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Finite Laguerre geometries and generalized quadrangles*

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Abstract

In this paper we begin with a discussion of circle planes (or Benz planes) which are axiomatisations of the point, line and plane section geometry of quadrics in three-dimensional projective space. In the case of the Laguerre plane, the axiomatic version of a quadratic cone, we give a natural generalisation and in the finite case investigate its properties. In particular, we establish a connection between these Laguerre geometries and generalized quadrangles.

1 Circle planes

The study of *circle planes*, sometimes referred to as *Benz planes* (see [4]), is motivated by the study of quadrics in the projective space $PG(3,\mathbb{F})$ over the field $\mathbb{F}$. In particular, we consider three different quadrics.

Elliptic quadric: the unruled quadric with canonical equation $f(X_0,X_1)+X_2X_3=0$, where f is an irreducible quadratic form over $\mathbb{F}$.

Quadratic cone: formed by taking a cone with point vertex over an irreducible quadric (conic) in a plane and with canonical equation $X_0^2+X_1X_2=0$.

Hypebolic quadric: with two rulings of lines forming a grid and with canonical equation $X_0X_1+X_2X_3=0$.

Circle planes axiomatise the point, line and plane section geometry of these three quadrics. As quadrics they have a number of common properties. First, any three points, no two on a line, contained on the quadric, span a plane that intersects the quadric in an irreducible plane quadric. Secondly, given an irreducible plane section C of a quadric $\mathcal{Q}$ and a point $P\in C$ there is a unique line ℓ tangent to C at P (in the plane of C). Given any point $Q\in\mathcal{Q}\setminus\{C\}$ not collinear to P the plane $\langle Q,\ell\rangle$ meets $\mathcal{Q}$ in the unique irreducible plane section of $\mathcal{Q}$ containing Q and touching C at P. Thirdly, if the quadric $\mathcal{Q}$ contains lines, then every line meets every irreducible plane section of $\mathcal{Q}$ in a unique point. Further, the lines occur in rulings where the lines in a ruling partition the points of $\mathcal{Q}$ and lines in distinct rulings intersect in a unique point.

We formalise these geometric properties of the quadrics in the definition of a circle plane, where the concept of a *circle* takes the place of an irreducible plane section of a quadric.

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Definition 1.1 (Circle plane). *A* Circle plane $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{C})$ *is an incidence structure of* points, lines *and* circles, *respectively, such that the following axioms are satisfied.*

C1 *Three pairwise non-collinear points are incident with a unique circle.*

C2 *For any circle C, point $P \in C$ and point Q not collinear to P there is a unique circle D incident with Q and touching C at P (that is, C and D are incident with exactly one common point P).*

C3 *The lineset $\mathcal{L}$ is partitioned into* parallel classes, *a point is incident with one line in each parallel class and a line meets a circle in a unique point.*

C4 *Two non-parallel lines intersect in a unique point.*

C5 *Every circle is incident with at least three points and there exists more than one circle.*

For more on circle planes see [5].

Note that we will often refer to a point or a line as the set of points with which they are incident.

The touching axiom C2 has an important consequence. If P is a point on a circle C then any other two circles touching C at P must also touch each other at P, or else axiom C2 is violated. Hence the circles on a fixed point P are partitioned into sets which touch pairwise at P and partition the points of $\mathcal{S}$ not collinear with P. Such a set of circles is called a *pencil.*

Definition 1.2. *A circle plane with* 0, 1 *or* 2 *parallel classes is called a* Inversive *(or* Möbius*) plane, a* Laguerre *plane or a* Minkowski *plane, respectively.*

Naturally, the quadrics in $\mathrm{PG}(3, \mathbb{F})$ give examples of circle planes with the elliptic quadric giving rise to a Möbius or inversive plane, the quadratic cone, without its vertex, to a Laguerre plane and the hyperbolic quadric to a Minkowski plane. In addition, there are other sets of points in projective space that have properties similar enough to those of the quadrics in $\mathrm{PG}(3, \mathbb{F})$ that we may use them to define circle planes. In particular we consider ovoids.

Definition 1.3 ([16], see [8]). *An* ovoid $\mathcal{O}$ *of a projective space of dimension two or greater is a non-empty set of points such that:*

1. No three points are collinear.

2. The tangent lines (that is, lines meeting $\mathcal{O}$ in a single point) at a point $P \in \mathcal{O}$ form a hyperplane.

