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Global Existence for a Quasilinear System Arising in Shape Memory Alloys

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1 Introduction

This paper is based on the result of [12]. We consider the following initial-boundary value problem in quasi-linear thermoelasticity $(TE)_n$:

$$\mathbf{u}_{tt} + \Delta^2\mathbf{u} - \nu\Delta\mathbf{u}_t = \nabla\cdot(G(\theta)H_{,\nabla\mathbf{u}}(\nabla\mathbf{u}) + \overline{H}_{,\nabla\mathbf{u}}(\nabla\mathbf{u})), \tag{1.1}$$

$$[1-\theta G''(\theta)H(\nabla\mathbf{u})]\theta_t - \Delta\theta = \theta G'(\theta)\partial_t H(\nabla\mathbf{u}) + \nu|\nabla\mathbf{u}_t|^2 \quad \text{in } \Omega_T, \tag{1.2}$$

$$\mathbf{u} = \Delta\mathbf{u} = \nabla\theta\cdot\mathbf{n} = 0 \quad \text{on } S_T, \tag{1.3}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \mathbf{u}_1, \quad \theta(0,\cdot) = \theta_0 \geq 0 \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\Omega_T := (0,T)\times\Omega$, $S_T = [0,T)\times\partial\Omega$, and $\mathbf{n}$ is unit outward normal to $\partial\Omega$. Let $\mathbf{u} = (u_i) \in \mathbb{R}^n$ denote the displacement vector, θ the absolute temperature and $F \in \mathbb{R}$ is called the elastic energy density.

We use the following notation

$$f_t = \frac{\partial f}{\partial t}, \quad f_j = \frac{\partial f}{\partial x_j}, \quad \nabla\mathbf{u} = (u_{i,j}), \quad F_{,\nabla\mathbf{u}} = \left(\frac{\partial F}{\partial u_{i,j}}\right)$$

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

In this article, we consider the following structure of the elastic energy density: (A) $G(\theta)$, $H(\nabla\mathbf{u})$ and $\overline{H}(\nabla\mathbf{u})$ satisfy the following conditions.

(i) $G \in C^3(\mathbb{R},\mathbb{R})$ is as follows:

$$G(\theta) = \begin{cases} C_1\theta & \text{if } \theta\in[0,\theta_1] \\ \varphi(\theta) & \text{if } \theta\in[\theta_1,\theta_2] \\ C_2\theta^r & \text{if } \theta\in[\theta_2,\infty), \end{cases}$$

where $\varphi \in C^3(\mathbb{R}, \mathbb{R})$, $\varphi'' \leq 0$ and C_1 and C_2 are positive constants for some fixed θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < \infty$. We extend G defined on $\mathbb{R}$ as an odd function.

(ii) $H \in C^3(\mathbb{R}^{n^2}, \mathbb{R})$ satisfies that $H(\nabla \mathbf{u}) \geq 0$, where $\mathbb{R}^{n^2}$ denotes the set of symmetric second order tensors in $\mathbb{R}^d$.

(iii) $\overline{H} \in C^3(\mathbb{R}^{n^2}, \mathbb{R})$ satisfies that $\overline{H}(\nabla \mathbf{u}) \geq -C_3$, where C_3 are some real number.

(iv) $H(\nabla \mathbf{u})$ and $\overline{H}(\nabla \mathbf{u})$ satisfy the following growth conditions:

$$
\begin{aligned}
&|H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_1-1}, && |\overline{H}_{,\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_2-1},\\
&|H_{,\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_1-2}, && |\overline{H}_{,\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_2-2},\\
&|H_{,\nabla\mathbf{u}\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_1-3}, && |\overline{H}_{,\nabla\mathbf{u}\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq C|\nabla\mathbf{u}|^{K_2-3}
\end{aligned}
$$

for large $|\nabla \mathbf{u}|$.

Here we note that the regularity assumption for $H(\nabla \mathbf{u})$ and $\overline{H}(\nabla \mathbf{u})$ assures that there exists a positive constant M such that

$$
\begin{aligned}
&|H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})| + |H_{,\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| + |H_{,\nabla\mathbf{u}\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})|\\
&\quad + |\overline{H}_{,\nabla\mathbf{u}}(\nabla\mathbf{u})| + |\overline{H}_{,\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| + |\overline{H}_{,\nabla\mathbf{u}\nabla\mathbf{u}\nabla\mathbf{u}}(\nabla\mathbf{u})| \leq M
\end{aligned}
$$

for small $|\nabla \mathbf{u}|$.

For the related results, we refer to [11] and [12]. Our main result of this paper is as follows.

Theorem 1.1. *(i) Let $5 < p \leq q < \infty$. The exponents r, K_1 and K_2 satisfy the following conditions*

$$
0 \leq r < \frac{5}{6}, \qquad 0 \leq K_1, K_2 < 6, \qquad 6r + K_1 < 6. \tag{1.5}
$$

Then, for any $T > 0$ and $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times B^{2-2/q}_{q,q} =: U(p,q)$, there exists at least one solution $(\mathbf{u}, \theta)$ to (1.1)–(1.4) satisfying

$$
(\mathbf{u}, \theta) \in W^{4,2}_p(\Omega_T) \times W^{2,1}_q(\Omega_T) =: V_T(p,q).
$$

Moreover, if we assume $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$
\theta \geq \theta_* \exp(-\omega t) \qquad \text{in } \Omega_T.
$$

(ii) Let $4 < p \leq q < \infty$ and assume that

$$
0 \leq r < 1, \qquad 0 \leq K_1, K_2 < \infty. \tag{1.6}
$$

Then for the two-dimensional system $(TE)_2$ the same conclusion as in (i) holds.

Here, we have used and will be used the following function spaces.

- $L^p(\Omega_T) = L^p_T L^p = L^p(0,T;L^p(\Omega))$ is the standard Lebesgue space. We often use the notation $L^p(\Omega_I) = L^p_I L^p$ for some interval I.

- $W^{2l,l}_p(\Omega_T)$ is the Sobolev space equipped with the norm

$$\|u\|_{W^{2l,l}_p(\Omega_T)} := \sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \|D^r_t D^\alpha_x u\|_{L^p(\Omega_T)},$$

where $D_t := i\frac{\partial}{\partial t}$, $D^\alpha_x = \prod_{\alpha=\alpha_1+\alpha_2+\alpha_3} D^{\alpha_k}_k$ and $D_k := i\frac{\partial}{\partial x_k}$ for multi index $\alpha = (\alpha_i)^n_{i=1}$.

- $H^j(\Omega) := W^j_2(\Omega)$, where W^j_p is the Sobolev space equipped with the norm $\|u\|_{W^j_p(\Omega)} := \sum_{|\alpha|\le j} \|D^\alpha_x u\|_{L^p(\Omega)}$.

- $B^s_{p,q} = B^s_{p,q}(\Omega)$ is the Besov space. Namely, $B^s_{p,q} := [L^p(\Omega), W^j_p(\Omega)]_{s/j,q}$, where $[X,Y]_{s/j,q}$ is the real interpolation space. For more details we refer to [1] by Adams and Fournier.

- $C^{\alpha,\alpha/2}(\Omega_T)$ is the Hölder space: the set of all continuous functions in Ω_T satisfying Hölder condition in x with exponent α and in t with exponent $\alpha/2$.

