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On the Solvability of System of Linear Equations on a Commutative Semigroup

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Abstract

This paper deals with the solvability of a system of linear operator equations in the linear space. In particular, we provide necessary and sufficient conditions under which certain kinds of differential and difference equations are solvable.

1. Introduction

Let G be a semigroup of commuting linear operators on a linear space S. The solvability of a system of equations $l_i f = \phi_i$; $i = 1, \ldots, r$ (1), where $l_i \in G$, $\phi_i \in S$, was considered by Dahmen and Miccheli (in [1], *On multivariate E-splines,* Adv. Math, 76, 1989). In theirs studies, the compability conditions $l_i\phi_i = l_i\phi_j$; $i \neq j$ (2) are neccessary for the system (1) to have a solution in S. However, in general, they do not provide sufficient conditions. In [4] (*Solvability of systems of linear operator equations*, Proc. of AMS, 120, 1994), R. Q. Jia, S. Riemenschneider and Z. Shen gave some kinds of conditions on the operators l_i so that the compability conditions guarantee sufficiently for solvability of (1). In this report, we give some another conditions on operators l_i so that the compability condition will be sufficient for this. Actualy, we give some generalizations for the results of [4].

2. Solvability of a system of linear operator equations

Let X be a linear space on a field $\mathcal{F}$. The operator $D \in L(X)$ is called right invertible if there is an operator $R \in L_0(X)$ satisfies $RX \subset \operatorname{dom} D$ and $DR = I$. We call R a right inverse of D and denote by $R(X)$ the set of all right invertible operators. The set of all right inverses of $D \in R(X)$ is denoted by $\mathcal{R}_D$.

The operator $L \in L(X)$ is called left invertible if there is an operator $L^* \in L(X)$ such that $LX \subset \operatorname{dom} L^*$ and $L^*L = I$.
We call L^* a left inverse of L and denote by $\mathcal{L}(X)$ the set of all left invertible operators.

The set of all left inverses of $L \in \mathcal{L}(X)$ is denoted by $\mathcal{L}_L$.

The operator $A \in L(X)$ is called generalized invertible if there exists an operator $B \in L(X)$ so that $\operatorname{Im} A \subset \operatorname{dom} A$, $\operatorname{Im} B \subset \operatorname{dom} A$ and $ABA = A$ on $\operatorname{dom} A$.
We call B a generalized inverse of A and denote by $\mathcal{W}(X)$ the set of all generalized invertible operators. The set of all generalized inverses of A is denoted by $\mathcal{W}_A$.

Definition 2.1. *([8]) Let G be a semigroup of commuting operators on a linear space S with the group operation of position. We say that G possesses $s-$dimensional additivity if for any $s+1$ elemens $l_1, l_1', l_2, \ldots, l_s \in G$,*

$$\dim(\ker(\mathrm{l_1 l_1'}, \mathrm{l_2}, \ldots, \mathrm{l_s})) = \dim(\ker(\mathrm{l_1}, \mathrm{l_2}, \ldots, \mathrm{l_s})) + \dim(\ker(\mathrm{l_1'}, \mathrm{l_2}, \ldots, \mathrm{l_s})),$$

where $\ker(\tilde{\mathrm{l_1}}, \ldots, \tilde{\mathrm{l_s}}) := \{\mathrm{f} \in \mathrm{S}, \tilde{\mathrm{l_j}}\mathrm{f} = 0, \mathrm{j} = 1, \ldots, \mathrm{s}\}$.

Definition 2.2. *([8]) Let M be a subspace of S. An element $l \in G$ is said to be nilpotent on M if for any $\phi \in M$, there exists a positive integer m such that $l^m\phi = 0$ (m may depend on ϕ). We say that M is compatible with G if the following conditions are satisfied:*

1. *M is invariant under G, i.e., for any $l \in G, l(M) \subset M$.*
2. *For any $l \in G, l\,|_M$ is either invertible or nilpotent.*

Theorem 2.1 ([4] by R. Q. Jia, S. Riemenschneider, Z. Shen). *Let G be a semigroup of commuting operators on a linear space S, and let $l_1, \ldots, l_r$ be elements in G. Assume that one of them, say l_1, is invertible on S. Then for given $\phi_1, \ldots, \phi_r$ in S, the system of equations*

$$l_j f = \phi_j;\ j = 1, \ldots, r, \tag{2.1}$$

has a solution in S if and only if the following compatibility conditions hold

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r. \tag{2.2}$$

We can state another theorem that is equivalent Theorem 2.1

Theorem 2.2. *Let G be a semigroup of commuting operators on a linear space S, and let $l_1, \ldots, l_r$ be elements in G. Assume that one of the operators l_i, $l_il_1, l_il_1l_2, \ldots, l_il_1 \cdots l_r$ is invertible on S for the $i = 1, 2, \ldots, r$. Then for given $\phi_1, \ldots, \phi_r$ in S, the system of equations*

$$l_j f = \phi_j;\ j = 1, \ldots, r,$$

has a solution in S if and only if the following compatibility conditions hold

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r.$$

Theorem 2.3. *Let G be a semigroup of commuting operators on a linear space S, and let $l_1, \ldots, l_r \in G$. Suppose that there exist $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathfrak{F}$ so that $l = \alpha_1 l_1 + \alpha_2 l_2 + \cdots \alpha_r l_r \in G$ is invertible on S. If the following compatibility conditions hold*

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r, \tag{2.3}$$

for given $\phi_1, \ldots, \phi_r$ in S, then the system of equations

$$l_j f = \phi_j;\ j = 1, \ldots, r, \tag{2.4}$$

has a solution in S.