Note that in two dimensions an ovoid is usually called an *oval.*

For an ovoid in three dimensions or greater any hyperplane of the space that is *not* tangent to the ovoid must intersect the ovoid in an ovoid of that hyperplane.

We now have the following constructions of circle planes from ovoids.

Inversive plane: from the points, lines and non-tangent plane sections of an ovoid in three dimensions.

Laguerre plane: construct a cone in three dimensions from the a point vertex over an oval in a plane and then take the non-vertex points, lines and plane sections not containing the vertex.

In all the quadric models for circles planes we have a local projective structure in the sense that for any point P on the quadric $\mathrm{PG}(3,q)/P$ is a projective plane. Also, if we remove the tangent plane to the quadric at the point P, then in the quotient space we have an affine plane. (Note that in this context a tangent plane meets the quadric in just P in the elliptic case, a line in the quadratic cone case and two lines in the hyperbolic case.) This local or internal structure is preserved by generalising to circle planes.

Definition 1.4. *If $\mathcal{S} = (\mathcal{P}, \mathcal{C}, \mathcal{L})$ is a circle plane, then the* derived plane *at $P \in \mathcal{P}$ is the point-line geometry $\mathcal{S}_P$ with*

Points: *points of $\mathcal{S}$ not collinear with P.*

Lines: *Circles not incident with P and lines of $\mathcal{S}$ not incident with P.*

Incidence: *Inherited from $\mathcal{S}$.*

Since three pairwise non-collinear points define a unique circle in $\mathcal{S}$ we have that two points define a unique line in $\mathcal{S}_P$ (whether they be collinear or non-collinear in $\mathcal{S}$). The touching axiom for $\mathcal{S}$ implies that the circles of $\mathcal{S}$ incident with P fall into parallel classes as lines of $\mathcal{S}_P$. Also the lines of $\mathcal{S}$ not incident with P form a parallel class of lines in $\mathcal{S}_P$. Note also that any circle of $\mathcal{S}$ not incident with P meets any circle incident with P in at most two points (axiom C1 guarantees this for any two distinct circles). Hence we have the following important theorem.

Theorem 1.5. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{C}, \mathcal{L})$ be a circle plane and $P \in \mathcal{P}$.*

1. *$\mathcal{S}_P$ is an affine plane and we denote its projective completion $\overline{\mathcal{S}_P}$.*
2. *If C is a circle of $\mathcal{S}$ not incident with P, then in $\overline{\mathcal{S}_P}$ the points of $\mathcal{S}_P$ incident with C plus the point of $\overline{\mathcal{S}_P}$ that is the parallel class of $\mathcal{S}_P$ corresponding to the lines of $\mathcal{S}$ not incident with P, form an oval of $\overline{\mathcal{S}_P}$.*

Note that in all of the models for circle planes mentioned the derived affine plane is classical.

This relationship between circle planes, affine planes and ovals allows many interesting classification/characterisation results for circle planes to be proved and tells us much about the structure of circle planes (for examples see [7]).

2 Finite Laguerre planes, ovoids and Laguerre geometries

We now turn our attention to finite circle planes and in particular the finite Laguerre planes and investigate the links to ovoids in finite projective spaces. This in turn will suggest a natural generalisation of Laguerre planes.

If a finite Laguerre plane $\mathcal{S}$ has a line incident with a finite number n of points, then it is straightforward to show that each line is incident with n points, there are $n+1$ lines, each circle has $n+1$ points, there are n^2+n points in total and every derived plane has order n. The number n is called the *order* of $\mathcal{S}$.

Theorem 2.1 ([16]). *Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(n,q)$, q a prime power.*

1. *$|\mathcal{O}| = q^{n-1}+1$.*
2. *$n = 2$ or 3.*

Both parts of the above theorem can be proved by straightforward counting arguments. Given the above theorem we shall often employ the term *oval* to refer a two-dimensional ovoid and *ovoid* to refer to a three-dimensional ovoid.

It is known that if q is odd that the ovoids are always classical, that is an elliptic quadric if $n = 3$ and an irreducible conic if $q = 2$. In the case where q is even there are non-classical examples of both ovoids and ovals.