For completeness we recall also the uniqueness result which follows by repeating the arguments of the corresponding result in [9, Section 6]

Theorem 1.2. *In addition to assumptions of Theorem 1.1, suppose that* $F(\nabla \mathbf{u},\theta) \in C^4(\mathbb{R}^{n^2}\times\mathbb{R}^+,\mathbb{R})$. *Then the solution* $(\mathbf{u},\theta) \in V_T(p,q)$ *to* (1.1)–(1.4) *constructed above is unique.*

We prove Theorem 1.1 by using the Leray-Schauder fixed point principle. The key estimates are the maximal regularity estimate for (1.1), and the classical energy estimate and the parabolic De Giorgi method for (1.2). In general, the derivative of a solution is less regular than the right-hand side of the corresponding equation. However, for parabolic equations such a loss of regularity does not occur, as in the case of elliptic equations. The estimate ensuring this regularity is called the maximal regularity. For more precise information on the maximal regularity, we refer to [2] and for more recent topics of the maximal L^p-regularity we refer to [4]. Since the maximal regularity theory is limited to linear parabolic equations, we cannot use it directly for the quasilinear equation (1.2). To obtain the higher order a priori estimates we also use the classical energy methods and the parabolic De Giorgi method (see [6], [7]). Using these methods we can show the Hölder continuity of θ. By virtue of such regularity, we arrive at the estimate in higher Sobolev norm.

Throughout this paper C and Λ are positive constants independent of time T and depending on time T, respectively. In particular, we may use Λ instead of $\Lambda(\|(u_0, u_1, \theta_0)\|_X)$ for some X if there is no danger of confusion.

Remark. We can obtain the same result for the system replacing Δ in (1.1) with Q defined by

$$Q\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}),$$

where correspondingly we have to replace $\nabla\mathbf{u}$ on the system with the shear strain tensor $\epsilon = (\nabla\mathbf{u} + {}^T\nabla\mathbf{u})/2$ (see [12]).

2 Preliminaries

In this section, we present some auxiliary results which will be used in the subsequent sections.

Lemma 2.1 (Maximal Regularity). (i) *Let $p \in (1, \infty)$. Denote by $\mathbf{u}$ the solution of the linear problem*

$$\begin{cases} \mathbf{u}_{tt} + \Delta^2\mathbf{u} - \nu\Delta\mathbf{u}_t = \nabla \cdot f & \text{in } \Omega_T, \\ \mathbf{u} = \Delta\mathbf{u} = 0 & \text{on } S_T, \\ \mathbf{u}(0,\cdot) = u_0, \quad \mathbf{u}_t(0,\cdot) = u_1 & \text{in } \Omega. \end{cases}$$

Then the following estimates hold

$$\|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)} \le C(\|\mathbf{u}_0\|_{B_{p,p}^{4-\frac{2}{p}}} + \|\mathbf{u}_1\|_{B_{p,p}^{2-\frac{2}{p}}} + \|\nabla \cdot f\|_{L^p(\Omega_T)}) \tag{2.1}$$

for any $(\mathbf{u}_0, \mathbf{u}_1) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p}$ and $\nabla \cdot f \in L^p(\Omega_T)$, and

$$\|\nabla\mathbf{u}\|_{W_p^{2,1}(\Omega_T)} \le C(\|\mathbf{u}_0\|_{B_{p,p}^{3-\frac{2}{p}}} + \|\mathbf{u}_1\|_{B_{p,p}^{1-\frac{2}{p}}} + \|f\|_{L^p(\Omega_T)}) \tag{2.2}$$

for any $(\mathbf{u}_0, \mathbf{u}_1) \in B_{p,p}^{3-2/p} \times B_{p,p}^{1-2/p}$ and $f \in L^p(\Omega_T)$.

(ii) *Let $q \in (1, \infty)$. Assume that $\rho(x)$ is Hölder continuous in $\overline{\Omega}$ such that $\inf_\Omega \rho > 0$. Denote by θ the solution of the linear problem*

$$\begin{cases} \theta_t - \rho\Delta\theta = g & \text{in } \Omega_T, \\ n \cdot \nabla\theta = 0 & \text{on } S_T, \\ \theta(0,x) = \theta_0(x) & \text{in } \Omega. \end{cases}$$

Then the following estimate holds

$$\|\theta\|_{W_q^{2,1}(\Omega_T)} \le C(\|\theta_0\|_{B_{q,q}^{2-\frac{2}{q}}} + \|g\|_{L^q(\Omega)}) \tag{2.3}$$

for any $\theta_0 \in B_{q,q}^{2-2/q}$, where C depends on $\inf_\Omega \rho$.

For the proof of (i) we refer to [10, Lemma 2.1, Proposition 2.4], and (ii) is the particular case of [5, 3.2 Examples A), 2)]. Next, we recall the useful space-time embedding lemma.

Lemma 2.2 (Embedding [6, Lemma II.3.3]). *Let $f \in W_p^{2l,l}(\Omega_T)$. Then, for $l \in \mathbb{Z}^+$ and multi index α, it follows that*

$$\|D_t^r D_x^\alpha f\|_{L^q(\Omega_T)} \le C\delta^{l-\psi}\|f\|_{W_p^{2l,l}(\Omega_T)} + C\delta^{-\psi}\|f\|_{L^p(\Omega_T)}, \tag{2.4}$$

provided $q \ge p$ and $\psi := r + \frac{|\alpha|}{2} + \frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \le l$. If $\varphi := r + \frac{|\alpha|}{2} + \frac{n+2}{2p} < l$, then

$$\|D_t^r D_x^\alpha f\|_{L^\infty(\Omega_T)} \le C\delta^{l-\varphi}\|f\|_{W_p^{2l,l}(\Omega_T)} + C\delta^{-\varphi}\|f\|_{L^p(\Omega_T)}, \tag{2.5}$$

moreover, $D_t^r D_x^\alpha f$ is Hölder continuous. Here, $\delta \in (0, \min(T, \zeta^2)]$, ζ is the altitude of the cone in the statement of the cone condition satisfied by Ω.

Lemma 2.3. *Let φ be given in (A)-(i). Then the function $\varphi(s)$ satisfies*

$$\varphi(s) - s\varphi'(s) \ge 0 \tag{2.6}$$

for any $s \in [\theta_1, \theta_2]$

Proof. Putting $f(s) = \varphi(s) - s\varphi'(s)$, we have $f'(s) = -s\varphi''(s) \ge 0$ and $f(\theta_1) = 0$. Then $f(s) = \varphi(s) - s\varphi'(s) \ge 0$ in $[\theta_1, \theta_2]$. □

To show Theorem 1.1 we apply the Leray-Schauder fixed point principle. We recall it here in one of its equivalent formulations for the reader's convenience .

Theorem 2.4 (Leray-Schauder Fixed Point Principle [3]). *Let X be a Banach space. Assume that $\Phi : [0,1] \times X \to X$ is a map with the following properties.*

(L1) For any fixed $\tau \in [0,1]$ the map $\Phi(\tau, \cdot) : X \to X$ is compact.

(L2) For every bounded subset $\mathcal{B}$ of X, the family of maps $\Phi(\cdot, \xi) : [0,1] \to X$, $\xi \in \mathcal{B}$, is uniformly equicontinuous.

(L3) $\Phi(0, \cdot)$ has precisely one fixed point in X.

(L4) There is a bounded subset $\mathcal{B}$ of X such that any fixed point in X of $\Phi(\tau, \cdot)$ is contained in $\mathcal{B}$ for every $0 \le \tau \le 1$.

Then $\Phi(1, \cdot)$ has at least one fixed point in X.

