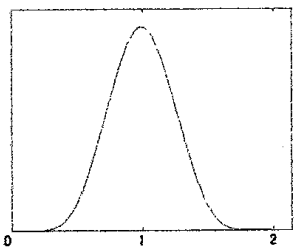



(E4), $\operatorname{supp} f = [0,2]$




(E5), $\operatorname{supp} f = [0,3]$

FIGURE 3. Solution of the equations (E1)–(E5).




(E6) and (E7), $\operatorname{supp} f = [0,2]$.

FIGURE 4. Solution of the equation (E6) and (E7).

We have considered application of the function (E1) by using "Quark theory" established by H.Trebel. Next section, we introduce the result of application. This application is considered with Yoshihiro Sawano who belongs to Tokyo University.

## 1. RESULT OF APPLICATION

**Definition 1.1.** Let $\rho > 1$.

Let $0 < p, q \leq \infty$ and $s > \sigma_p$. $B^s_{pq}(\mathbb{R})$ is a set of Schwartz distributions $f$ for which $f$ can be written as

$$f = \sum_{\beta \in \mathbb{N}_0^d} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \lambda^{\beta}_{\nu,m} 2^{-\nu(s-d/p)} (2^\nu x - m)^\beta \phi(2^\nu x - m) \tag{1.1}$$

where, the finction $\phi$ satisfies $\phi'(x) = 2\phi(2x+1) - 2\phi(2x-1)$.

**Theorem 1.1.** *The equation*

$$f'(x) = f(x-1)$$

*can solve explicty with following initial data*

$$f|_{[0,1]} = \left( \sum_{\beta \in \mathbb{N}_0} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}:\, Q_{\nu,m} \cap [0,1] \neq \emptyset} \lambda^{\beta}_{\nu,m} 2^{-\nu(s-d/p)} (2^\nu x - m)^\beta \phi(2^\nu x - m) \right)_{x \in [0,1]}.$$

*The solution $f|_{[1,2]}$ can be written as follows;*

$$\begin{aligned} f(x) &= f(1) + \sum_{\beta \in \mathbb{N}_0} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}:\, Q_{\nu,m} \cap [0,1] \neq \emptyset} 2^{-\nu} \lambda^{\beta}_{\nu,m} \phi^{\beta *}_{\nu,m}(x-1) \\ &\quad + \left(\int_{\mathbb{R}} x^\beta \phi(x)\, dx\right) \sum_{\nu \in \mathbb{N}_0} \sum_{2 \leq 2^\nu} \left( \sum_{l=0}^{m-2} \lambda^0_{\nu,l} 2^{-\nu} \right) \phi(2^\nu x - m - 1). \end{aligned}$$

*where*

$$\begin{aligned} \phi^{\beta *}_{\nu,m}(x) &:= \left( \sum_{\gamma=0}^{\beta} (-1)^\gamma \binom{\beta}{\gamma} (2^\nu x - m)^{\beta-\gamma} I_\gamma(\phi(2^\nu x - m)) \right) \\ &\quad - 2^{-\nu(s-\frac{n}{p})} \left(\int_{\mathbb{R}} x^\beta \phi(x)\, dx\right) \sum_{l=2}^{\infty} \phi(2^\nu(x-l) - m), \end{aligned}$$

$$I_\beta(\phi)(x) = \sum_{j_{\beta+1}=0}^{\infty} \sum_{j_\beta=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} 2^{\frac{\beta(\beta+1)}{2}} \phi\left( \frac{x - 2^{\beta+1} + 1}{2^{\beta+1}} - \sum_{\gamma=1}^{\beta+1} \frac{j_\gamma}{2^{\gamma-1}} \right)$$

*and*

$$I_0(\phi)(x) = \sum_{j=0}^{\infty} \phi\left( \frac{x-1-2j}{2} \right).$$

## 2. Acknowledgement

The author would like to thank Professor Eiichi Nakai for his useful comments.

## References


[1] P. O. Frederickson, *Global solutions to certain nonlinear functional differential equations.* J. Math. Anal. Appl. 33 (1971), 355–358.

[2] P. O. Frederickson, *Dirichlet series solutions for certain functional differential equations.* Japan-United States Seminar on Ordinary Differential and Functional Equations (Kyoto, 1971), 249–254. Lecture Notes in Math., Vol. 243, Springer, Berlin, 1971.

[3] T. Kato, *Asymptotic behavior of solutions of the functional differential equation* $y'(x) = ay(\lambda x) + by(x)$, Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972), Academic Press, New York, 1972, 197–217.

[4] T. Kato and J. B. McLeod, *The functional-differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. 77 (1971), 891–937.

[5] T. Yoneda, *Spline functions and n-periodic points*, Transactions of the Japan Society for Industrial and Applied Mathematics (Japanese), 15 (2005), 245–252.

[6] T. Yoneda, *On the functional-differential equation of advanced type* $f'(x) = af(2x)$ *with* $f(0) = 0$, to appear in Journal of Mathematical Analysis and Applications.

[7] T. Yoneda, *On the functional-differential equation of advanced type* $f'(x) = af(\lambda x)$, $\lambda > 1$ *with* $f(0) = 0$, preprint.



Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

*E-mail address*: bkaoj300@rinku.zaq.ne.jp