We have seen that a cone over an oval gives rise to a Laguerre geometry, so now we will investigate the geometry of a cone $\mathcal{K}$ (with point vertex V) over an ovoid. In particular, we will consider the geometry of points, lines and ovoidal hyperplane sections (that is, where the hyperplane does not contain V) of $\mathcal{K}$. Firstly, there is a unique line on any non-vertex point. Secondly, any three pairwise non-collinear points define a plane of $\mathrm{PG}(4, q)$ which intersects $\mathcal{K}$ in an oval and is contained in q hyperplane sections of $\mathcal{K}$ not containing V. This number is *finite* as we are now working in a finite projective space. Finally, any ovoidal hyperplane section has a unique tangent plane at every point. The hyperplane about this tangent plane, not containing V, give a set of q ovoidal hyperplane sections that meet pairwise in a common point. That is, the tangent plane property of the ovoid means that the geometry of $\mathcal{K}$ satisfies the touching axiom as in circle planes. This geometry prompts the following generalisation of Laguerre planes where we modify only the first axiom.

Definition 2.2 (Laguerre geometry). *A* Laguerre geometry $\mathcal{S} = (\mathcal{P}, \mathcal{C}, \mathcal{L})$ *is an incidence structure of* points, lines *and* circles, *respectively, such that the following axioms are satisfied.*

L1 *Three pairwise non-collinear points are incident with a constant (finite) number s of circles.*

L2 *For any circle C, point $P \in C$ and point Q not collinear to P there is a unique circle D incident with Q and touching C at P (that is, C and D are incident with exactly one common point P).*

L3 *A line meets a circle in a unique point.*

L4 *Every circle is incident with at least three points and there exists more than one circle.*

We will say that $\mathcal{S}$ is *finite* if $\mathcal{P}, \mathcal{C}, \mathcal{L}$ are finite sets. (Note that $\mathcal{P}, \mathcal{L}$ finite implies $\mathcal{C}$ is finite.) If a line of a finite Laguerre geometry $\mathcal{S}$ is incident with n points, then all lines of $\mathcal{S}$ are incident with n points. The parameter s is the number of circles of $\mathcal{S}$ on three pairwise non-collinear points. We introduce a parameter ℓ to denote the number of lines of $\mathcal{S}$, that is $\ell = |\mathcal{L}|$. Given these parameters for $\mathcal{S}$ it is straightforward to count that there are $n\ell$ points in total, ℓ points incident with a circle, $n^3 s$ circles in total, ns circles incident with two given non-collinear points and $n^2 s$ circles incident with a given point.

If a Laguerre geometry $\mathcal{S}$ has parameters n, s, ℓ, then for the case $s = 1$ we have exactly the Laguerre planes which implies that $\ell = n + 1$ and the geometry $\mathcal{S}_P$ is an affine plane. If, however, $s > 1$, what can we say about the concept analogous to that of the derived plane and what can we say about the relationship between the parameters?

If we consider our motivating example of a cone with point vertex and base an ovoid $\mathcal{O}$ of $\mathrm{PG}(3, q)$, then we have a Laguerre geometry $\mathcal{S}$ with parameters $n = s = q$ and $\ell = |\mathcal{O}| = q^2 + 1$. For a point P of $\mathcal{S}$ consider extending the concept of the geometry $\mathcal{S}_P$ as used for Laguerre planes. This geometry has points the points of $\mathcal{S}$ not collinear with P. Lines are of two types, firstly circles of $\mathcal{S}$ incident with P and secondly lines of $\mathcal{S}$ not incident with P. In this case the first type of line is incident with q^2 points of $\mathcal{S}_P$, while the second type of line is incident with q points of $\mathcal{S}_P$. This is clearly an unsatisfactory definition so in the definition that follows we omit the second type of line.