3 Proof of Theorem 1.1 (Existence)

We only prove the existence theorem in three-dimensional case. We apply Theorem 2.4 to the map Φ_τ from $V_T(p,q)$ into $V_T(p,q)$,

$$\Phi_\tau : (\bar{\mathbf{u}}, \bar{\theta}) \to (\mathbf{u}, \theta), \qquad \tau \in [0,1],$$

defined by means of the following initial-boundary value problems:

$$\begin{aligned}
&\mathbf{u}_{tt} + \Delta^2\mathbf{u} - \nu\Delta\mathbf{u}_t = \tau\nabla\cdot[G(\bar{\theta})H_{,\nabla\mathbf{u}}(\nabla\bar{\mathbf{u}}) + \overline{H}_{,\nabla\mathbf{u}}(\nabla\bar{\mathbf{u}})], \\
&\theta_t - \Delta\theta = \tau\left\{\bar{\theta}G''(\bar{\theta})\theta_t H(\nabla\mathbf{u}) + \bar{\theta}G'(\bar{\theta})\partial_t H(\nabla\mathbf{u}) + \nu|\nabla\mathbf{u}_t|^2\right\} && \text{in } \Omega_T, \\
&\mathbf{u} = \Delta\mathbf{u} = \nabla\theta\cdot\mathbf{n} = 0 && \text{on } S_T, \\
&\mathbf{u}(0,\cdot) = \tau\mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \tau\mathbf{u}_1(x), \quad \theta(0,\cdot) = \tau\theta_0 && \text{in } \Omega.
\end{aligned}$$

A fixed point of $\Phi_\tau(1,\cdot)$ in $V_T(p,q)$ is the desired solution of the system $(TE)_3$. Therefore to prove the existence statement it is sufficient to check that the map Φ_τ satisfies assumptions $(L1)$–$(L4)$ of Theorem 2.4. We can check assumptions $(L1)$, $(L2)$ and $(L3)$ in the same way as that in [8, Section 3]. Then it is sufficient to check the assumption $(L4)$, namely, to derive a priori bounds for a fixed point of the solution map Φ_τ. Without loss of generality we may set $\tau = 1$. Hence from now on our purpose is to obtain a priori bounds for $(TE)_3$. To this end we prepare several lemmas.

Lemma 3.1 (Energy Conservation Law). *Assume that $\theta \geq 0$ a.e. in Ω_T, $K_2 \leq 6$ and $6r + K_1 \leq 6$. Then for any $t \in [0,T]$ a smooth solution of (1.1)–(1.4) satisfies*

$$\|\theta(t)\|_{L^1(\Omega)} + \|\mathbf{u}_t(t)\|_{L^2(\Omega)} + \|\Delta\mathbf{u}(t)\|_{L^2(\Omega)} \leq C(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{H^2\times L^2\times L^1}). \tag{3.1}$$

Proof. Multiplying (1.1) by $\mathbf{u}_t$ and integrating the resulting equation with respect to the space variable, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_t\|_{L^2}^2 + \frac{1}{2}\|\Delta\mathbf{u}\|_{L^2}^2 + \int_\Omega \overline{H}(\nabla\mathbf{u})dx\right) + \nu\int_\Omega|\nabla\mathbf{u}_t|^2dx + \int_\Omega G(\theta)\partial_t H(\nabla\mathbf{u})dx = 0.$$

Integrating (1.2) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega\theta dx = \nu\int_\Omega|\nabla\mathbf{u}_t|^2dx + \int_\Omega\theta G'(\theta)\frac{\partial}{\partial t}H(\nabla\mathbf{u})dx + \int_\Omega\theta G''(\theta)\theta_t H(\nabla\mathbf{u})dx.$$

Combining these equalities, we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_t\|_{L^2}^2 + \frac{1}{2}\|\Delta\mathbf{u}\|_{L^2}^2 + \int_\Omega\theta dx + \int_\Omega\overline{H}(\nabla\mathbf{u})dx\right) \\
&= \int_\Omega\left(\theta G'(\theta)\frac{\partial}{\partial t}H(\nabla\mathbf{u}) + \theta G''(\theta)\theta_t H(\nabla\mathbf{u}) - G(\theta)\frac{\partial}{\partial t}H(\nabla\mathbf{u})\right)dx \\
&= -\frac{d}{dt}\int_\Omega\overline{G}(\theta)H(\nabla\mathbf{u})dx,
\end{aligned}$$

where $\overline{G}(\theta) = G(\theta) - \theta G'(\theta)$. Consequently, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_t\|_{L^2}^2 + \frac{1}{2}\|\Delta \mathbf{u}\|_{L^2}^2 + \int_\Omega \theta dx + \int_\Omega \overline{H}(\nabla \mathbf{u})dx + \int_\Omega \overline{G}(\theta)H(\nabla \mathbf{u})dx\right) = 0.$$

Here we recall that $\theta \geq 0$ and $H(\nabla \mathbf{u}) \geq 0$. By the structure of $G(\theta)$ the function $\overline{G}(\theta)$ is as follows:

$$\overline{G}(r) = \begin{cases} 0 & \text{if } \theta \in [0, \theta_1], \\ \varphi(\theta) - \theta\varphi'(\theta) & \text{if } \theta \in [\theta_1, \theta_2], \\ C_2(1-r)\theta^r & \text{if } \theta \in [\theta_2, \infty). \end{cases}$$

Since from Lemma 2.3 we have $\overline{G}(\theta) \geq 0$. Consequently, it follows from (A)–(iii) that

$$\frac{1}{2}\|\mathbf{u}_t(t)\|_{L^2}^2 + \frac{1}{2}\|\mathbf{u}(t)\|_{H^2}^2 + \|\theta(t)\|_{L^1} \leq \frac{1}{2}\|\mathbf{u}_0\|_{H^2}^2 + \frac{1}{2}\|\mathbf{u}_1\|_{L^2}^2 + \|\theta_0\|_{L^1} + C_3|\Omega|$$
$$+ \int_\Omega |\overline{H}(\nabla \mathbf{u}_0)|dx + \int_{\{\theta_2 \geq \theta_0 \geq \theta_1\}\cap\Omega} [\varphi(\theta_0) - \theta_0\varphi'(\theta_0)]H(\nabla \mathbf{u}_0)dx + C_2(1-r)\int_{\{\theta_0 > \theta_2\}\cap\Omega} \theta_0^r H(\nabla \mathbf{u}_0)dx.$$

Since the smooth function $\varphi(s) - s\varphi'(s)$ is bounded for $s \in [\theta_1, \theta_2]$, we have

$$\begin{aligned}\int_{\{\theta_2 \geq \theta_0 \geq \theta_1\}\cap\Omega} [\varphi(\theta_0) - \theta_0\varphi'(\theta_0)]H(\nabla \mathbf{u}_0)dx &\leq C\int_\Omega |\nabla \mathbf{u}_0|^{K_1}dx \\ &\leq C\|\mathbf{u}_0\|_{H^2}^{K_1}\end{aligned}$$

for $K_1 \leq 6$,

$$\begin{aligned}\int_{\{\theta_0 > \theta_2\}\cap\Omega} \theta_0^r H(\nabla \mathbf{u}_0)dx &\leq C\|\theta_0\|_{L^1}^r \|\nabla \mathbf{u}_0\|_{L^{\frac{K_1}{1-r}}}^{K_1} \\ &\leq C\|\theta_0\|_{L^1}^r \|\mathbf{u}_0\|_{H^2}^{K_1}\end{aligned}$$

for $6r + K_1 \leq 6$ and

$$\int_\Omega |\overline{H}(\nabla \mathbf{u}_0)|dx \leq \|\mathbf{u}_0\|_{H^2}^{K_2}$$

for $K_2 \leq 6$. Hence we conclude the assertion. □

Lemma 3.2. *Assume that $\theta \geq 0$ a.e. in Ω_T and (1.5) holds. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2 =: U_3$, the solution $(\mathbf{u}, \theta)$ to (1.1)–(1.4) satisfies*

$$\|\nabla \mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} + \|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2} \leq \Lambda, \tag{3.2}$$

where Λ depends on T and $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_3}$. Moreover we have

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega_T)} + \|\theta\|_{L^{10/3}(\Omega_T)} \leq \Lambda. \tag{3.3}$$

