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The existence of infinitely many tight Euclidean designs having certain parameters *

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1 Introduction

The concept of spherical design was introduced by Delsarte, Goethals and Seidel [8] in 1977 for finite sets in the unit sphere S^{n-1} (in the Euclidean space $\mathbb{R}^n$) . It measures how much the finite set approximates the sphere S^{n-1} with respect to the integral of polynomial functions. The exact definition is given as follows.

Definition 1.1. *Let t be a positive integer. A finite nonempty subset $X\subseteq S^{n-1}$ is called a* spherical t-design *if the following condition holds:*

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x)=\frac{1}{|X|}\sum_{x\in X} f(x), \tag{1}$$

for any polynomial $f(x)\in\mathbb{R}[x_1,x_2,\ldots,x_n]$ of degree at most t, where $\sigma(x)$ is the $O(n)$-invariant measure on S^{n-1} and $|S^{n-1}|$ is the area of the sphere S^{n-1}.

The concept of spherical t-design was generalized by Neumaier and Seidel [12] in the following two ways: (i) to drop the condition that it is on a sphere, (ii) to allow weight. The new concept is called *Euclidean t-design.* This concept is closely related to the cubature formulae in numerical analysis and approximation theory, and a similar concept such as rotatable design has already existed also in mathematical statistics (see, e.g., [6, 11].)

Recently, Bannai and Bannai [4], slightly modified the Neumaier and Seidel's definition of Euclidean t-design by dropping the assumption of excluding the origin. We will review the definition below.

Let X be a finite set in $\mathbb{R}^n$, $n\geq 2$. Let $\{r_1,\ r_2,\ \ldots,\ r_p\}=\{\|x\|,\ x\in X\}$, where $\|x\|$ is a norm of x defined by standard inner product in $\mathbb{R}^n$ and r_i is possibly 0. For each i, we define $S_i=\{x\in\mathbb{R}^n,\ \|x\|=r_i\}$, the sphere of radius r_i centered at 0. We say that X is supported by the p concentric spheres $S_1,\ S_2,\ \ldots,\ S_p$. If $r_i=0$, then $S_i=\{0\}$. Let $X_i=X\cap S_i$, for $1\leq i\leq p$. Let $\sigma_i(x)$ be the $O(n)$-invariant measure on the unit sphere $S^{n-1}\subseteq\mathbb{R}^n$. We consider the measure $\sigma_i(x)$ on each S_i so that $|S_i|=r_i^{n-1}|S^{n-1}|$, with $|S_i|$ is the surface area of S_i. We associate a positive real valued function w on X, which is called a *weight* of X. We define $w(X_i)=\sum_{x\in X_i} w(x)$. Here if

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$r_i = 0$, then we define $\frac{1}{|S_i^{n-1}|}\int_{S_i^{n-1}} f(x)d\sigma_i(x) = f(0)$, for any function $f(x)$ defined on $\mathbb{R}^n$. Let $S = \bigcup_{i=1}^{p} S_i^{n-1}$. Let $\varepsilon_S \in \{0,\ 1\}$ be defined by

$$\varepsilon_S = \begin{cases} 1, & 0 \in S \\ 0, & 0 \notin S \end{cases}.$$

We give some more notation we use. Let $\mathrm{Pol}(\mathbb{R}^n) = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be the vector space of polynomials in n variables $x_1,\ x_2, \ldots,\ x_n$. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathrm{Pol}(\mathbb{R}^n)$ spanned by homogeneous polynomials of degree l. Let $\mathrm{Harm}(\mathbb{R}^n)$ be the subspace of $\mathrm{Pol}(\mathbb{R}^n)$ consisting of all harmonic polynomials. Let $\mathrm{Harm}_l(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$. Then we have $\mathrm{Pol}_l(\mathbb{R}^n) = \oplus_{i=0}^{l}\mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathrm{Pol}_l^*(\mathbb{R}^n) = \oplus^{l}_{\substack{i\equiv 0(2),\\ 0\le i\le l}}\mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathrm{Pol}(S)$, $\mathrm{Pol}_l(S)$, $\mathrm{Hom}_l(S)$, $\mathrm{Harm}(S)$, $\mathrm{Harm}_l(S)$ and $\mathrm{Pol}_l^*(S)$ be the sets of corresponding polynomials restricted to the union S of p concentric spheres. For example $\mathrm{Pol}(S) = \{f|_S,\ f \in \mathrm{Pol}(\mathbb{R}^n)\}$.