Definition 2.3. *If $\mathcal{S} = (\mathcal{P}, \mathcal{C}, \mathcal{L})$ is a Laguerre geometry and $P \in \mathcal{P}$ define $\mathcal{S}^P$ the* internal structure *of $\mathcal{S}$ at P to be the incidence geometry with*

Points: *Points of $\mathcal{S}$ not collinear with P,*
Hyperplanes: *Circles of $\mathcal{S}$ incident with P, and*
Incidence: *Inherited from $\mathcal{S}$.*

In the case of a Laguerre plane, $\mathcal{S}^P$ is a projective plane with an incident point-line pair removed. In the case of the Laguerre geometry $\mathcal{S}$ arising from a cone with base an ovoid in $\mathrm{PG}(3, q)$, if P is a point of $\mathcal{S}$, then the circles on P come from the hyperplanes of $\mathrm{PG}(4, q)$ on P but not on the vertex of the cone, while each point of $\mathcal{S}$ not collinear with P spans a distinct line with P. Looking in the quotient space $\mathrm{PG}(4, q)/P$ the geometry $\mathcal{S}^P$ is the geometry of $\mathrm{PG}(3, q)$ with a point-plane pair removed (corresponding in $\mathrm{PG}(4, q)$ to the line PV and the hyperplane meeting the cone in PV). These examples prompt the following definition.

Definition 2.4. *A Laguerre geometry $\mathcal{S} = (\mathcal{P}, \mathcal{C}, \mathcal{L})$ has* classical internal structure *at $P \in \mathcal{P}$ if $\mathcal{S}^P$ is a projective space with a point-hyperplane pair removed.*

We now return to the question of the relationship between the parameters n, s and ℓ of a Laguerre geometry $\mathcal{S}$. The first question is: do n and s determine ℓ? The following example shows that this is not the case. Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(3, q)$ and let $\overline{\mathcal{O}} \subset \mathcal{O}$ such that there is a unique tangent plane to $\overline{\mathcal{O}}$ at each point. In this context by tangent plane at $P \in \overline{\mathcal{O}}$ we mean a plane incident with P and no other point of $\overline{\mathcal{O}}$. If we form a cone in $\mathrm{PG}(4, q)$ with point vertex and base $\overline{\mathcal{O}}$, then the geometry of (non-vertex) points, lines and hyperplane sections of this cone is a Laguerre geometry with $n = s = q$ and $\ell = |\overline{\mathcal{O}}|$. The unique tangent plane at every point of $\overline{\mathcal{O}}$ ensures that the touching axiom is still valid. So in this construction we see that a given n, s may give rise to Laguerre geometries with various values of ℓ. More generally, this construction will work if we take the cone over a set of points in $\mathrm{PG}(n, q)$, $n \geq 3$, such that no three are collinear and there is a unique tangent plane at each point.

The following theorem gives us a definite relationship between n, s and ℓ.

Theorem 2.5. *Let $\mathcal{S}$ be a finite Laguerre geometry with parameters n, s, ℓ. Then $n + 1 \leq \ell \leq ns + 1$. Equality in the lower bound means $s = 1$, $\ell = n + 1$ and $\mathcal{S}$ is a Laguerre plane which is equivalent to $\mathcal{S}$ having intersection sizes between two circles being $0, 1, 2$. In the case $\ell = ns + 1$ we have $s + 1 | n^3 - n$, $n \geq s$ and the possible intersection sizes for two circles are $0, 1, s + 1$. Further, in the $\ell = ns + 1$ case, there exist disjoint circles if and only if $n > s$.*

Proof. To start we observe that the total number of points is $n\ell$, the total number of circles is $n^3 s$, the number of circles on a pair of non-collinear points is ns and the number of circles on a given point is $n^2 s$.

Now let C be a fixed circle of $\mathcal{S}$ and P a fixed point of C. There are exactly $n - 1$ circles meeting C in exactly P and hence $n^2 s - n$ circles meeting C in P and at least one other point. For any point $Q \in C \setminus \{P\}$ there are $ns - 1$ circles meeting C in at least P and Q. Hence there are at most $(ns - 1)(\ell - 1)$ circles meeting C in P and at least one other point. Hence, $n^2 s - n \leq (ns - 1)(\ell - 1)$ which is equivalent $\ell \geq n + 1$. From the count it follows that we have equality if and only if the possible intersection sizes between two circles are $0, 1, 2$. From the inequality we have equality if and only if $s = 1$ (since ℓ must be integral) and $\ell = n + 1$, that is $\mathcal{S}$ is a Laguerre plane.