Proof. Remark that $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{H^2\times L^2\times L^1} \le C\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_3}$ (see [1]). From the Gagliardo-Nirenberg inequality and Lemma 3.1 it follows that

$$\|\nabla \mathbf{u}\|_{L^{5p}(\Omega_T)} \le C\left\| \|\nabla \mathbf{u}\|^{\frac{4}{5}}_{L^6(\Omega)} \|\nabla \mathbf{u}\|^{\frac{1}{5}}_{W^2_p(\Omega)} \right\|_{L^{5p}_T} \le C\|\nabla \mathbf{u}\|^{\frac{1}{5}}_{W^{2,1}_p(\Omega_T)} \tag{3.4}$$

and

$$\begin{aligned} \|\theta\|_{L^{8/3}(\Omega_T)} &\le C\left\| \|\theta\|^{\frac{1}{4}}_{L^1(\Omega)} \|\theta\|^{\frac{3}{4}}_{H^1(\Omega)} \right\|_{L^\infty_T} \\ &\le C\|\theta\|^{\frac{1}{4}}_{L^\infty_T L^1} \|\theta\|^{\frac{3}{4}}_{L^2_T H^1} \\ &\le \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2})^{\frac{3}{4}}. \end{aligned} \tag{3.5}$$

It follows from (3.4) that

$$\|\overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^{16/5}(\Omega_T)} \le \Lambda\|\nabla \mathbf{u}\|^{K_2-1}_{L^{16}(\Omega_T)} \le \Lambda\|\nabla \mathbf{u}\|^{\frac{K_2-1}{5}}_{W^{2,1}_{16}(\Omega_T)} \le \frac{1}{4}\|\nabla \mathbf{u}\|_{W^{2,1}_{16}(\Omega_T)} + \Lambda$$

for $K_2 \in [1,6)$, and

$$\|\overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^{16/5}(\Omega_T)} \le M|\Omega_T|^{\frac{5}{16}} \le \Lambda$$

for $K_2 \in [0,1)$.

We first consider the case of $K_1 \ge 1$. Applying the growth condition and the Young inequality, we have

$$\begin{aligned} \|G(\theta)H_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^{\frac{16}{5}}(\Omega_T)} &\le \|\theta\|^r_{L^{\frac{8}{3}}(\Omega_T)} \|\nabla \mathbf{u}\|^{K_1-1}_{L^{\frac{16(K_1-1)}{5-6r}}(\Omega_T)} \\ &\quad + \sup_{\theta\in[0,\theta_2]} |G(\theta)| \|\nabla \mathbf{u}\|^{K_1-1}_{L^{\frac{16(K_1-1)}{5}}(\Omega_T)} \\ &\le \Lambda\|\theta\|^r_{L^{\frac{8}{3}}(\Omega_T)} \|\nabla \mathbf{u}\|^{K_1-1}_{L^{16}(\Omega_T)} + \Lambda\|\nabla \mathbf{u}\|^{K_1-1}_{L^{16}(\Omega_T)} \end{aligned}$$

for $6r + K_1 \le 6$ (and $K_1 \le 6$). Then we have

$$\begin{aligned} &\|\theta\|^r_{L^{8/3}(\Omega_T)} \|\nabla \mathbf{u}\|^{K_1-1}_{L^{16}(\Omega_T)} + \|\nabla \mathbf{u}\|^{K_1-1}_{L^{16}(\Omega_T)} \\ &\quad \le \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{3r/4} \|\nabla \mathbf{u}\|^{(K_1-1)/5}_{W^{2,1}_{16/5}(\Omega_T)} + \Lambda\|\nabla \mathbf{u}\|^{(K_1-1)/5}_{W^{2,1}_{16/5}(\Omega_T)} \\ &\quad \le \frac{1}{4}\|\nabla \mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{15r}{4(6-K_1)}} + \Lambda \end{aligned}$$

for $6r + K_1 < 6$ (and $K_1 < 6$). From the maximal regularity (2.2) it follows that

$$\begin{aligned} \|\nabla \mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} &\le C\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_3} + C\|G(\theta)H_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^{16/5}(\Omega_T)} \\ &\quad + C\|\overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^{16/5}(\Omega_T)} \\ &\le C\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_3} + \Lambda + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{15r}{4(6-K_1)}}. \end{aligned} \tag{3.6}$$

Next, multiplying (1.2) by θ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \\
&= \int_\Omega \theta^2 G''(\theta)\theta_t H(\nabla\mathbf{u})dx + \int_\Omega \theta^2 G'(\theta)\partial_t H(\nabla\mathbf{u})dx + \nu\int_\Omega \theta|\nabla\mathbf{u}_t|^2 dx \\
&= \int_\Omega G_2'(\theta)\theta_t H(\nabla\mathbf{u})dx + \int_\Omega G_2(\theta)\partial_t H(\nabla\mathbf{u})dx \\
&\quad + 2\int_\Omega \overline{G}_2(\theta)\partial_t H(\nabla\mathbf{u})dx + \nu\int_\Omega \theta|\nabla\mathbf{u}_t|^2 dx \\
&= \frac{d}{dt}\int_\Omega G_2(\theta)H(\nabla\mathbf{u})dx + 2\int_\Omega \overline{G}_2(\theta)\partial_t H(\nabla\mathbf{u})dx + \nu\int_\Omega \theta|\nabla\mathbf{u}_t|^2 dx,
\end{aligned} \tag{3.7}$$

where $G_2(\theta) = \theta^2 G'(\theta) - \overline{G}_2(\theta)$ and $\overline{G}_2(\theta) = 2\int_0^\theta sG'(s)ds$. Noting that

$$G_2(\theta) = \frac{C_2 r(r-1)}{r+1}\theta^{r+1} \le 0 \qquad \text{and} \qquad \overline{G}_2(\theta) = \frac{2C_2 r}{r+1}\theta^{r+1} \qquad \text{for } \theta \ge \theta_2,$$

and

$$\sup_{\theta\in[0,\theta_2]}|G_2(\theta)| + \sup_{\theta\in[0,\theta_2]}|\overline{G}_2(\theta)| =: M < \infty,$$

we have

$$\begin{aligned}
-\int_\Omega G_2(\theta)H(\nabla\mathbf{u})dx &= -\int_{\Omega\cap\{\theta\ge\theta_2\}} G_2(\theta)H(\nabla\mathbf{u})dx - \int_{\Omega\cap\{\theta_1\le\theta\le\theta_2\}} G_2(\theta)H(\nabla\mathbf{u})dx \\
&\ge -M\int_\Omega |H(\nabla\mathbf{u})|dx.
\end{aligned}$$

Hence integrating (3.7) with respect to time variable, we obtain

$$\begin{aligned}
\frac{1}{2}\|\theta\|_{L_T^\infty L^2}^2 + \|\nabla\theta\|_{L^2(\Omega_T)}^2 \le \frac{1}{2}\|\theta_0\|_{L^2}^2 + \|\overline{G}_2(\theta)\partial_t H(\nabla\mathbf{u})\|_{L^1(\Omega_T)} + \nu\|\theta|\nabla\mathbf{u}_t|^2\|_{L^1(\Omega_T)} \\
+ M\sup_{t\in[0,T]}\int_\Omega |H(\nabla\mathbf{u}(t))|dx + \int_\Omega |G_2(\theta_0)H(\nabla\mathbf{u}_0)|dx.
\end{aligned}$$