With the notation mentioned above, we define a Euclidean t-design as follows.

Definition 1.2. *Let X be a finite set with a weight function w and let t be a positive integer. Then (X, w) is called a* Euclidean t-design *in $\mathbb{R}^n$ if the following condition holds:*

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i^{n-1}|}\int_{S_i^{n-1}} f(x)d\sigma_i(x) = \sum_{x\in X} w(x)f(x),$$

for any polynomial $f(x) \in \mathrm{Pol}(\mathbb{R}^n)$ of degree at most t.

Let X be a Euclidean $2e$-design in $\mathbb{R}^n$. Then it is known that $|X| \ge \dim(\mathrm{Pol}_e(S))$. Let X be an antipodal $(2e+1)$-design in $\mathbb{R}^n$. Then it is also known that $|X^*| \ge \dim(\mathrm{Pol}_e^*(S))$. Here X^* is an antipodal half part of X satisfying $X^* \cup (-X^*) = X$ and $X^* \cap (-X^*) = \{0\}$ or $\emptyset$. Although better lower bounds are proved in [9] and [12], $\dim(\mathrm{Pol}_e(S))$ and $\dim(\mathrm{Pol}_e^*(S))$ are considered to be very natural. We define the following tightness for the Euclidean designs (c.f. [4, 5]).

Definition 1.3. *Let X be a Euclidean $2e$-design supported by S. If $|X| = \dim(Pol_e(S))$ holds we call X* a tight $2e$-design on S. *Moreover if $\dim(Pol_e(S)) = \dim(Pol_e(\mathbb{R}^n))$ holds, then X is called* a tight Euclidean $2e$-design.

Definition 1.4. *Let X be an antipodal Euclidean $(2e+1)$-design supported by S. Assume $w(x) = w(-x)$ for any $x \in X$. If $|X^*| = \dim(Pol_e^*(S))$ holds, we call X* an antipodal tight $(2e+1)$-design on S. *Moreover if $\dim(Pol_e^*(S)) = \dim(Pol_e^*(\mathbb{R}^n))$ holds, then X is called* an antipodal tight Euclidean $(2e+1)$-design.

In Section 2, we give some more basic facts about the Euclidean designs. In Section 3, we give the definition of the strong non-rigidity of Euclidean designs. Our main theorem is Theorem 3.8, in which we show that the following known examples of tight Euclidean designs are strongly non-rigid: tight Euclidean 4-designs in $\mathbb{R}^2$, tight Euclidean 2-designs in $\mathbb{R}^n$ supported by one sphere, or equivalently, tight spherical 2-designs. We also show that antipodal tight spherical 3-designs in $\mathbb{R}^2$ in the sense of Euclidean design as well as antipodal tight Euclidean 5-designs in $\mathbb{R}^2$ are strongly non-rigid. The implication of these facts are the existence of infinitely many non-isomorphic tight Euclidean designs with the given strength.

The complete classification of tight Euclidean 2-designs in $\mathbb{R}^n$ is given in Section 4. We also show that any finite subset $X \subseteq \mathbb{R}^n$ of cardinality $n+1$ is a Euclidean 2-design if and only if X is a 1-inner product set with negative inner product value. Here we say $X \in \mathbb{R}^n$ is an e-inner product set if $|\{\langle x, y\rangle,\ x, y \in X, x \ne y\}| = e$ holds. We remark that $|X| \le \dim(\mathrm{Pol}_e(\mathbb{R}^n)) = \binom{n+e}{e}$ holds for any e-inner product set X in $\mathbb{R}^n$.

2 Basic facts on Euclidean designs

The following theorem gives a condition which is equivalent to the definition of Euclidean t-designs.

Theorem 2.1 (Neumaier-Seidel). *Let X be a finite nonempty subset in $\mathbb{R}^n$ with weight function w. Then the following (1) and (2) are equivalent:*

(1) X is a Euclidean t-design.

(2) $\sum_{u\in X} w(u)\|u\|^{2j}\varphi(u) = 0$, for any polynomial $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$ with $1 \le l \le t$ and $0 \le j \le \lfloor \frac{t-l}{2} \rfloor$.

We will use the condition (2) of Theorem 2.1 in what follows. Theorem 2.1 implies the following proposition.