Now let the set of circles meeting C in P and at least one other point be $\{C_1, C_2, \ldots, C_{ns^2-n}\}$ and $t_i = |(C_i \cap C) \setminus \{P\}|$. Then $\sum_{i=1}^{ns^2-n} t_i = (\ell-1)(ns-1)$ and the average value of the t_i is $\bar{t} = (\ell-1)(ns-1)/(n^2s-n)$. If we count triples (Q, R, C') such that P, Q, R are distinct, pairwise non-collinear points, $C' \neq C$ and $P, Q, R \subset C \cap C'$, then we have $\sum_{i=1}^{n^2s-n} t_i(t_i-1) = (\ell-1)(\ell-2)(s-1)$. So now if we calculate $\sum_{i=1}^{n^2s-n}(t_i-\bar{t})^2 = (\ell-1)(n-1)(ns+1-\ell)$ we obtain the inequality $\ell \leq ns+1$.

If we have $\ell = ns+1$, then by the above any two circles that intersect in at least two points intersect in exactly $\bar{t}+1 = s+1$ points. In this case, if for a fixed circle C, we count the triples (P, Q, C') with $P \neq Q$ and $P, Q \in C \cap C'$, then we see that the number of circles, distinct from C, meeting C in at least two (and hence exactly $s+1$ points) is $(ns+1)ns(ns-1)/(s^2+s)$. Since this is an integer we have $s+1 | n(n^2s^2-1)$ and hence $s+1 | n^3-n$. Now the total number of circles not touching C is $n^3s - \ell(n-1) - 1 = n^3s - (ns+1)(n-1) - 1$ which is an upper bound for the number of circles meeting C in at least two points. Hence $n^3s - (ns+1)(n-1) - 1 \geq (ns+1)n(ns-1)/(s+1)$, which simplifies to $(n-s)(n-1) \geq 0$ and hence $n \geq s$. If we have equality, then by the count there are no circles disjoint to C and the possible intersection sizes between circles is 1 and $s+1$. □

Note that from the above theorem we have that $\ell = n+1$ if and only if the circle intersection sizes are $0, 1, 2$ which is the case if and only if $s = 1$, that is we are in the Laguerre plane case.

Corollary 2.6. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{C})$ be a finite Laguerre geometry with parameters n, s, ℓ. If $\mathcal{S}$ has a classical internal structure at some point P, then either*

1. *$s = 1$, $\ell = n+1$ and $\mathcal{S}^P$ is a projective plane with a point-line pair removed; or*
2. *$s = n$, $\ell = n^2+1$ and $\mathcal{S}^P$ is a projective 3-space with a point-plane pair removed.*

Proof. Since $\mathcal{S}^P$ is classical it follows that distinct circles of $\mathcal{L}$ on Y intersect in a constant number. By the proof of Theorem 2.5 we have that $\ell = ns+1$ and that number of points of the affine space must be $n\ell - n = n^2s$. The projective space giving rise to $\mathcal{S}^P$ has order n and so $n^2s = n^k$ for some integer k. By Theorem 2.5 we also have that $n \geq s$ so the only possibilities are $s = 1$ and $k = 2$ or $s = n$ and $k = 3$. □

3 Generalized quadrangles

Now we introduce a particular class of point-line geometries, the generalized quadrangles, and show their connection to Laguerre geometries. Generalized quadrangles were introduced by Tits in [15].

Definition 3.1. *A* generalized quadrangle *(GQ) is an incidence geometry of points and lines satisfying:*

GQ1 *Two points are incident with at most one line.*

GQ2 *For a non-incident point line pair (P, m) there is a unique point of m collinear with P.*

GQ3 *No point is collinear with all others.*

The dual structure of a GQ (that is, swapping the labels of "points" and "lines") is also a GQ. If a GQ has a line incident with a finite number of points, then all lines are incident with exactly this number of points and dually for lines. A *finite* GQ has an order (s,t) for s,t finite such that there are $s+1$ points incident with a line and $t+1$ lines incident with a point. For a comprehensive introduction to finite GQs see [10].

GQs are exactly the rank 2 polar spaces and so as examples we have the non-singular quadrics and Hermitian varieties in finite projective spaces that contain lines but no higher dimensional subspaces.

The first non-classical construction of GQs comes from Tits (see [8]). Let $\mathcal{O}$ be an ovoid in $H = \mathrm{PG}(n,q)$, $n = 2,3$ embedded in $\Sigma = \mathrm{PG}(n+1,q)$. Then the following structure $T_n(\mathcal{O})$ is a GQ of order (q, q^{n-1}). Points are of three types: (i) the points of $\Sigma \setminus H$; (ii) the n-dimensional subspaces of Σ meeting H in a tangent space to $\mathcal{O}$; and (iii) a formal point (∞). Lines are of two types: (a) lines of Σ, not in H, meeting H in a point of $\mathcal{O}$; and (b) the points of $\mathcal{O}$. Incidence is natural plus (∞) is incident with all lines of type (b).