By (3.4), (3.5) and the assumptions we have

$$\begin{aligned}
\|\theta^{r+1}\partial_t H(\nabla\mathbf{u})\|_{L^1(\Omega_T)} &\le \Lambda\|\theta\|_{L^{8/3}(\Omega_T)}^{r+1}\|\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)}\|\nabla\mathbf{u}\|_{L^{16}(\Omega_T)}^{K_1-1} \\
&\le \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L_T^\infty L_2})^{\frac{3(r+1)}{4}}\|\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)}^{1+\frac{K_1-1}{5}},
\end{aligned}$$

$$\begin{aligned}
\|\theta|\nabla\mathbf{u}_t|^2\|_{L^1(\Omega_T)} &\le C\|\theta\|_{L^{\frac{8}{3}}(\Omega_T)}\|\nabla\mathbf{u}_t\|_{L^{\frac{16}{5}}(\Omega_T)}^2 \\
&\le \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L_T^\infty L_2})^{\frac{3}{4}}\|\nabla\mathbf{u}_t\|_{\frac{16}{5}(\Omega_T)}^2,
\end{aligned}$$

$$\int_\Omega |H(\nabla \mathbf{u}(t))|dx \le C\|\mathbf{u}(t)\|_{H^2}^{K_1} \le \Lambda$$

and

$$\begin{aligned}\|\theta_0^{r+1}H(\nabla \mathbf{u}_0)\|_{L^1(\Omega)} &\le C\|\theta_0\|_{L^2(\Omega)}^{r+1}\|\nabla \mathbf{u}_0\|^{K_1}_{L^{\frac{2K_1}{1-r}}(\Omega)}\\ &\le C\|\theta_0\|_{L^2(\Omega)}^{r+1}\|\mathbf{u}_0\|^{K_1}_{H^2(\Omega)}.\end{aligned}$$

Consequently we arrive at

$$\begin{aligned}\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} &\le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3})\\ &+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3(r+1)}{4}}\|\nabla \mathbf{u}\|^{\frac{4}{5}+\frac{K_1}{5}}_{W^{2,1}_{16/5}(\Omega_T)}\\ &+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3}{4}}\|\nabla \mathbf{u}_t\|^2_{L^{\frac{16}{5}}(\Omega_T)}\end{aligned}\tag{3.8}$$

Substituting (3.6) into (3.8), we have

$$\begin{aligned}\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} &\le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3})\\ &+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3(r+1)}{4}}\left(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3} + \|\nabla\theta\|^{\frac{15r}{4(6-K_1)}}_{L^2(\Omega_T)}\right)^{\frac{4}{5}+\frac{K_1}{5}}\\ &+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3}{4}}\left(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3} + \|\nabla\theta\|^{\frac{15r}{4(6-K_1)}}_{L^2(\Omega_T)}\right)^2.\end{aligned}$$

Here from the assumption $6r + K_1 < 6$ it follows that

$$\frac{3(r+1)}{4} + \frac{15r}{4(6-K_1)}\left(\frac{4}{5}+\frac{q}{5}\right) = \frac{30r + 3(6-K_1)}{4(6-K_1)} < 2,$$
$$\frac{3}{4} + \frac{30r}{4(6-K_1)} < 2.$$

Thus we obtain

$$\|\theta\|_{L^\infty_T L^2} + \|\nabla\theta\|_{L^2(\Omega_T)} \le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3}) + \Lambda\|\nabla\theta\|^{1-}_{L^2(\Omega_T)}.$$

Here we use $p-$ to denote a number less than p. Hence by the Young inequality we have

$$\|\theta\|_{L^\infty_T L^2} + \frac{1}{2}\|\theta\|_{L^2(\Omega_T)} \le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3}).$$

Substituting the above inequality into (3.6), we also obtain the following

$$\|\nabla \mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} \le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3}).$$

Next, we consider the case of $0 \le K_1 \le 1$ and $0 \le r < 5/6$. In this case it follows that

$$|H_{,\nabla\mathbf{u}}(\nabla \mathbf{u})| \le C < \infty.$$

From an argument similar to the above we have

$$\begin{aligned}\|\nabla\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} &\le \|(\mathbf{u}_0,\mathbf{u}_1,0)\|_{U_3} + \|G(\theta)H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^{16/5}(\Omega_T)} \\ &\le \|(\mathbf{u}_0,\mathbf{u}_1,0)\|_{U_3} + C\|\theta\|^r_{L^{\frac{16r}{5}}(\Omega_T)} + C\sup_{\theta\in[0,\theta_2]} G(\theta) \\ &\le \|(\mathbf{u}_0,\mathbf{u}_1,0)\|_{U_3} + \Lambda\|\theta\|^r_{L^{\frac{8}{3}}(\Omega_T)} + C.\end{aligned} \tag{3.9}$$

Noting that

$$\|\theta^{r+1}\partial_t H(\nabla\mathbf{u})\|_{L^1(\Omega_T)} \le \Lambda\|\theta\|^{r+1}_{L^{8/3}(\Omega_T)}\|\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)},$$

we obtain

$$\begin{aligned}\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} &\le \|\theta_0\|^2_{L^2} + \|\theta^{r+1}\partial_t H(\nabla\mathbf{u})\|_{L^1(\Omega_T)} + \|\theta|\nabla\mathbf{u}_t|^2\|_{L^1(\Omega_T)} \\ &\quad + M\sup_{t\in[0,T]}\int_\Omega |H(\nabla\mathbf{u}(t))|dx + \int_\Omega |G_2(\theta_0)H(\nabla\mathbf{u}_0)|dx \\ &\le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3}) + \Lambda\|\theta\|^{r+1}_{L^{8/3}(\Omega_T)}\|\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)} + C\|\theta\|_{L^{8/3}(\Omega_T)}\|\mathbf{u}\|^2_{W^{2,1}_{16/5}(\Omega_T)} \\ &\le \Lambda(\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_3}) + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2})^{3(2r+1)/4}.\end{aligned}$$

Since $3(2r+1)/4 < 2$, we obtain the desired estimate (3.2).

The estimate (3.3) follows with the help of the embeddings

$$\|\nabla\mathbf{u}\|_{L^\infty(\Omega_T)} \le \Lambda\|\nabla\mathbf{u}\|_{W^{2,1}_{16/5}(\Omega_T)}$$

and of the inequality

$$\|\theta\|_{L^{10/3}(\Omega_T)} \le C\left\|\|\theta\|^{2/5}_{L^2(\Omega)}\|\theta\|^{3/5}_{H^1(\Omega)}\right\|_{L^{10/3}_T} \le C\|\theta\|^{2/5}_{L^\infty_T L^2}\|\theta\|^{3/5}_{L^2H^1}.$$

This completes the proof. □

Lemma 3.3. *Assume that $\theta \ge 0$ a.e. in Ω_T and (1.5) holds. Then for any $(\mathbf{u}_0,\mathbf{u}_1,\theta_0) \in B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1 = U_4$ the following estimate holds*

$$\|\nabla\mathbf{u}\|_{W^{2,1}_4(\Omega_T)} + \|\nabla\theta\|_{L^\infty_T L^2} + \|\theta\|_{W^{2,1}_2(\Omega_T)} \le \Lambda,$$

where constant Λ depends on T and $\|(\mathbf{u}_0,\mathbf{u}_1,\theta_0)\|_{U_4}$. Moreover, we have

$$\|\nabla\theta\|_{L^{10/3}(\Omega_T)} + \|\theta\|_{L^{10}(\Omega_T)} + \|\Delta\mathbf{u}\|_{L^{20}(\Omega_T)} \le \Lambda.$$

Proof. Remark that $U_4 \hookrightarrow U_3$. Using (3.3) we have

$$\|G(\theta)H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^4(\Omega_T)} \le \begin{cases}\Lambda\|\theta\|^r_{L^{10/3}(\Omega_T)}\|\nabla\mathbf{u}\|^{K_1-1}_{L^\infty(\Omega_T)} \le \Lambda & \text{if } K_1 \ge 1,\\ \Lambda\sup|H_{,\nabla\mathbf{u}}|\|\theta\|^r_{L^{10/3}(\Omega_T)} \le \Lambda & \text{if } K_1 \le 1,\end{cases} \tag{3.10}$$

for $r \le 5/6$. Then from the maximal regularity (2.2) it follows that

$$\|\nabla \mathbf{u}\|_{W_4^{2,1}} \le C\|(\mathbf{u}_0, \mathbf{u}_1, 0)\|_{U_4} + C\|G(\theta)H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^4} \le \Lambda. \tag{3.11}$$