Proposition 2.2 ([4], Proposition 2.4). *Let (X, w) be a Euclidean t-design in $\mathbb{R}^n$. Then the following (1) and (2) hold:*

(1) Let λ be a positive real number and $X' = \{\lambda u,\ u \in X\}$. Then X' is also a Euclidean t-design with weight w' defined by $w' = w(\frac{1}{\lambda}u')$, $u' \in X'$.

(2) Let μ be a positive real number and $w'(u) = \mu w(u)$ for any $u \in X$. Then X is also a Euclidean t-design with respect to the weight w'.

We also need the proposition below in the subsequent sections.

Proposition 2.3 ([4], Lemma 1.8). *Let (X, w) be a tight Euclidean $2e$-design or antipodal tight Euclidean $(2e+1)$-design in $\mathbb{R}^n$. Then the weight function w is constant on each sphere.*

Let (X, w) be a finite weighted subset in $\mathbb{R}^n$. Let $S_1, S_2, \dots, S_p$ be the p concentric spheres supporting X and let $S = \bigcup_{i=1}^{p} S_i$.

For any $\varphi,\ \psi \in \mathrm{Harm}(\mathbb{R}^n)$, we define the following inner-product

$$\langle \varphi, \psi \rangle = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \varphi(x)\psi(x)d\sigma(x).$$

Let $h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$ and $\varphi_{l,1}, \dots, \varphi_{l,h_l}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner-product defined above. Then,

$$\left\{\|x\|^{2j},\ 0 \le j \le \min\left\{p-1, \left[\frac{e}{2}\right]\right\}\right\} \bigcup$$
$$\left\{\|x\|^{2j}\varphi_{l,i}(x),\ 1 \le l \le e,\ 1 \le i \le h_l,\ 0 \le j \le \min\left\{p - \varepsilon_S - 1, \left[\frac{e-l}{2}\right]\right\}\right\}$$

gives a basis of $\mathrm{Pol}_e(S)$.

Now, we are going to construct a more convenient basis of $\mathrm{Pol}_e(S)$ for our purpose. Let $\mathcal{G}(\mathbb{R}^n)$ be the subspace of $\mathrm{Pol}_e(S)$ spanned by $\{\|x\|^{2j},\ 0 \le j \le p-1\}$. Let $\mathcal{G}(X) = \{g|_X,\ g \in \mathcal{G}(\mathbb{R}^n)\}$. Then $\{\|x\|^{2j},\ 0 \le j \le p-1\}$ is a basis of $\mathcal{G}(X)$. We define an inner-product $\langle -, - \rangle_l$ on $\mathcal{G}(X)$ by

$$\langle f, g \rangle_l = \sum_{x\in X} w(x)\|x\|^{2l} f(x)g(x), \text{ for } 1 \le l \le e. \tag{2}$$

We apply the Gram-Schmidt method to the basis $\{\|x\|^{2j},\ 0 \le j \le p-1\}$ to construct an orthonormal basis

$$\{g_{l,0}(x),\ g_{l,1}(x),\ \dots,\ g_{l,p-1}(x)\}$$

of $\mathcal{G}(X)$ with respect to the inner-product $\langle -, - \rangle_l$. We can construct them so that for any l the following holds:

$g_{l,j}(x)$ is a linear combination of $1,\ \|x\|^2,\ \dots,\ \|x\|^{2j}$, with $\deg(g_{l,j}) = 2j$, for $0 \le j \le p-1$.

For example, we can express $g_{l,0}(x)$ as

$$g_{l,0}(x) \equiv \frac{1}{\sqrt{a_l}}, \text{ with } a_l = \sum_{x \in X} w(x)\|x\|^{2l}. \tag{3}$$

Now we are ready to give a new basis for $\mathrm{Pol}_e(S)$. Let us consider the following sets:

$$\begin{aligned}
\mathcal{H}_0 &= \left\{g_{0,j} \;\middle|\; 0 \le j \le \min\left\{p-1, \left[\frac{e}{2}\right]\right\}\right\}, \\
\mathcal{H}_l &= \left\{g_{l,j}\varphi_{l,i} \;\middle|\; 0 \le j \le \min\left\{p-\varepsilon_S-1, \left[\frac{e-l}{2}\right]\right\},\ 1 \le i \le h_l\right\}, \text{ for } 1 \le l \le e.
\end{aligned}$$

Then $\mathcal{H} = \bigcup_{l=0}^{e} \mathcal{H}_l$ is a basis of $\mathrm{Pol}_e(S)$.