Proof. System (2.4) is equivalent the system

$$\begin{cases} l_1 f &= \phi_1 \\ l_2 f &= \phi_2 \\ \vdots & \vdots \\ l f &= x_1\phi_1 + \cdots + x_r\phi_r \\ \vdots & \vdots \\ l_r f &= \phi_r \end{cases} \tag{2.5}$$

Since l is invertible on S and $l_j\phi_k = l_k\phi_j$, $1 \leqslant j \leqslant k \leqslant r$, we obtain

$$l_i(x_1\phi_1 + \cdots + x_r\phi_r) = l\phi_i \tag{2.6}$$

Thus, the compability conditions is satisfied, the proof is complete. □

Example 2.1. Let S be a linear space of all polynomials (it can be n variables $x, y, \dots, z$), theirs degrees are not exceeded n. Consider two operators determined in S as follows

$$l_1 = I - xD_x \ ,$$
$$l_2 = xD_x \ .$$

Note that l_1, l_2 is not invertible, because of $l_1(x) = 0, l_2(c) = 0$. Obviously, $l_1l_2 = l_2l_1 = xD - xDxD - x^2D^2$. Althought l_1, l_2 are not invertible, but $l = l_1 + l_2$ is invertible on S, provided $l \equiv I$. So in this case, the assumption of Theorem 2.1 is not satisfied, but the assumption of Theorem 2.3 is.

Theorem 2.4. *Let G be a semigroup of commuting operators on a linear space S, and let $l_1, \dots, l_r$ be elements in G. Assume that one of them, say l_1, is generalized invertible on S. Then for given $\phi_1, \dots, \phi_r$ in S, the system of equations*

$$l_j f = \phi_j;\ j = 1, \dots, r, \tag{2.7}$$

has a solution in S if and only if the following conditions hold: $\forall l^* \in \mathcal{W}_{l_1}$, there exists $z \in \ker l_1$ such that

$$(I - l_1l^*)\phi_1 = 0,\ \phi_i = l_i(l^*\phi_1 + z),\ \forall i = 2, \dots r. \tag{2.8}$$

Proof. 1. *Necessity.* Suppose that the system (2.7) has a solution $f \in S$. From $l_1 f = \phi_1$ it implies $\phi_1 \in \operatorname{Im} l_1$. There exist $f_0 \in S$ such that

$$\phi_1 = l_1 f_0 \ . \tag{2.9}$$

We then have

$$\phi_1 = l_1l^*l_1f_0 = l_1l^*\phi_1, \quad \text{for } l^* \in \mathcal{W}_{l_1}. \tag{2.10}$$

Then, $(I - l_1l^*)\phi_1 = 0$. Hence, $l_1 f = l_1l^*\phi_1$ i.e. $l_1(f - l^*\phi_1) = 0$. So we receive $f - l^*\phi_1 = z$ for any $z \in \ker l_1$, i.e. $f = l^*\phi_1 + z$. Thus,

$$\phi_i = l_i(l^*\phi_1 + z),\ \ 1 \leqslant i \leqslant r.$$

2. *Sufficiency*: If $(I - l_1l^*)\phi_1 = 0$ then $\phi_1 = l_1l^*\phi_1$. Clearly, $f = l^*\phi_1 + z$ is a solution of (2.7).

□

Two immediate consequences of Theorem 2.4 as follows

Corollary 1. *Let G be a semigruop of commuting operators on a linear space S, and let $l_1, \dots, l_r$ be elements in G. Suppose that one of them, say l_1, is right invertible on S. Then for given $\phi_1, \dots, \phi_r$ in S, the system of equations*

$$l_jf = \phi_j;\ j = 1, \dots, r,$$

has a solution in S if and only if the following conditions hold: For any $R \in \mathcal{R}_{l_1}$, there exists $z \in ker l_1$ such that

$$\phi_i = l_i(R\phi_1 + z)\ \forall i = 2, \dots r.$$

Corollary 2. *Let G be a semigroup of commuting operators on a linear space S, and let $l_1, \dots, l_r$ be elements in G. Suppose that one of them, say l_1, is left invertible on S. Then for given $\phi_1, \dots, \phi_r$ in S, the system of equations*

$$l_jf = \phi_j;\ j = 1, \dots, r,$$

has a solution in S if only if the following conditions hold

$$(I - l_1l^*)\phi_1 = 0,\ \phi_i = l_il^*\phi_1,\ \ l^* \in \mathcal{L}_{l_1},\ \ i = 2, \dots r.$$

Comment 1. If the operators l_1 in Theorem 2.4, in Corallary 1 as well in Corollary 2 are invertible, then the conditions stated there turn out the compability conditions of Theorem 2.1. So, at the present we also call all of them compability conditions.

Now we recall another theorem

Theorem 2.5 ([4], R. Q. Jia, S. Riemenschneider, Z. Shen). *Let G be a semigroup of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspace M and N which are invariant under G. Moreover, assume that $l_1, l_2, \dots, l_s \in G$ are nilpotent on M and they have the property* $\dim(\ker(l_1, l_2, \dots, l_s)) < \infty$. *Let $r \in \{1, 2, \dots, s\}$. Then for given $\phi_1, \dots, \phi_r \in M$, the system of equations*

$$l_jf = \phi_j;\ j = 1, \dots, r,$$

has a solution in M if and only if the following compatibility conditions hold

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r.$$

Here are our generalizations:

Theorem 2.6. *Let G be a ring (under the addition and coposition) of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspace s M and N which are invariant under G. Assume* *that $l_1, l_1+l_2, \ldots, l_1+l_2+\cdots+l_s \in G$ are nilpotent on M and they have the property* $\dim(\ker(l_1, l_2, \ldots, l_s)) < \infty$. *Let $r \in \{1, \ldots, s\}$. Then for given $\phi_1, \ldots, \phi_r \in M$, the system of equations*

$$l_j f = \phi_j;\ j = 1, \ldots, r, \tag{2.11}$$

has a solution in M if and only if the following compatibility conditions hold

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r. \tag{2.12}$$

Proof. Obviously, (2.11) is equivalently to the following system

$$\begin{cases} l_1 f & = \phi_1 \\ (l_1+l_2)f & = \phi_1+\phi_2 \\ \vdots & \vdots \\ (l_1+\cdots+l_r)f & = \phi_1+\cdots+\phi_r \end{cases} \tag{2.13}$$

Put $l_j' = \sum\limits_{i=1}^{j} l_i$, $\phi_j' = \sum\limits_{i=1}^{j}\phi_i$, $(1 \leqslant i \leqslant r)$. Then (2.13) becomes

$$\begin{cases} l_1' f & = \phi_1' \\ l_2' f & = \phi_2' \\ \vdots & \vdots \\ l_r' f & = \phi_r' \end{cases} \tag{2.14}$$

We prove the system (2.14) sastisfies the assumption of Theorem 2.5. We have

$$\begin{aligned} l_j'\phi_k' &= \sum_{i=1}^{j} l_i \sum_{m=1}^{k}\phi_m = \sum_{i=1}^{j}\sum_{m=1}^{k} l_i\phi_m \\ &= \sum_{i=1}^{j}\sum_{m=1}^{k} l_m\phi_i = l_k'\phi_j' , \quad 1 \leqslant j \leqslant k \leqslant r. \end{aligned}$$

Thus, the system (2.14) has a solution in S. This means that (2.11) also has solution in S. □

Comment 2. From the nilpotence of $l_1, l_2, \ldots l_r$ it follows that the operator $l_1+l_2+\cdots l_s$ is so. However, the operators $l_1 = I + D_x, l_2 = -I$ are not nilpotent, but $l_1 + l_2 = D_x$ is nilpotent. So the assumption of Theorem 2.6 is more weakly than what of Theorem 2.5.

Theorem 2.7. *Let G be a semigroup of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspaces M and N which are invariant under G. Moreover, assume that $l_1, l_2, \ldots l_s \in G$,* $\dim(\ker(l_1, l_2, \ldots, l_s)) < \infty$ *and satisfy: for any $g \in M$ and for any pair of operators (l_i, l_j) $(1 \leqslant i \leqslant j \leqslant s)$ there exist $m_i, m_j \in \mathbb{N}$ such that $l_i^{m_i} l_j^{m_j} g = 0$. Take $r \in \{1, \ldots, s\}$. For given $\phi_1, \ldots, \phi_r \in M$, the following compatibility conditions hold*

$$l_j\phi_k = l_k\phi_j,\ 1 \leqslant j < k \leqslant r. \tag{2.15}$$

Then, the system

$$l_j f = \phi_j;\ j = 1, \ldots, r, \tag{2.16}$$

has a solution in M.

Proof. By induction on r.

Step 1. We consider the special case when $r = s$, $\phi_2 = \cdots = \phi_s = 0$. We have

$$\phi_1 \in \ker(l_2, \ldots, l_s). \tag{2.17}$$

There exists $m_1, m_2 \in \mathbb{N}^*$ such that

$$l_1^{m_1} l_2^{m_2} \phi_1 = 0. \tag{2.18}$$

Put $H' := \ker(l_1^{m_1} l_2^{m_2}, l_2, \ldots, l_s)$.
By (2.17) and (2.18), we obtain

$$\phi_1 \in H' . \tag{2.19}$$

Write

$$H := \ker(l_1^{m_1+1} l_2^{m_2}, l_2, \ldots, l_s) ,$$
$$H'' := \ker(l_1, l_2, \ldots, l_s) .$$

It is easy to see that $H'' \subseteq H' \subseteq H$.
We prove $l_1(H) \subseteq H'$. For some $x \in H$ we have

$$\begin{cases} l_1^{m_1+1} l_2^{m_2} x &= 0 \\ l_2 x &= 0 \\ \vdots & \vdots \\ l_s x &= 0. \end{cases} \tag{2.20}$$

Then

$$\begin{cases} l_1^{m_1} l_2^{m_2} (l_1 x) &= 0 \\ l_2 (l_1 x) &= 0 \\ \vdots & \vdots \\ l_s (l_1 x) &= 0. \end{cases}$$

It implies $l_1 x \in H'$. Hence, $l_1(H) \subseteq H'$. We prove $\ker(l_1 |_H) \equiv H'$. Suppose that $x \in \ker(l_1 |_H)$, i.e. $x \in H$ and $l_1(x) = 0$. It is easy to see $x \in H''$. If $x \in H'' \subseteq H$, then $l_1(x) = 0$. Hence, l_1 maps H to H' and its kernel is exactly H''. By assumption $\dim H < \infty$, we obtain

$$\mathrm{dimH} = \mathrm{dimH''} + \mathrm{diml_1(H)}. \tag{2.21}$$

Provided G possesses s–dimensional additivity semigroup, then

$$\mathrm{dimH} = \mathrm{dimH''} + \mathrm{dimH'} . \tag{2.22}$$

We then have

$$\dim l_1(H) = \dim H' \text{ or } l_1(H) = H'.$$

Hence, l_1 maps H onto H'. Because of $\phi_1 \in H'$, there exists $f \in H$ so that

$$l_1 f = \phi_1 \ .$$

It is clearly $l_j f = 0\ ,\quad \forall j = 2, \ldots, s$. Put $f = f_1 + f_2$, $f_1 \in M$, $f_2 \in N$. Then