Geometrically, we can see a link between this construction of Tits and Laguerre geometries if we consider *dualising* the Tits construction in $\mathrm{PG}(n+1,q)$. The important observation that makes this useful is that if $(n,q) \neq (2,2^h)$ for some $h \geq 1$, then under a duality of $\mathrm{PG}(n,q)$ the points of an ovoid $\mathcal{O}$ of $\mathrm{PG}(n,q)$ are mapped to a set of hyperplanes which are the set of tangent hyperplanes to an ovoid equivalent to $\mathcal{O}$ ([11, 12, 1, 9, 13]). So if $\mathcal{O}$ is an ovoid in $H = \mathrm{PG}(n,q)$, $n = 2,3$ embedded in $\Sigma = \mathrm{PG}(n+1,q)$, then dualising Σ $\widehat{H}$ is a point, the points of $\mathcal{O}$ become hyperplanes on $\widehat{H}$ that are in fact the hyperplanes obtained by taking the span of $\widehat{H}$ with the tangent planes to an ovoid equivalent to $\mathcal{O}$ in a n-dimensional subspace not containing $\widehat{H}$. That is, the points of $\mathcal{O}$ become tangent planes to a cone with an ovoid equivalent to $\mathcal{O}$ as a base, with lines the dual of the tangent planes to $\mathcal{O}$. Hence, if $\mathcal{K}$ is such a cone with vertex V we may represent the construction of Tits above by: a point of type (i) $\leftrightarrow$ ovoidal hyperplane section of $\mathcal{K}$, a point of type (ii) $\leftrightarrow$ a point of $\mathcal{K} \setminus \{K\}$, $(\infty) \leftrightarrow V$, a line of type (a) $\leftrightarrow$ an $(n-1)$-dimensional subspace contained in a tangent plane to $\mathcal{K}$, but not containing V and finally a line of type (b) $\leftrightarrow$ a hyperplane tangent to $\mathcal{K}$ and hence a line of $\mathcal{K}$. Note that if we take a $(n-1)$-dimensional subspace not containing V and contained in a tangent plane to $\mathcal{K}$, the hyperplanes on the subspace intersect $\mathcal{K}$ in a pencil of the Laguerre geometry $\mathcal{S}$ constructed from $\mathcal{K}$. In this way the above correspondence is actually a correspondence between the geometry of $T_n(\mathcal{O})$, removing the point (∞) and the geometry of $\mathcal{S}$.

The above discussion raises the question: when does a Laguerre geometry give rise to a GQ? In the above connection circles of the Laguerre geometry are points of the GQ and the pencils of circles of the Laguerre geometry correspond to lines of the GQ. So given axiom GQ2 a Laguerre geometry must at least satisfy:

(GQ) Three pairwise touching circles have a common point (of contact).

Given this condition we can determine when a Laguerre geometry corresponds to a GQ and conversely. For the following theorem we note that a *triad* of a GQ is a set of three pairwise non-collinear points and a *centre* of a triad is a point collinear with all three points of the triad.

Theorem 3.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{C})$ be a finite Laguerre geometry with parameters n, ℓ, s satisfying axiom (GQ). Consider the incidence structure which has points of three types: (i) points of $\mathcal{S}$; (ii) circles of $\mathcal{S}$; and (iii) a formal point (∞). The lines are of two types: (a) lines of $\mathcal{S}$; and (b) pencils of circles. Incidence is that inherited from $\mathcal{S}$, plus (∞) is incident with all lines of type (a) and a point of $\mathcal{S}$ is incident with all pencils for which it is the common point of* *contact. This incidence structure is a GQ, denoted* GQ($\mathcal{S}$) *if and only if* $\mathcal{S}$ *satisfies (GQ) and* $\ell = ns + 1$*, in which case the GQ has order* (n, ns)*.*