Proposition 2.4. *If (X, w) is a tight $2e$-design on S, then the following (1) and (2) hold:*

(1) The weight function of X satisfies

$$\sum_{\substack{1 \le l \le e,\\ 0 \le j \le \min\{p-\varepsilon_S-1, [\frac{e-l}{2}]\}}} \|u\|^{2l} g_{l,j}^2(u) Q_l(1) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}^2(u) = \frac{1}{w(u)}, \text{ for all } u \in X. \tag{4}$$

(2) For any distinct points $u,\ v \in X$, we have

$$\sum_{\substack{1 \le l \le e,\\ 0 \le j \le \min\{p-\varepsilon_S-1, [\frac{e-l}{2}]\}}} \|u\|^l \|v\|^l g_{l,j}(u) g_{l,j}(v) Q_l\left(\frac{\langle u, v\rangle}{\|u\|\|v\|}\right) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u) g_{0,j}(v) = 0. \tag{5}$$

Here $\langle u, v\rangle$ is the standard inner-product in Euclidean space $\mathbb{R}^n$ and $Q_l(u)$ is the Gegenbauer polynomial of degree l. Moreover, for the case $e = 1$ the converse is also true, namely, if (1) and (2) hold, then X is a tight 2-design on S.

3 Rigidity of spherical and Euclidean designs

We call a spherical t-design *non-rigid* (resp. *rigid*) if it cannot be (resp. can be) deformed locally keeping the property that it is a spherical t-design. The exact definition is given as follows (c.f. [2]).

Definition 3.1. *A spherical t-design $X = \{x_i,\ 1 \le i \le N\} \subseteq S^{n-1}$ is called* non-rigid *or* deformable *in $\mathbb{R}^n$ if for any $\varepsilon > 0$ there exists another spherical t-design $X' = \{x'_i,\ 1 \le i \le N\} \subseteq S^{n-1}$ such that the following two conditions hold:*

(1) $\|x_i - x'_i\| < \varepsilon$, for $1 \le i \le N$; and

(2) there is no any transformation $g \in O(n)$, with $g(x_i) = x'_i$, for $1 \le i \le N$.

Motivated by the above definition and Proposition 2.2, we define a similar concept of rigidity and non-rigidity for Euclidean t-design, depending upon whether the designs can be transformed to each other by orthogonal transformations, scaling, or adjustment of the weight functions. In the definition below, $O^*(n) = \langle O(n), g_\lambda, g^\mu\rangle$ denotes a group generated by an orthogonal group $O(n)$, a scaling g_λ of X :

$$\left\{\begin{array}{ccc} g_\lambda : (X, w) & \longrightarrow & (X', w') \\ x & \longmapsto & x' = \lambda x \\ w'(x') = w(x) & & \end{array}\right. ,$$

and an adjustment g^μ of weight function w :

$$\begin{cases} g^\mu : (X, w) & \longrightarrow & (X', w') \\ x & \longmapsto & x' := x \\ w'(x') = \mu w(x) & & \end{cases} .$$

Definition 3.2. *A Euclidean t-design $X = (\{x_i\}_{i=1}^N, w) \subseteq \mathbb{R}^n$ is called* non-rigid *or* deformable *in $\mathbb{R}^n$ if for any $\varepsilon > 0$ there exists another Euclidean t-design $X' = (\{x_i'\}_{i=1}^N, w') \subseteq \mathbb{R}^n$ such that the following two conditions hold:*

(1) $\|x_i - x_i'\| < \varepsilon$, and $|w(x_i) - w'(x_i')| < \varepsilon$, for $1 \leq i \leq N$; and

(2) there is no any transformation $g \in O^*(n)$, with $g(x_i) = x_i'$ for $1 \leq i \leq N$.

It is well known that any tight spherical t-design is rigid, because the possible distances of any two points in the design are finitely many in number and determined by only n and t (see Theorem 5.11 and 5.12 in [8]). A natural question is whether tight spherical t-designs are rigid as Euclidean t-designs. We have the proposition below.

Proposition 3.3. *Any tight spherical $2e$-design is rigid as a Euclidean design, for $e \geq 2$.*

On the other hand, as we will show later, any tight spherical 2- and 3-design are non-rigid as Euclidean designs.