$$l_1 f_1 = \phi_1\ ,\quad l_j f_1 = 0\ ,\quad \forall j = 2, \ldots, s\ .$$

Thus, $f_1 \in M$ is a solution of system (2.15).
Step 2. The case $r = 1$: The system (2.15) becomes

$$l_1 f = \phi_1\ ,$$

then l_1 is nilpotent on M. In this case, it turn out the case of Theorem 2.6.
Step 3. Assume that for any $r > 1$, there exists $f_1 \in M$ so that

$$l_i f_1 = \phi_i\ ,\quad \forall i = 1, \ldots, r-1\ . \tag{2.23}$$

We prove that the system

$$l_i f = \phi_i\ ,\quad \forall i = 1, \ldots, r \tag{2.24}$$

has a solution in M. Write $g = l_r f_1$, then $g \in M$. We obtain

$$l_i g = l_i(l_r f_1) = l_r(l_i f_1) = l_r \phi_i = l_i \phi_r\ ,\quad \forall i = 1, \ldots, r.$$

It follows

$$l_i(\phi_r - g) = 0\ ,\quad \forall i = 1, \ldots, r-1\ . \tag{2.25}$$

On the other hand, $\phi_r - g \in M$ so for any $j \in \{r+1, \ldots, s\}$ there exists m_{1j} , $m_j \in \mathbb{N}^*$ such that

$$l_1^{m_{1j}} l_j^{m_j}(\phi_r - g) = 0\ ,\quad \forall j = r+1, \ldots, s\ . \tag{2.26}$$

Consider the following system

$$\begin{cases} l_i h &= 0, \quad i = 1, \ldots, r-1 \\ l_r h &= \phi_r - g \\ l_1^{m_{1j}} l_j^{m_j} h &= 0\ ,\quad j = r+1, \ldots, s \end{cases} \tag{2.27}$$

In this case, this system turns out the case os Step 1. So the system (2.27) has solution $h \in M$. Put $f = f_1 + h$. We have

$$l_i f = l_i f_1 + l_i h = \phi_i\ ,\quad i = 1, \ldots, r-1\ ,$$
$$l_r f = l_r f_1 + l_r h = g + \phi_r - g = \phi_r.$$

Hence, f is a solution of system (2.15). □

Corollary 3. *Let G be a semigroup of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspaces M and N which are invariant under G. Moreover, assume that $l_1, l_2, \dots, l_s \in G$, $\dim(\ker(l_1, l_2, \dots, l_s)) < \infty$ and satify: for $k \in \{1, \dots, s\}$ fixed, $g \in M$ and for any pair of opetators $l_{i_1}, \dots, l_{i_k})$ $(1 \leqslant i_1 \leqslant \dots \leqslant i_k \leqslant s)$ there exist $m_{i_1}, \dots, m_{i_k} \in \mathbb{N}$ such that*

$$l_{i_1}^{m_{i_1}} \dots l_{i_k}^{m_{i_k}} g = 0.$$

Let $r \in \{1, \dots, s\}$. For given $\phi_1, \dots, \phi_r \in M$, the following compatibility conditions hold

$$l_j \phi_k = l_k \phi_j, \ 1 \leqslant j < k \leqslant r.$$

Then, the system

$$l_j f = \phi_j; \ j = 1, \dots, r,$$

has a solution in M.

Theorem 2.8. *Let G be a semigroup of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspaces M and N which are invariant under G. Moreover, assume that $l_1, l_2, \dots, l_s \in G$, $\dim(\ker(l_1, l_2, \dots, l_s)) < \infty$ and that they are all idempotency on M, i.e., for some $g \in M$ and for any l_i $(i \in \{1, \dots, s\})$ there exists $m_i \in \mathbb{N}$ such that*

$$l_i^{m_i} g = l_i g.$$

Take $r \in \{1, \dots, s\}$. For given $\phi_1, \dots, \phi_r \in M$, the following compatibility conditions hold

$$l_j \phi_k = l_k \phi_j, \ 1 \leqslant j < k \leqslant r. \tag{2.28}$$

Then, the system

$$l_j f = \phi_j; \ j = 1, \dots, r, \tag{2.29}$$

has a solution in M.

Proof. By induction on r .
Step 1. We consider the case when $r = s$, $\phi_2 = \dots = \phi_s = 0$. We have

$$\phi_1 \in \ker(l_2, \dots, l_s). \tag{2.30}$$

There exists $m_1 \in \mathbb{N}^*$ such that

$$l_1^{m_1} \phi_1 = l_1 \phi_1.$$

Equivalently,

$$(l_1^{m_1} - l_1)\phi_1 = 0. \tag{2.31}$$

So we have

$$\phi_1 \in H' , \tag{2.32}$$

where

$$H^{'} := \ker(l_1^{m_1} - l_1, l_2, \dots, l_s).$$

Put

$$H := \ker(l_1^{m_1+1} - l_1^2, l_2, \dots, l_s),$$
$$H^{''} := \ker(l_1, l_2, \dots, l_s).$$

It is clearly that $H^{''} \subseteq H^{'} \subseteq H$. We prove $l_1(H) \subseteq H^{'}$. For any $x \in H$ we have

$$\begin{cases} l_1^{m_1+1} l_1^2 x &= 0 \\ l_2 x &= 0 \\ \vdots & \vdots \\ l_s x &= 0. \end{cases}$$

Equivalently,

$$\begin{cases} (l_1^{m_1} - l_1)(l_1 x) &= 0 \\ l_2(l_1 x) &= 0 \\ \dots & \dots \\ l_s(l_1 x) &= 0. \end{cases}$$