Multiplying (1.2) by θ_t and integrating over Ω_T, we get

$$\begin{aligned}
\|\theta_t\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|\nabla\theta\|^2_{L^\infty_T L^2} &\le \frac{1}{2}\|\theta_0\|^2_{H^1} + \iint_{\Omega_T} \theta_t^2 \theta G''(\theta) H(\nabla\mathbf{u}) dxdt \\
&\quad + \iint_{\Omega_T} \theta_t \theta G'(\theta)\partial_t H(\nabla\mathbf{u}) dxdt + \iint_{\Omega_T} \theta_t |\nabla \mathbf{u}_t|^2 dxdt \\
&\le \frac{1}{2}\|\theta_0\|^2_{H^1} + C\|\theta_t\|_{L^2(\Omega_T)}\|\theta^r H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^4} + C\|\theta_t\|_{L^2}\|\nabla\mathbf{u}_t\|^2_{L^4} \\
&\le \frac{1}{2}\|\theta_0\|^2_{H^1} + \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_4})\|\theta_t\|_{L^2(\Omega_T)} \\
&\le \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_4}) + \frac{1}{2}\|\theta_t\|^2_{L^2(\Omega_T)},
\end{aligned}$$

where we applied (3.10) and (3.11). Therefore we arrive at

$$\|\nabla\mathbf{u}\|_{W_4^{2,1}(\Omega_T)} + \|\theta_t\|_{L^2(\Omega_T)} + \|\nabla\theta\|_{L^\infty_T L^2} \le \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_4}). \tag{3.12}$$

Next multiplying (1.2) by $\frac{-\Delta\theta}{1-\theta G''(\theta)H(\nabla\mathbf{u})}$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2_{L^2} + \int_\Omega \frac{|\Delta\theta|^2}{1-\theta G''(\theta)H(\nabla\mathbf{u})} dx \\
&\le \int_\Omega \frac{\Delta\theta}{1-\theta G''(\theta)H(\nabla\mathbf{u})}\left(\theta G'(\theta)\partial_t H(\nabla\mathbf{u}) + \nu|\nabla\mathbf{u}_t|^2\right) dx.
\end{aligned}$$

Here we recall that

$$1 \le 1 - \theta G''(\theta)H(\nabla\mathbf{u}) \le 1 + M\Lambda,$$

where $0 \le \sup_{\theta\ge 0}(-\theta G''(\theta)) =: M < \infty$. Then integrating with respect to time variable, we conclude that

$$\begin{aligned}
&\|\nabla\theta(t)\|^2_{L^2} + \frac{2}{1+\Lambda M}\|\Delta\theta\|^2_{L^2(\Omega_T)} \\
&\le \|\nabla\theta_0\|^2_{L^2} + \frac{1}{1+\Lambda M}\|\Delta\theta\|^2_{L^2(\Omega_T)} + (1+\Lambda M)\left\|\theta G'(\theta)\partial_t H(\nabla\mathbf{u}) + |\nabla\mathbf{u}_t|^2\right\|^2_{L^2(\Omega_T)} \\
&\le \Lambda + \frac{1}{1+\Lambda M}\|\Delta\theta\|^2_{L^2(\Omega_T)} + \Lambda\|\theta^r H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^4(\Omega_T)}\|\nabla\mathbf{u}_t\|_{L^4(\Omega_T)} + \Lambda\|\nabla\mathbf{u}_t\|^2_{L^4(\Omega_T)} \\
&\le \Lambda + \frac{1}{2(1+\Lambda M)}\|\Delta\theta\|^2_{L^2(\Omega_T)}
\end{aligned}$$

due to (3.10) and (3.11). Consequently we obtain the first assertion.

With the help of Lemma 2.2, we also obtain

$$\|\nabla\theta\|_{L^{10/3}(\Omega_T)} + \|\theta\|_{L^{10}(\Omega_T)} + \|\Delta\mathbf{u}\|_{L^{20}(\Omega_T)} \le \Lambda(\|\theta\|_{W_2^{2,1}(\Omega_T)} + \|\nabla\mathbf{u}\|_{W_4^{2,1}(\Omega_T)}) \le \Lambda,$$

which completes the proof. □

Lemma 3.4. *Let $p \in [20/9, 10/3]$ and assume that $\theta \geq 0$ a.e. in Ω_T and (1.5) holds. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1 =: U_5(p)$, the solution $(\mathbf{u}, \theta)$ to (1.1)–(1.4) satisfies*

$$\|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)} \leq \Lambda,$$

where Λ depends on T and $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_5(p)}$.

Proof. Since the embedding $B_{p,p}^{4-\frac{2}{p}} \hookrightarrow B_{4,4}^{\frac{5}{2}}$ holds for any $\frac{20}{9} \leq p$, by the Lemma 3.3 we have

$$\begin{aligned}\|\nabla \mathbf{u}\|_{W_4^{2,1}(\Omega_T)} + \|\theta\|_{W_2^{2,1}(\Omega_T)} &\leq \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{B_{4,4}^{5/2} \times B_{4,4}^{1/2} \times H^1}) \\ &\leq \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1}).\end{aligned}$$

For any $p \leq \frac{10}{3}$ we have

$$\begin{aligned}\|\nabla \cdot (G(\theta) H_{,\nabla \mathbf{u}}(\nabla \mathbf{u}))\|_{L^p(\Omega_T)} &\leq \Lambda \|\nabla\theta\|_{L^{10/3}(\Omega_T)} \|G'(\theta)\|_{L^\infty(\Omega_T)} \|H_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \\ &\quad + \Lambda \|\theta\|^r_{L^{10}(\Omega_T)} \|\Delta \mathbf{u}\|_{L^{20}(\Omega_T)} \|H_{,\nabla \mathbf{u}\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \\ &\leq \Lambda\end{aligned}$$

and

$$\|\nabla \cdot \overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^p(\Omega_T)} \leq \Lambda \|\Delta \mathbf{u}\|_{L^{20}(\Omega_T)} \|\overline{H}_{,\nabla \mathbf{u}\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \leq \Lambda,$$

thanks to Lemmas 3.2 and 3.3. Then by the maximal regularity (2.1) we have

$$\begin{aligned}\|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)} &\leq C\|(\mathbf{u}_0, \mathbf{u}_1, 0)\|_{U_5(p)} + C\|\nabla \cdot (G(\theta) H_{,\nabla \mathbf{u}}(\nabla \mathbf{u}))\|_{L^p(\Omega_T)} \\ &\quad + C\|\nabla \cdot \overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^p(\Omega_T)}) \\ &\leq \Lambda.\end{aligned}$$

This completes the proof. □

Lemma 3.5. *Let $l > 2$ be integer and $p \in (1, \infty)$. Assume that $\theta \geq 0$ a.e. in Ω_T and (1.5) holds. Then for any $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in B_{10/3,10/3}^{17/5} \times B_{10/3,10/3}^{7/5} \times (L^l \cap H^1) =: U_6(l)$, the solution $(\mathbf{u}, \theta)$ to (1.1)–(1.4) satisfies*

$$\|\theta\|_{L_T^\infty L_x^l} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(\mathbf{u}_0, u_1, \theta_0)\|_{U_6(l)})$. Moreover, if $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U_6(\infty)$ we have

$$\|\theta\|_{L^\infty(\Omega_T)} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_6(\infty)})$, and for $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in (B_{p,p}^{3-2/p} \cap B_{10/3,10/3}^{17/5}) \times (B_{p,p}^{1-2/p} \cap B_{10/3,10/3}^{7/5}) \times (L^\infty \cap H^1) =: U_7(p)$ it holds that

$$\|\nabla \mathbf{u}\|_{W_p^{2,1}(\Omega_T)} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U_7(p)})$