Now, let us consider the following two examples of tight Euclidean 4-designs in $\mathbb{R}^2$ given by Bannai and Bannai [4] and also antipodal tight Euclidean 5-designs in $\mathbb{R}^2$ given in Bannai [5].

Example 3.4 (see [4]). *Let $X(r) = X_1 \cup X_2(r)$, where $X_1 = \left\{(1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}$ and $X_2(r) = \left\{(-r,0), \left(\frac{r}{2}, \frac{\sqrt{3}}{2}r\right), \left(\frac{r}{2}, -\frac{\sqrt{3}}{2}r\right)\right\}$. Let $w(x) = 1$ for $x \in X_1$ and $w(x) = \frac{1}{r^3}$ for $x \in X_2(r)$. If $r \neq 1$, then $X(r)$ is a tight Euclidean 4-design.*

Example 3.5 (see [5]). *Let $X(r) = X_1 \cup X_2(r)$ where $X_1 = \{(\pm 1, 0), (0, \pm 1)\}$ and $X_2 = \left\{\left(\pm\frac{r}{\sqrt{2}}, \pm\frac{r}{\sqrt{2}}\right)\right\}$. Let $w(x) = 1$ for $x \in X_1$ and $w(x) = \frac{1}{r^4}$ for $x \in X_2(r)$. If $r \neq 1$, then $X(r)$ is an antipodal tight Euclidean 5-design.*

In both examples above, we can easily see that if we move all the points on $X_2(r)$ simultaneously by changing the radius r while the other points remain sitting on the original position, the resulting designs are again Euclidean designs of the same type. This kind of transformation is not contained in the group $O^*(n)$ since $X(r)$ and $X(r')$ are not similar to each other for any $r \neq r'$. Hence the designs are non-rigid.

In the deformation explained above, all points on the same sphere move to the new one. One natural question is, what will happen if we deform X so that some two points from the same sphere move to distinct two spheres? This question bring us to the notion of strong non-rigidity, a special kind of non-rigidity.

Definition 3.6 (strong non-rigidity). *Let $X = (\{x_i\}_{i=1}^N, w)$ be a Euclidean t-design in $\mathbb{R}^n$. If X satisfies the following condition we say X is* strongly non-rigid *in $\mathbb{R}^n$:*
For any $\varepsilon > 0$ there exists a Euclidean t-design $X' = (\{x_i'\}_{i=1}^N, w')$ such that the following two conditions hold:

(1) $\|x_i - x_i'\| < \varepsilon$ and $|w(x_i) - w'(x_i')| < \varepsilon$, for any $1 \leq i \leq N$; and

(2) There exist distinct i, j satisfying $\|x_i\| = \|x_j\|$ and $\|x_i'\| \neq \|x_j'\|$.

Remark 3.7. *It is clear that any strongly non-rigid Euclidean t-design is non-rigid, since the condition (ii) above implies that the transformation:*

$$x_i \longmapsto x_i', \ 1 \leq i \leq N,$$

is not contained in $O^*(n)$.

Here is our main theorem.

Theorem 3.8. *The following tight Euclidean t-designs are strongly non-rigid:*

(1) Tight spherical 2-designs in S^{n-1} considered as tight Euclidean 2-designs.

(2) Antipodal tight spherical 3-designs in S^1 considered as tight Euclidean 2-designs.

(3) Tight Euclidean 4-designs in $\mathbb{R}^2$ supported by 2 concentric spheres.

(4) Antipodal tight Euclidean 5-designs in $\mathbb{R}^2$ supported by 2 concentric spheres.

Theorem 3.8 implies the following corollary.

Corollary 3.9. *There are infinitely many tight Euclidean designs of the following type:*

(1) 2-designs in $\mathbb{R}^n$ supported by $p = 2, 3, \ldots, n+1$ concentric spheres, respectively.

(2) Antipodal 3-designs in $\mathbb{R}^2$ supported by 2 concentric spheres.

(3) 4-designs in $\mathbb{R}^2$ supported by 3 and 4 concentric spheres.

(4) Antipodal 5-designs in $\mathbb{R}^2$ supported by 3 and 4 concentric spheres.

Corollary 3.9 says about the existence of quite plenty of tight Euclidean t-designs, contrary to the initial guess made by Neumaier and Seidel and also Delsarte and Seidel respectively in [12] and [9]. We remark here that antipodal tight Euclidean 3-designs in $\mathbb{R}^n$ have been completely classified in [5].

We may prove Theorem 3.8 using the implicit function theorem described below.