Since, $l_1 x \in H^{'}$. Now, we prove $\ker(l_1 |_H) = H^{''}$. Indeed, if $x \in H$ and $l_1 x = 0$ then $x \in H^{''}$. Hence, $\ker(l_1 |_H) = H^{''}$.
By the assumption $\dim \mathrm{H} < \infty$ we then have

$$\mathrm{dimH} = \mathrm{dimH}^{''} + \mathrm{diml_1(H)}.$$

By the $s-$dimensional additivity of G, we obtain

$$\mathrm{dimH} = \mathrm{dimH}^{''} + \mathrm{dimH}^{'}.$$

From (48) and (49) it together implies

$$\mathrm{diml_1(H)} = \mathrm{dimH}^{'}.$$

Moreover,

$$l_1(H) \subseteq H^{'} \quad \text{and} \quad \mathrm{dimH}^{'} < \infty.$$

So

$$l_1(H) = H^{'}. \tag{2.33}$$

Hence, l_1 maps H onto $H^{'}$. Since (2.32) and (2.33), there exists $f \in H$ so that

$$l_1 f = \phi_1 \text{ , and}$$
$$l_j f = 0 \text{ , } \quad \forall j = 2, \dots, s.$$

Put $f = f_1 + f_2$, where $f_1 \in M$, $f_2 \in N$. By the assumption that M, N are invariant under G, we have

$$\begin{cases} l_1 f_1 &= \phi_1 \ , \\ l_j f_1 &= 0 \ , \quad \forall j = 2, \dots, s \ . \end{cases} \tag{2.34}$$

Hence, f_1 is a solution of the system (2.29) in the case when $r = s$, $\phi_2 = \cdots = \phi_s = 0$.
Step 2: The case $r = 1$. (2.29) becomes

$$l_1 f = \phi_1 \ .$$

There exists $m_1 \in \mathbb{N}^*$ such that

$$l_1^{m_1} \phi_1 = l_1 \phi_1.$$

It follows

$$\phi_1 \in H_1^{'},$$

where

$$H_1^{'} := \ker (l_1^{m_1} - l_1) \ .$$

Write

$$H_1 : = \ker (l_1^{m_1+1} - l_1^2) \ ,$$
$$H_1^{''} : = \ker (l_1).$$

Applying the method in *Step1*, we obtain

$$l_1(H_1) = H_1^{'} \quad \text{and} \ker (l_1 \mid_{H_1}) = H_1^{''}. \tag{2.35}$$

Since $\phi_1 \in H_1^{'}$ and (2.35) there exists $f \in H_1$ such that

$$l_1 f = \phi_1 \ .$$

Put $f = f_1 + f_2$, where $f_1 \in M$,$f_2 \in N$. Because of M, N are invariant under G, then $l_1 f_1 = \phi_1$.
Step 3: Suppose that for $r > 1$, there exists $f_1 \in M$ such that

$$l_i f_1 = \phi_i \ , \quad \forall i = 1, \dots, r-1 \ .$$

We prove the system

$$l_i f = \phi_i \ , \quad \forall i = 1, \dots, r$$

has a solution in M. Put $g = l_r f_1$. Then $g \in M$ and

$$l_i g = l_i(l_r f_1) = l_r(l_i f_1) = l_r \phi_i = l_i \phi_r \ , \quad i = 1, \dots, r.$$

Hence,

$$l_i(\phi_r - g) = 0 \ , \quad \forall i = 1, \dots, r-1 \ . \tag{2.36}$$

Because of $\phi_r - g \in M$, so for any $j = r+1, \dots, s$ there exists $m_j \in \mathbb{N}^*$ so that

$$l_j^{m_j}(\phi_r - g) = l_j(\phi_r - g).$$

Equivalently,

$$(l_j^{m_j} - l_j)(\phi_r - g) = 0 \; , \quad \forall j = r+1, \dots, s \; . \tag{2.37}$$

Consider the following system

$$\begin{cases} l_i h & = 0 \; , \quad i = 1, \dots, r-1 \\ l_r h & = \phi_r - g \\ (l_1^{m_j} - l_j)h & = 0 \; , \quad j = r+1, \dots, s \end{cases} \tag{2.38}$$

The last system turn out the case of Step 1. So, the system (2.38) has a solution $h \in M$. Put $f = f_1 + h$, then $f \in M$ and

$$l_i f = l_i f_1 + l_i h = \phi_i \; , \quad \forall i = 1, \dots, r-1,$$
$$l_r f = l_r f_1 + l_r h = g + \phi_r - g = \phi_r .$$

Therefore, f is a solution of system (2.29). □

Conjecture. Theorems 2.5, 2.6, 2.7, 2.8 are still true in the case when $\dim(\ker(l_1, l_2, \dots, l_s)) = \infty$.

Example 2.2. Let S is a linear space of the functions, which is generated by: $\{\sin x$, $\cos x$, $e^y\}$ (x, y are two variables) on $\mathbb{R}$. It is clearly that, $S = M \oplus N$ where M is generated by $\{\sin x \, , \cos x\}$, N is generated by $\{e^y\}$. Denote by G a linear comutative semigroup generated by the differentials D_x, D_y on S. It is easy to see that M and N are invariant under G and $D_x|_M$, $D_y|_M$ also are idempotency on M.