Proof. We can deduce that

$$\frac{1}{l}\frac{d}{dt}\|\hat{\theta}\|_{L^l}^l+(l-1)\int_\Omega \theta^{l-2}|\nabla\theta|^2dx=\int_\Omega \overline{G}_l(\theta)\partial_t H(\nabla\mathbf{u})dx+\nu\int \theta^{l-1}|\nabla\mathbf{u}_t|^2dx, \quad (3.13)$$

where we set $G_l(\theta)=\theta^l G'(\theta)-\overline{G}_l(\theta)$, $\overline{G}_l(t)=l\int_0^\theta s^{l-1}G'(s)ds$ and

$$\hat{\theta}=\theta\left(1-\frac{lG_l(\theta)H(\nabla\mathbf{u})}{\theta^l}\right)^{1/l}\geq\theta. \quad (3.14)$$

Since $\|H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^\infty(\Omega_T)}=\Lambda<\infty$ from (3.3), we have

$$\begin{aligned}\left|\int_\Omega \overline{G}_l(\theta)\partial_t H(\nabla\mathbf{u})dx\right| &\leq C\|\theta^{l-1}\|_{L^1(\Omega)}\|\theta\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}_t\|_{L^\infty(\Omega)}\|H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^\infty(\Omega)}\\ &\leq \Lambda\|\theta\|_{L^l(\Omega)}^{l-1}\|\theta\|_{H^2(\Omega)}\|\nabla\mathbf{u}_t\|_{L^\infty(\Omega)}.\end{aligned}$$

Therefore, we conclude from (3.13) that

$$\frac{1}{l}\frac{d}{dt}\|\hat{\theta}\|_{L^l(\Omega)}^l\leq\Lambda\|\nabla\mathbf{u}_t\|_{L^\infty(\Omega)}\|\theta\|_{H^2(\Omega)}\|\theta\|_{L^l(\Omega)}^{l-1}+C\|\nabla\mathbf{u}_t\|_{L^\infty(\Omega)}^2\|\theta\|_{L^l(\Omega)}^{l-1}. \quad (3.15)$$

Here note that $\partial_t\|\hat{\theta}\|_{L^l(\Omega)}^l=l\|\hat{\theta}\|_{L^l(\Omega)}^{l-1}\partial_t\|\hat{\theta}\|_{L^l(\Omega)}$ and that from the Sobolev embedding and Lemma 3.4

$$\begin{aligned}\|\nabla\mathbf{u}_t\|_{L_T^2L^\infty}\leq\Lambda\|\nabla\mathbf{u}_t\|_{L_T^2W_{10/3}^1}\leq\Lambda\|u\|_{W_{10/3}^{4,2}(\Omega_T)}\leq\Lambda,\\ \|\theta\|_{L_T^2H^2}\leq\|\theta\|_{W_2^{2,1}(\Omega_T)}\leq\Lambda,\end{aligned}$$

where Λ is independent of l. Thus, integrating (3.15) with respect to time variable, we obtain

$$\begin{aligned}\|\hat{\theta}\|_{L_T^\infty L^l}&\leq\|\hat{\theta}_0\|_{L^l}+\Lambda\|\nabla\mathbf{u}_t\|_{L_T^2L^\infty}\|\theta\|_{L_T^2H^2}+\Lambda\|\nabla\mathbf{u}_t\|_{L_T^2L^\infty}^2\\ &\leq\Lambda+\|\hat{\theta}_0\|_{L^l}\end{aligned}$$

Since we have $\hat{\theta}_0\leq\theta_0(1+lM\Lambda)^{1/l}$, the desired result can be obtained. For the $W_p^{2,1}$-norm of $\nabla\mathbf{u}$, we have

$$\begin{aligned}\|\nabla\mathbf{u}\|_{W_p^{2,1}(\Omega_T)}\leq C\|(\mathbf{u}_0,\mathbf{u}_1,0)\|_{U_7(p)}+\Lambda\|\theta\|_{L^\infty(\Omega_T)}^r\|H_{,\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^\infty(\Omega_T)}\\ +\Lambda\|\overline{H}_{\nabla\mathbf{u}}(\nabla\mathbf{u})\|_{L^\infty(\Omega_T)}\leq\Lambda\end{aligned}$$

for $p\in(1,\infty)$, by virtue of the maximal regularity (2.2). This completes the proof. □

The same procedure as in [8, Section 6] yields that $\theta\in C^{\alpha,\alpha/2}(\overline{\Omega_T})$ for some Hölder exponent $0<\alpha<1$ depending on T, $\sup_\Omega\theta_0$ and $\|\theta\|_{L^\infty(\Omega_T)}$. Essentially the proof relies on the classical parabolic De Giorgi method. For more precise information of this method we refer to [6, Chapter II, §7] and [7, Chapter VI, §12]. Here we note that $\nabla\mathbf{u}$ is Hölder continuous because of Lemma 2.2.

Lemma 3.6 ([8, Lemma 6.1]). *Assume that* $k = \sup_\Omega \theta_0 < \infty$. *Suppose that*

$$\|\nabla \mathbf{u}\|_{W_s^{2,1}(\Omega_T)} + \|\theta\|_{W_2^{2,1}(\Omega_T)} + \|\theta\|_{L^\infty(\Omega_T)} \leq \Lambda \tag{3.16}$$

holds for any $s \in (1, \infty)$. *Then* $\theta \in \mathrm{C}^{\alpha,\alpha/2}(\overline{\Omega_T})$ *with Hölder exponent* $\alpha \in (0,1)$ *depending on* Λ *and* k.

Lemma 3.7. *Assume that* (3.16) *holds. Then for any* $(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in U(p,q)$ *and* $5 < p, q < \infty$ *we have*

$$\|(\mathbf{u}, \theta)\|_{V_T(p,q)} = \|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)} + \|\theta\|_{W_q^{2,1}(\Omega_T)} \leq \Lambda,$$

where Λ *depends on* $\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_{U(p,q)}$ *and* T.

Proof. By using Lemma 3.6 we have θ is Hölder continuous. For brevity of notation we denote $1 - \theta G''(\theta) H(\nabla \mathbf{u})$ by $c_0(\nabla \mathbf{u}, \theta)$, and $\theta G'(\theta)\partial_t H(\nabla \mathbf{u}) + \nu |\nabla \mathbf{u}_t|^2$ by $R(\nabla \mathbf{u}, \theta)$. Then the equation (1.2) can be rewritten as

$$c_0(\nabla \mathbf{u}_0, \theta_0)\theta_t - \Delta\theta = (c_0(\nabla \mathbf{u}_0, \theta_0) - c_0(\nabla \mathbf{u}, \theta))\theta_t + R(\nabla \mathbf{u}, \theta).$$

It follows from the assumptions that

$$\begin{aligned} \|R(\nabla \mathbf{u}, \theta)\|_{L^q(\Omega_T)} &\leq C\|\theta\|^r_{L^\infty(\Omega_T)} \|H_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \|\nabla \mathbf{u}_t\|_{L^q(\Omega_T)} + C\|\nabla \mathbf{u}_t\|^2_{L^{2q}(\Omega_T)} \\ &\leq \Lambda. \end{aligned}$$

From Hölder continuity it follows that

$$\|c_0(\nabla \mathbf{u}_0, \theta_0) - c_0(\nabla \mathbf{u}, \theta)\|_{L^\infty(\Omega_{T_1})} \leq K T_1{}^{\frac{\alpha}{2}},$$

where K is Hölder constant independent of T_1. Here $T_1 \ll T$ will be determined later.