Let X be a tight Euclidean t-design in $\mathbb{R}^n$. Let $|X| = N$, $X = \{u_i,\ 1 \le i \le N\}$ and $u_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ for $1 \le i \le N$. Let $w(u_i)$ be the weight of u_i, for $1 \le i \le N$. Then we consider $(u_{i,1}, u_{i,2}, \ldots, u_{i,n}, w(u_i),\ 1 \le i \le N)$ as a vector $\eta = (\eta_1, \eta_2, \ldots, \eta_{(n+1)N}) \in \mathbb{R}^{(n+1)N}$ whose entries are given by $u_{i,1}, u_{i,2}, \ldots, u_{i,n}, w(u_i)$, for $1 \le i \le N$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_{(n+1)N}) \in \mathbb{R}^{(n+1)N}$ be the vector variable whose entries are defined by $(x_{i,1}, x_{i,2}, \ldots, x_{i,n}, w(x_i),\ 1 \le i \le N)$. Then η is a common zero point of a given set of polynomials $f_1(\xi), f_2(\xi), \ldots, f_K(\xi)$ in the vector variable ξ (c.f. Theorem 2.1 (2)). Let $I = \{i,\ 1 \le i \le (n+1)N\}$ and $I' \subseteq I$. We denote by J' the Jacobian

$$J' = \left(\frac{\partial f_i}{\partial \xi_k}\right)_{\substack{1 \le i \le K,\\ k \in I \setminus I'}}.$$

Assume $|I \setminus I'| = K$ and that $\mathrm{rank}(J') = K$ holds at η. We may assume $I \setminus I' = \{1, 2, \ldots, K\}$ by reordering the components of the vectors ξ and η. Let $\xi' = (\xi_i,\ i \in I')$ and $\eta' = (\eta_i,\ i \in I')$. Then the implicit function theorem tells us that there exist unique continuously differentiable function $\Psi(\xi') = (\psi_i(\xi'),\ i \in I \setminus I')$ satisfying the following conditions:

(1) For any $1 \le j \le K$,

$$f_j(\psi_1(\xi'), \psi_2(\xi'), \ldots, \psi_K(\xi'), \xi') = 0$$

holds in some small neighborhood of η'.

(2) $\psi_i(\eta') = \eta_i$, for any $1 \le i \le K$.

Let $\xi_i = \psi_i(\xi')$, for $1 \le i \le K$. Then for any ξ' in a small neighborhood of η', $X' = \{\xi_i,\ i \in I\}$ is a Euclidean t-design. Since $\psi_i(\xi')$, $1 \le i \le K$, are continuous function of ξ', we can make $|\xi_i - \eta_i| < \varepsilon$ for any given positive real number ε. For example, if X is a tight Euclidean $2e$-design and I' contains all the indices corresponding to the variables $w_1, w_2, \ldots, w_N$, then we can make every point in X' having distinct weight values. Since, by Proposition 2.3, a tight Euclidean $2e$-design X' must have constant weight on each sphere which support X', every point of X' must be on the different spheres.

4 Tight Euclidean 2-designs in $\mathbb{R}^n$

In the previous section we have shown that tight spherical 2-designs in $\mathbb{R}^n$ are strongly non-rigid and hence there exist infinitely many (non-isomorphic) tight Euclidean 2-designs in $\mathbb{R}^n$ supported by $2, 3, \ldots, n+1$ concentric spheres, respectively. The aim of this section is to give the complete classification of tight Euclidean 2-designs in $\mathbb{R}^n$.

By Proposition 2.4 (2) and the fact that in $\mathbb{R}^n$ the Gegenbauer polynomial of degree 1 satisfies

$$Q_1(y) = ny,$$

we obtain

$$\langle u, v\rangle = -\frac{a_1}{na_0}, \text{ for any distinct vectors } u, v \in X.$$

Therefore every tight Euclidean 2-design X is a 1-inner product set with negative inner product value $-\frac{a_1}{na_0}$. In general, a subset $X \subseteq \mathbb{R}^n$ is called e-inner product set if

$$|\{\langle x, y\rangle,\ x, y \in X,\ x \neq y\}| = e$$

holds. The cardinality of e-inner product set in $\mathbb{R}^n$ is known to be bounded from above by $\binom{n+e}{e}$ (see [7]). In particular, a 1-inner product set is bounded above by $n+1$ which is attained by regular simplices which is also tight spherical 2-designs and tight Euclidean 2-designs at the same time.