Theorem 2.9. *Let G be a semigroup of commuting operators on a linear space S and let $l_1, \dots, l_r \in G$. Suppose that $l_i(K) \supseteq K$ for any subspace K of S, $i = 1, \dots, r$. Assume that the following compatibility conditions hold*

$$l_j \phi_k = l_k \phi_j, \; 1 \leqslant j < k \leqslant r. \tag{2.39}$$

Then, the system

$$l_i f = \phi_i ; \; \phi_i \in S, \; i = 1, \dots, r \tag{2.40}$$

has a solution in S.

Proof. By induction on r.
Step 1. The case $r = 1$. The system (2.40) becomes

$$l_1 f = \phi_1 \; .$$

There exists $f \in S$ such that

$$l_1 f = \phi_1 \; .$$

Step 2. Assume that for $r > 1$, system of $r-$equations

$$l_i f = \phi_i \ , \quad i = 1, \ldots, r-1$$

has a solution f_1 in S. We prove the system of equations

$$l_i f = \phi_i \ , \quad i = 1, \ldots, r$$

has solution in S. Put $l_r f_1 = g \in S$. We then have

$$l_i g = l_i l_r f_1 = l_r l_i f_1 = l_r \phi_i = l_i \phi_r, \quad i = 1, \ldots, r-1.$$

It implies

$$l_i(\phi_r - g) = 0, \quad \forall i = 1, \ldots, r-1. \tag{2.41}$$

Hence,

$$\phi_r - g \in H,$$

where

$$H := \ker (l_1, \ldots, l_{r-1}) \ .$$

It is easy to verify that H is a subspace of S . There exists $h \in H$ so that $\phi_r - g = l_r h$. Put $f = h + f_1$. So we have

$$l_i f = l_i h + l_i f_1 = \phi_i \ , \quad \forall i = 1, \ldots, r-1,$$
$$l_r f = l_r h + l_r f_1 = \phi_r - g + g = \phi_r.$$

Hence, f is a solution of (2.40) in S. □

Theorem 2.10. *Let G be a semigroup of commuting operators on a linear space S which possesses $s-$dimentional additivity. Suppose that S is a direct sum of two subspaces M and N which are invariant under G. Let $l_1, \ldots, l_r \in G$ so that $l_i(K) = K$, where K is an arbitrary subspace of S that is invariant under G. Assume that the following compatibility conditions hold*

$$l_j \phi_k = l_k \phi_j, \ 1 \leqslant j < k \leqslant r. \tag{2.42}$$

Then the system

$$l_i f = \phi_i; \ \phi_i \in M, \ i = 1, \ldots, r \tag{2.43}$$

has a solution in M.

Proof. By induction on r.

The case $r = 1$. Then the system (2.43) becomes

$$l_1 f = \phi_1, \ \phi_1 \in M.$$

There exists $f \in M$ such that $l_1 f = \phi_1$. Assume that for $r > 1$ the system

$$l_i f = \phi_i, \ i = 1, \ldots, r-1$$

has a solution f in M. We prove the system

$$l_i f = \phi_i,\ i = 1, \dots, r$$

has solution in M. Put $l_r f_1 = g \in M$, then

$$l_i g = l_i l_r f_1 = l_r l_i f_1 = l_r \phi_1 = l_i \phi_r,\ \forall i = 1, \dots, r-1.$$

It implies

$$l_i(\phi_r - g) = 0,\ \forall i = 1, \dots, r-1.$$

Equivalently,

$$\phi_r - g \in H,$$

where

$$H := \ker(l_1, \dots, l_{r-1}).$$

It is easy to verify that H is a subspace of S and that H is invariant under l_i for $i = 1, \dots, r-1$. So we have $l_r(H) = H$. There exist an $h \in H$ such that $\phi_r - g = l_r h$. It follows $l_i h = 0,\ i = 1, \dots, r-1$. Put $f = h + f_1$. Then

$$l_i f = l_i h + l_i f_1 = \phi_i,\ \forall i = 1, \dots, r-1,$$
$$l_r f = l_r h + l_r f_1 = \phi_r - g + g = \phi_r.$$

Therefore, f is a solution of (2.43) in S. □

Example 2.3. Denote by S a space of the functions which are generated by $\{e^{x+y}, \sin(x+y), \cos(x+y)\}$ on $\mathbb{R}$. Assume that D_x, D_y are two differential operators on S, G is the semigroup generated by $\{D_x, D_y\}$. This is easy to verify that G is a commutative semigruop and $S = M \bigoplus N$, where M is generated by $\{e^{x+y}\}$ and N is generated by $\{\sin(x+y), \cos(x+y)\}$. It is easy to see that S has the subspaces O, M, N, S which are invariant under G.

3. Solvability of differential and difference equations

In this section, we prove that certain systems of differential and difference equations are solvable provided the compatibility conditions. From now on, we denote by $\mathcal{K}$ an algebraic closed field of characteristic zero, and the ring of polynomials in s indeterminates $X_1, \dots, X_s$ over $\mathcal{K}$ is denoted by $K[X_1, \dots, X_s] = \prod(K^s)$.

For a given ideal I of $K[X_1, \dots, X_s]$ we denote by $V(I)$ the (affine) algebraic variety determined by I:

$$V(I) := \{a \in K^s : p(a) = 0 \ \text{ for all } \ p \in I\}.$$

An algebraic variety V is said to be reducible if it can be represented as a union of two algebraic varieties, both different from V. Otherwise, we call V an irreducible algebraic variety. Given an algebraic variety V, denote by $I(V)$ the ideal of all polynomials which vanish on V. The ring

$$K[V] := K[X_1, \dots, X_s]/I(V)$$

is called the coordinate ring of V. If V is irreducible, then $K[V]$ is an integral domain. If it is in case, the quotient field of $K[V]$ is called the field of rational functions on V, and denoted by $K(V)$.