Conversely, let $\mathcal{G} = (\mathcal{P}, \mathcal{B})$ *be a generalized quadrangle of order* $(\bar{s}, \bar{t})$ *with point* (∞) *such that each triad of* S *with* (∞) *as a centre has exactly* $s + 1$ *centres for some* s*. Then the incidence structure* $\mathcal{S} = (\{(\infty)^\perp \setminus \{(\infty)\}, \{\ell \in \mathcal{B} : (\infty) \mathrm{I} \ \ell\}, \{X^\perp \cap (\infty)^\perp : X \in \mathcal{P} \setminus (\infty)^\perp\})$ *where incidence is induced by that of* $\mathcal{G}$ *is a Laguerre geometry with* $s = \bar{t}/\bar{s}$*,* $n = \bar{s}$*,* $\ell = \bar{t} + 1$ *and satisfying (GQ).*

Proof. Given a Laguerre geometry satsifying (GQ) the incidence structure given has the property that given a non-incident line pair (P, m) there is at most one line incident with P and concurrent with m. The only case in which this number is possibly less than one is where P is a circle of $\mathcal{S}$ and m is a pencil of circles of $\mathcal{S}$ whose base point is not contained in the circle P. Counting we see that there are $\ell(n-1)$ circles touching P and $(n\ell - \ell)ns$ pencils whose base point is not on P. Since each circle touching P is contained in $\ell - 1$ pencils whose base point is not on P to be a GQ we require that $\ell(n-1)(\ell-1) = (n\ell - \ell)ns$, that is $\ell = ns + 1$. □

We can extract a couple cases of special interest from this theorem.

Theorem 3.3. *Let* $\mathcal{S}$ *be a Laguerre geometry with parameters* ℓ, n, s *such that* $\ell = ns + 1$*.*
1. *If* $\mathcal{S}$ *satisfies (GQ), then* $s + 1 | n + 1$ *and hence if* $s = 1$*,* n *is odd.*
2. *If* $n = s$*, then* $\mathcal{S}$ *satisfies (GQ).*

Proof. Since $\ell = ns + 1$ the possible intersection numbers for circles are $0, 1, s + 1$. Let C and C' be two circles of $\mathcal{S}$ touching at the point P. Let Q be any point of $C' \setminus \{P\}$, let R be the point of C collinear with Q and consider the circles touching C' at Q. Each of these $n - 1$ circles meets $C \setminus \{P, R\}$ in either 1 or $s + 1$ points and each point of $C \setminus \{P, R\}$ is contained in a unique such circle. Hence $a(s + 1) + b = ns - 1$ where $a + b \leq n - 1$ and $a, b \geq 0$.

If $\mathcal{S}$ satisfies (GQ), then $b = 0$ and we have $s + 1 | ns - 1$ or equivalently $s + 1 | n + 1$.

If $n = s$, then $a(s + 1) + b = s^2 - 1$ and $a + b \leq s - 1$, from which it follows that $a \geq s - 1$. Hence $a = s - 1$, $b = 0$ and $\mathcal{S}$ satisfies (GQ). □

Remark 3.4. *Note that the converse part of Theorem 3.2 also follows from [10, 1.7.1].*

What we see from the above theorem is that a Laguerre geometry with $n = s$ and $\ell = ns + 1$ always gives rise to a GQ of order (s, s^2). Conversely, every GQ of order $(\bar{s}, \bar{s}^2)$ has the property that every triad has $\bar{s} + 1$ centres ([6]) and so from *every* point of such a GQ there arises a Laguerre geometry with parameters $n = s = \bar{s}$ and $\ell = \bar{s}^2 + 1$.

There are many examples of GQs of order (s, s^2), for instance the GQs $T_3(\mathcal{O})$ for $\mathcal{O}$ an ovoid of PG$(3, q)$, as mentioned earlier, the dual of GQs arising from the flock of a quadratic cone, as well as certain translation generalized quadrangles (a generalisation of the $T_3(\mathcal{O})$ construction). See [14] for more details. So we have many constructions of Laguerre geometries from these GQs and interestingly in *most* cases the Laguerre geometry in question does not have classical internal structure at any point.

A GQ of order (s, s^2) that has a point where the associated Laguerre geometry has a classical internal structure at a point is exactly a GQ that is usually referred to as having *Property (G)*. The papers [2, 3] investigate this connection and prove many strong characterisation results of such GQs, although the idea of Laguerre geometries is never explicitly mentioned in these papers.

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