For any positive real numbers $R_1, R_2, \ldots, R_n$, we define a function f_k of k variables $R_1, R_2, \ldots, R_k$ by the recurrence relation as follows:

$$\begin{cases} f_1 &= R_1, \\ f_k &= f_{k-1}(1+R_k) - \prod_{i=1}^{k-1}(1+R_i), \text{ for } 2 \leq k \leq n. \end{cases} \tag{6}$$

Then we have the following theorem.

Theorem 4.1. *Let $X = \{x_k,\ 1 \leq k \leq n+1\}$ be an $(n+1)$-subset in $\mathbb{R}^n$. Let also $R_k = \|x_k\|^2$, for $1 \leq k \leq n+1$. If X is a 1-inner product set satisfying*

$$\langle x, y\rangle = -1, \textit{ for any distinct } x, y \in X, \tag{7}$$

then the following two conditions hold:

(1) $f_k > 0$, *for* $1 \leq k \leq n$,

(2) $1 + R_{n+1} = \dfrac{\prod_{i=1}^{n}(1+R_i)}{f_n}$.

Conversely, if the conditions (1) and (2) hold, then there exists 1-inner product set $X = \{x_k,\ 1 \leq k \leq n+1\} \subseteq \mathbb{R}^n$ satisfying the condition (7).

In view of Proposition 2.4, we have the theorem below.

Theorem 4.2. *$(X, w) \subseteq \mathbb{R}^n$ is a tight Euclidean 2-design if and only if (X, w) is a weighted 1-inner product set in $\mathbb{R}^n$ of negative inner-product value.*

The theorem 4.1 above enables us to derive the complete classification of 1-inner product set having negative inner product value, while the last theorem guarantees the our 1-inner product set is nothing but the tight Eucldiean 2-designs. Hence, from the above two theorem we have:
(Up to the action of an orthogonal trasformation $O(n)$) Any tight Euclidean 2-designs $X = \{x_k,\ 1 \leq k \leq n+1\} \subseteq \mathbb{R}^n$ is of the following form:

$$\begin{aligned} x_1 &= (\sqrt{R_1},\ 0,\ 0,\ \ldots,\ 0), \\ x_k &= (b_1,\ b_2,\ b_3,\ \ldots,\ b_{k-1},\ x_{k,k},\ 0,\ \ldots,\ 0), \\ &\quad \text{for } 2 \leq k \leq n; \text{ and} \\ x_{n+1} &= (b_1,\ b_2,\ b_3,\ \ldots,\ b_n), \end{aligned}$$

where b_k and $x_{k,k} > 0$ are determined recursively by

$$b_1 = -\frac{1}{\sqrt{R_1}},$$
$$x_{k,k} = \sqrt{\frac{f_k}{f_{k-1}}}, \text{ for } 2 \leq k \leq n,$$
$$b_k = -\frac{\prod_{i=1}^{k-1}(1+R_i)}{f_{k-1}x_{k,k}}, \text{ for } 2 \leq k \leq n,$$

and weight function given by
$$w(x) = \frac{1}{1+\|x\|^2}, \ x \in X.$$

5 Concluding Remarks

(1) Neumaier and Seidel and also Delsarte and Seidel conjectured that the only tight Euclidean $2e$-designs in $\mathbb{R}^n$ are regular simplices (See [12, Conjecture 3.4] and [9, pp. 225]). Recently, Bannai and Bannai [4] has disproved this conjecture providing the example of Euclidean tight 4-designs in $\mathbb{R}^2$ supported by two concentric spheres, i.e., which are not regular simplices. However, constructing a tight Euclidean design is not so easy in general. In this paper we introduce a new notion of a strong non-rigidity of Euclidean t-designs. An alternative way, and in fact a very trivial way, to disprove the conjecture comes from the method we use to investigate the strong non-rigidity of the designs.

(2) Regarding the existence of tight Euclidean designs, we believe in the following conjecture:

Conjecture 5.1. *If a tight Euclidean $2e$-design or an antipodal tight Euclidean $(2e+1)$-design supported by more than* $\left\lceil \frac{e+\varepsilon_S}{2} \right\rceil + 1$ *concentric spheres exists, then there exist infinitely many tight Euclidean $2e$-designs or antipodal tight Euclidean $(2e+1)$-designs, respectively.*

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