Definition 3.3. *The dimension of an irreducible variety V,* $\dim(V)$, *is the transcendence degree of $K(V)$ over K. The dimension of a variety is the maximum of the dimensions of its irreducible components.*

A single point has dimension 0, and the dimension of any algebraic variety $V \subseteq K^s$ is at most s. Let $V \subseteq K^s$ be an algebraic variety. For a polynomial $f \in K[X_1, \dots, X_s]$, we define the variety $V_f := \{a \in V : f(a) = 0\}$. We say that f does not vanish on V if $V_f \neq V$.

Theorem 3.11. *(Dimension Theorem, [7]) Let V be an irreducible variety of dimension $n \geq 1$ in K^s. If a polynomial $f \in K[X_1, \dots, X_s]$ does not vanish on V and $V_f \neq \emptyset$, then every irreducible component of V_f has dimension $n - 1$.*

Theorem 3.12. *([7]) Suppose that every irreducible component of a variety U in K^s has the same dimension n ($n \geq 1$). Let $\theta = (\theta_1, \dots, \theta_s) \in U$. Then there exists an element $v = (v_1, \dots, v_s) \in K^s$ such that*

$$F_v := (X_1 - \theta_1)v_1 + \cdots + (X_s - \theta_s)v_s \in K[X_1, \dots, X_s]$$

does not vanish on any component of U. Consequently, every irreducible component of $U \bigcap V(F_v)$ has dimension $n - 1$.

Here are our generalization

Theorem 3.13. *Suppose that the minimum of the dimensions of irreducible components of a variety U in K^s is n ($n \geq 1$). Let $\theta = (\theta_1, \dots, \theta_s) \in U$. Then, there exists an element $v = (v_1, \dots, v_s) \in K^s$ such that*

$$F_v := (X_1 - \theta_1)v_1 + \cdots + (X_s - \theta_s)v_s \in K[X_1, \dots, X_s]$$

does not vanish on any component of U and $U \bigcap V(F_v)$ is a finite set.

Proof. Let $U = U_1 \bigcup \cdots \bigcup U_m$ be the decomposition of U into irreducible components with $U_i \not\subseteq U_j$ for all $i \neq j$. Suppose that the dimension of U_i is at smallest, it is n, $i = 1, \dots, m$. For any $j \in \{1, \dots, m\}$, we set

$$L_j := \{v = (v_1, \dots, v_s) \in K^s : F_v \text{ vanish on } U_j\},\ j = 1, \dots, m.$$

Clearly, L_j is a linear subspace of K^s. The maxnimum dimension of $L_j, (j = 1, \dots, m)$ is $s - n$. Thus, the set $K^s \backslash \bigcup_{j=1}^m L_j$ is nonempty. For any $v \in K^s$ $\dim U_j < n$, F_v does not vanish on any of U_j for each $j = 1, \dots, m$ and the set $U \bigcap V(F_v)$ is finite. □

Corollary 4. *Suppose that the minimum of the dimensions of every irreducible components of a variety U in K^s is n ($n \geq 1$). For any $\theta \in U$ there exists n polynomials $p_1, \dots, p_n$ of degree 1 such that they vanish at θ and such that $U \bigcap V(p_1, \dots, p_n)$ is a finite set.*

Theorem 3.14 ([4]). *Let $p_1, \dots, p_r$ ($r \leqslant s$) be polynomials on K^s. Assume that the variety $V(p_1, \dots, p_r)$ is either empty or each of its irreducible components has dimension*

$s - r$. Then for given polynomials (resp. exponential polynomials) $\phi_1, \dots, \phi_r$, the system of differential equations

$$p_j(D)f = \phi_j,\ j = 1, \dots, r,$$

has a polynomial (resp. exponential polynomial) solution f if and only if the following compatibility conditions hold:

$$p_j(D)\phi_k = p_k(D)\phi_j,\ 1 \leqslant j < k \leqslant r.$$

Corollary 5 ([4]). *Let $p_1, \dots, p_s$ be polynomials on K^s such that $V(p_1, \dots, p_s)$ is a finite set. The system of differential equations*

$$p_j(D)f = \phi_j,\ j = 1, \dots, s,$$

where the ϕ_j are exponential polynomials, has a solution in the space of exponential polynomials if and only if the following compatibility conditions hold:

$$p_j(D)\phi_k = p_k(D)\phi_j,\ 1 \leqslant j < k \leqslant s.$$

A generalization of Theorem 3.14 is

Theorem 3.15. *Let $p_1, \dots, p_r$ $(r \leqslant s)$ be polynomials on K^s. Assume that the variety $V(p_1, \dots, p_r)$ is either empty or the minimum dimension of its irreducible components is $s - r$. Then for given polynomials (resp. exponential polynomials) $\phi_1, \dots, \phi_r$, the system of differential equations*

$$p_j(D)f = \phi_j,\ j = 1, \dots, r, \tag{3.44}$$

has a polynomial (resp. exponential polynomial) solution f if and only if the following compatibility conditions hold

$$p_j(D)\phi_k = p_k(D)\phi_j,\ 1 \leqslant j < k \leqslant r.$$

Proof. Let S be the linear space of all exponential polynomials on K^s and $G := G_{\prod(K^s)}(D)$. Because of $e_\theta \prod(K^s)$ is invariant under G, so it is sufficiently to consider $\phi_j \in e_\theta \prod(K^s)$ for some $\theta = (\theta_1, \dots, \theta_s) \in K^s$. Let $M = e_\theta \prod(K^s)$, $\phi_1, \dots, \phi_r \in M$, we consider the solvability of the system (3.44) in M. If $p_j(\theta) \neq 0$ for some j, then $p_j(D)$ is invertible on M. Hence, the solvability of (3.44) follows immediately from Theorem 2.1.