Next we show that $1/c_0(\nabla \mathbf{u}, \theta)(x, T_2)$ is Hölder continuous with respect to the space variable for T_2 fixed in $[0, T]$. We remark that

$$\mathcal{G}(y) := yG''(y) \leq M$$

and $\mathcal{G} \in \mathrm{C}^1$ is Lipschitz continuous. Then we have

$$\begin{aligned} &\left|\frac{1}{c_0}(x, T_2) - \frac{1}{c_0}(x', T_2)\right| \\ &= \left|\frac{\mathcal{G}(\theta(x', T_2))H(\nabla \mathbf{u}(x', T_2)) - \mathcal{G}(\theta(x, T_2))H(\nabla \mathbf{u}(x, T_2))}{\{1 - \mathcal{G}(\theta(x, T_2))H(\nabla \mathbf{u}(x, T_2))\}\{1 - \mathcal{G}(\theta(x', T_2))H(\nabla \mathbf{u}(x', T_2))\}}\right| \\ &\leq |\{\mathcal{G}(\theta(x', T_2))H(\nabla \mathbf{u}(x', T_2)) - \mathcal{G}(\theta(x, T_2))H(\nabla \mathbf{u}(x', T_2))\} \\ &\qquad + \{\mathcal{G}(\theta(x, T_2))H(\nabla \mathbf{u}(x', T_2)) - \mathcal{G}(\theta(x, T_2))H(\nabla \mathbf{u}(x, T_2))\}| \\ &\leq |H(\nabla \mathbf{u}(x', T_2))|\,|\mathcal{G}(\theta(x', T_2)) - \mathcal{G}(\theta(x, T_2))| \\ &\qquad + |\mathcal{G}(\theta(x, T_2))|\,|H(\nabla \mathbf{u}(x', T_2)) - H(\nabla \mathbf{u}(x, T_2))| \\ &\leq \Lambda K|x - x'|^\alpha + CM|x - x'|^\alpha \\ &\leq \Lambda|x - x'|^\alpha, \end{aligned}$$

where Λ is independent of T_2. Therefore $[1/c_0(\nabla \mathbf{u}, \theta)](x, T_2)$ is Hölder continuous for any $T_2 \in [0, T]$. Moreover, we have $\sup_{\Omega_T}[1/c_0(\nabla \mathbf{u}, \theta)] \geq 1/(1 + M\Lambda)$. These assure that $\frac{1}{c_0(\nabla \mathbf{u}(T_2), \theta(T_2))}\Delta$ has the maximal regularity property according to (2.3). Hence, taking $T_1 = \left(\frac{1}{2\Lambda(K,M,T)K}\right)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} &\leq \Lambda(K, M, T)\|c_0(\nabla \mathbf{u}_0, \theta_0) - c_0(\nabla \mathbf{u}, \theta)\|_{L^\infty(\Omega_{T_1})}\|\theta_t\|_{L^q(\Omega_{T_1})} \\
&\quad + \Lambda(K, M, T)\|R(\nabla \mathbf{u}, \theta)\|_{L^q(\Omega_{T_1})} + C\|\theta_0\|_{B_{q,q}^{2-2/q}(\Omega)} \\
&\leq \frac{1}{2}\|\theta_t\|_{L^q(\Omega_{T_1})} + \Lambda + \Lambda\|\theta_0\|_{B_{q,q}^{2-2/q}(\Omega)},
\end{aligned}$$

which yields

$$\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} \leq \Lambda + \Lambda\|\theta_0\|_{B_{q,q}^{2-2/q}(\Omega)}.$$

Here we remark that

$$\|\theta(T_1)\|_{B_{q,q}^{2-2/q}} \leq C(T_1)\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} \leq C(T_1)(\Lambda + \Lambda\|\mathbf{u}_0\|_{B_{q,q}^{2-2/q}})$$

thanks to the embedding $W_q^{2,1}(\Omega_{T_1}) \hookrightarrow BUC([0, T_1], B_{q,q}^{2-\frac{2}{q}})$ (see [2]). Then similarly for the interval $[T_1, 2T_1]$ we have

$$\|\theta\|_{W_q^{2,1}(\Omega_{[T_1,2T_1]})} \leq \Lambda + \Lambda\|\mathbf{u}(T_1)\|_{B_{q,q}^{2-2/q}} \leq \Lambda + \Lambda\|\mathbf{u}_0\|_{B_{q,q}^{2-2/q}} \leq \Lambda.$$

Repeating the same operation, we obtain

$$\|\theta\|_{W_q^{2,1}(\Omega_{[kT_1,(k+1)T_1]})} \leq \Lambda.$$

Summing the inequalities from $k = 0$ to $k = m$ satisfying $(m + 1)T_1 > T$ and $mT_1 \leq T$, we conclude that

$$\|\theta\|_{W_q^{2,1}(\Omega_T)} \leq \Lambda.$$

Next we estimate the norm $\|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)}$. From Lemma 2.2 it follows that

$$\|\nabla\theta\|_{L^\infty(\Omega_T)} + \|\Delta \mathbf{u}\|_{L^\infty(\Omega_T)} \leq \Lambda$$

for $q > 5$. Therefore, by virtue of the maximal regularity (2.1) we have

$$\begin{aligned}
\|\mathbf{u}\|_{W_p^{4,2}(\Omega_T)} &\leq C\|(\mathbf{u}_0, \mathbf{u}_1, 0)\|_{U(p,q)} + C\|\nabla \cdot (G(\theta)H_{,\nabla \mathbf{u}}(\nabla \mathbf{u}))\|_{L^p(\Omega_T)} \\
&\quad + C\|\nabla \cdot \overline{H}_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^p(\Omega_T)} \\
&\leq C\|(\mathbf{u}_0, \mathbf{u}_1, 0)\|_{U(p,q)} + \Lambda\|\nabla\theta\|_{L^\infty(\Omega_T)}\|G'(\theta)\|_{L^\infty(\Omega_T)}\|H_{,\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \\
&\quad + \Lambda\|\theta\|_{L^\infty(\Omega_T)}^r\|\Delta \mathbf{u}\|_{L^\infty(\Omega_T)}\|H_{,\nabla \mathbf{u}\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \\
&\quad + \Lambda\|\Delta \mathbf{u}\|_{L^\infty(\Omega_T)}\|\overline{H}_{,\nabla \mathbf{u}\nabla \mathbf{u}}(\nabla \mathbf{u})\|_{L^\infty(\Omega_T)} \\
&\leq \Lambda(\|(\mathbf{u}_0, \mathbf{u}_1, 0)\|_{U(p,q)}),
\end{aligned}$$

which completes the proof.

□

Here we note that we assume that $\theta \geq 0$ in all the lemmas of this section. The non-negativity of θ is assured for the sufficiently smooth solution $(\mathbf{u}, \theta)$ such as $(\mathbf{u}, \theta) \in W_p^{4,2}(\Omega_T) \times L_T^\infty L^2$. Hence, we can not proceed the above arguments, directly. One of the solvents for this problem is the following. We first consider the truncated problem $(TE)_3^L$:

$$\mathbf{u}_{tt} + \Delta^2\mathbf{u} - \nu\Delta\mathbf{u}_t = \Gamma_L\Big(\nabla \cdot [G(\theta)H_{,\nabla\mathbf{u}}(\nabla\mathbf{u}) + \overline{H}_{,\nabla\mathbf{u}}(\nabla\mathbf{u})]\Big), \tag{3.17}$$

$$\theta_t - \Delta\theta = \theta G''(\theta)\theta_t H(\nabla\mathbf{u}) + \theta G'(\theta)\partial_t H(\nabla\mathbf{u}) + \nu|\nabla\mathbf{u}_t|^2 \quad \text{in } \Omega_T, \tag{3.18}$$

$$\mathbf{u} = \Delta\mathbf{u} = \nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } S_T,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{u}_t(0,\cdot) = \mathbf{u}_1, \quad \theta(0,\cdot) = \theta_0 \geq 0 \quad \text{in } \Omega,$$

where

$$\Gamma_L(x) = \begin{cases} x & \text{if } |x| \leq L, \\ L\frac{x}{|x|} & \text{if } |x| \geq L. \end{cases}$$

We construct the solution $(\mathbf{u}_L, \theta_L)$ for $L > 0$. Then the solution satisfies also the original system (1.1)–(1.4) for sufficiently large truncation size L because a priori estimates obtained in this section are independent of L. More precisely, we refer to [12].

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