If $p_j(\theta) = 0$ for all $j = 1, \dots, r$, then $\theta \in V(p_1, \dots, p_r)$. By applying Corollary 5 to $V(p_1, \dots, p_r)$, there exists polynomials $p_{r+1}, \dots, p_s$ of degree 1 such that they vanish at θ and the set

$$V(p_1, \dots, p_r, p_{r+1}, \dots, p_s)$$

is finite. In particular, all p_j are exactly $p_j(D)$, $j = 1, \dots, s$. Thus, an application of Theorem 2.2 gives the disired result. □

Let $V \subseteq K^s$ be an algebraic variety. A subset $U \subseteq V$ is said to be closed if U itself is an algebraic variety. A subset $O \subset V$ is said to be open if $V \backslash O$ is closed in V. Let O be a nonempty open set of an irreducible algebraic variety V. If a polynomial $f \in K[X_1, \dots, X_s]$ vanishes on O, then it must be vanished on V, (if otherwise, $V =$

$(V\backslash O)\bigcup V_f$ gives a decomposition of V with $V\backslash O \neq V$ and $V_f \neq V$. It contradicts to the irreducibility of V). Thus, $I(O) = I(V)$. Hence, the coordinate ring $K[O]$ is the same as the coordiate ring $K[V]$. Consequently, the quotient field $K(O)$ of $K[O]$ is the same as $K(V)$. In particular, this shows that dim(O) = dim (V) for any nonempty open subset O of an irreducible variety V. Moreover, if V is irreducible, then $K[O] = K[V]$ is an integral domain, hence O is also irreducible.

Let V be an algebraic variety and let $V = U_1\bigcap\cdots\bigcup U_m$ be a decomposition of V into its irreducible components. For an open subset $O \subseteq V$, $O\bigcap U_j$ is open in U_j for each $j = 1,\ldots,m$. If $O\bigcap U_j$ is nonempty, then $O\bigcap U_j$ is an irreducible component of O. Thus, after discarding some possible empty sets,

$$O = (O\bigcap U_1)\bigcup\cdots\bigcup(O\bigcap U_m),$$

which gives a decomposition of O into its irreducible components.

Theorem 3.16 ([4]). *Let $O \subseteq K^s$ be an open subset of an algebraic variety with all irreducible components having the same dimension n. Then for any $\theta \in O$ there exist n polynomials $p_1,\ldots,p_n$ of degree 1 such that they vanish at θ and such that $O\bigcap V(p_1,\ldots,p_n)$ is a finite set.*

Our generalization as follows

Theorem 3.17. *Let $O \subseteq K^s$ be an open subset of an algebraic variety with the minimum dimension of all irreducible components is n. Then for any $\theta \in O$ there exist n polynomials $p_1,\ldots,p_n$ of degree 1 such that they vanish at θ and such that $O\bigcap V(p_1,\ldots,p_n)$ is a finite set.*

Finally, we recall the theorem about difference equations.

Theorem 3.18 ([4]). *Let $p_1,\ldots,p_r$ $(r \leqslant s)$ be polynomials on K^s. Assume that the intersection of the variety $V(p_1,\ldots,p_r)$ with $(K\backslash\{0\})^s$ is either empty or each of its irreducible components has dimension $s-r$. Then for given polynomial sequences (resp. exponential polynomial sequences) $\phi_1,\ldots,\phi_r$, the system of difference equations*

$$p_j(\tau)f = \phi_j,\ j = 1,\ldots,r,$$

has a polynomial sequence (resp. exponential polynomial sequence) solution f if and only if the following compatibility conditions hold

$$p_j(\tau)\phi_k = p_k(\tau)\phi_j,\ 1 \leqslant j < k \leqslant r.$$

The following is our generalization

Theorem 3.19. *Let $p_1,\ldots,p_r$ $(r \leqslant s)$ be polynomials on K^s. Assume that the intersection of the variety $V(p_1,\ldots,p_r)$ with $(K\backslash\{0\})^s$ is either empty or the minimum dimension of its irreducible components is $s-r$. Then for given polynomial sequences (resp. exponential polynomial sequences) $\phi_1,\ldots,\phi_r$, the system of difference equations*

$$p_j(\tau)f = \phi_j,\ j = 1,\ldots,r,$$

has a polynomial sequence (resp. exponential polynomial sequence) solution f if and only if the following compatibility conditions hold

$$p_j(\tau)\phi_k = p_k(\tau)\phi_j,\ 1 \leqslant j < k \leqslant r.$$

Proof. Let S be the linear space of all exponential polynomial $K-$ sequences on $\mathbb{Z}^s$. Since $\theta^0 \prod(K^s)$ is invariant under G, then without loss of generality, we may assume

$$\phi_1, \ldots, \phi_r \in \theta^0 \prod(K^s) := M.$$

If $p_j(\theta) \neq 0$ for some j, then $p_j(\theta)$ is invertible on M, so the proof is implied by Theorem 2.1. Suppose that $p_j(\theta) = 0$ for all $j = 1, \ldots, r$. Then $\theta \in V(p_1, \ldots, p_r)$. The minimum dimension of irreducible components of $V(p_1, \ldots, p_r) \bigcap (K \backslash \{0\})^s$ is $s - r$. By Corollary 5, ther exist polynomials $p_{r+1}, \ldots, p_s$ of degree 1 such that they vanish at θ and the set

$$V(p_1, \ldots, p_r, p_{r+1}, \ldots, p_s) \bigcap (K \backslash \{0\})^s$$

is finite. By using Theorem 2.2, the proof is complete. □

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