数理解析研究所講究録 1476 巻 2006 年 34-42



On the structure of the twisted Grassmann graphs

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November 30, 2005

# 1 Introduction

A graph $\Gamma$ with diameter $d$ is said to be *distance-regular* if there are integers $b_i$ $(i=0,\cdots,d-1)$ and $c_i$ $(i=1,\cdots,d)$ such that for any two veritices $x$ and $y$ such that $d(x,y)=i$,

$$\begin{aligned} b_i &= \#\{z \mid z \text{ :vertex, } d(x,z)=i+1, d(y,z)=1\}, \\ c_i &= \#\{z \mid z \text{ :vertex, } d(x,z)=i-1, d(y,z)=1\}. \end{aligned}$$

Let $q$ be a prime power and $n,e$ be integers such that $n/2 \geq e \geq 2$. The Grassmann graph $J_q(n,e)$ is a graph on the $e$-dimensional subspaces in an $n$-dimensional vector space over the finite field $GF(q)$ where two $e$-dimensional subspaces are adjacent if and only if they intersect in a $(e-1)$-dimensional subspace. The Grassmann graph $J_q(n,e)$ is a distance-regular graph whose parameters are

$$b_i = q^{2i+1}\begin{bmatrix} e-i \\ 1 \end{bmatrix}\begin{bmatrix} n-e-i \\ 1 \end{bmatrix}, \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2.$$

where $\begin{bmatrix} m \\ 1 \end{bmatrix} = q^{m-1}+\cdots+q+1$.

The twisted Grassmann graphs $\tilde{J}_q(2e+1,e)$, which is constructed by E. van Dam and J. Koolen [1], is defined as follows: let $H$ be a hyperplane of the $(2e+1)$-dimensional vector space $V$ over $GF(q)$. Put

$$\begin{aligned} \mathcal{B}_1 &= \{W \text{ :subspace of } V \mid \dim W = e+1, W \not\subseteq H\}, \\ \mathcal{B}_2 &= \{W \text{ :subspace of } H \mid \dim W = e-1\}. \end{aligned}$$

The vertex set of $\tilde{J}_q(2e+1,e)$ is $\mathcal{B}_1 \cup \mathcal{B}_2$ and the adjacency is defined as follows: for $W_1, W_2 \in \mathcal{B}_1 \cup \mathcal{B}_2$,

$$W_1 \sim W_2 \text{ if and only if } \begin{cases} \dim(W_1 \cap W_2) = e & \text{if } W_1, W_2 \in \mathcal{B}_1, \\ \dim(W_1 \cap W_2) = e-2 & \text{if } W_1, W_2 \in \mathcal{B}_2, \\ \dim(W_1 \cap W_2) = e-1 & \text{otherwise .} \end{cases}$$

**Theorem 1** [1] *The twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ is distance-regular and its parameters are same as the Grassmann graph $J_q(2e+1,e)$. Moreover the automorphism group of the twisted Grassmann graph acts on the vertex set with two orbits $\mathcal{B}_1$ and $\mathcal{B}_2$.*

M. Tagami determined the automorphism group of $\tilde{J}_q(2e+1,e)$ and later J. Koolen showed another proof of the coincidence (see [3]).

**Theorem 2** *The automorphism group of $\tilde{J}_q(2e+1,e)$ is just $P\Gamma L(2e+1,q)_H$.*

Let $\mathcal{X} = (X,\{R_i\}_{0\le i\le d})$ be a commutative association scheme. Suppose that $\mathcal{X}$ is Q-polynomial. For $i \in \{0,\cdots,d\}$, let $A_i$ be a matrix indexed by $X$ defined as follows: for two vertices $x,y$,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y)\in R_i, \\ 0 & \text{if } (x,y)\notin R_i, \end{cases}$$

Fix a vertex $x$. For $i\in\{0,\cdots,d\}$, let $E_i^* = E_i^*(x)$ be a diagonal matrix indexed by the vertex set of $\Gamma$ defined by, for each vertex $y$,

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } d(x,y)=i, \\ 0 & \text{otherwise} \end{cases}$$

The algebra $T=T(x)$ generated by $A_0,\cdots,A_d$ and $E_0^*,\cdots,E_d^*$ over the complex field is called the *Terwilliger algebra with respect to $x$*. For an irreducible $T$-module $W$, if for any $i\in\{0,\cdots,d\}$, $\dim(E_i^*W)\le 1$, we say that $W$ is *thin*, and if any irreducible $T$-module is thin, we say $T$ is *thin*. Every Terwilliger algebra $T$ has a thin module $T\mathbf{1}$ where $\mathbf{1}$ is an all-one vector. This module satisfies $\dim(E_i^*W)=1$ for any $i$. If an irreducible $T$-module $W$ has an integer $j$ of $\{0,\cdots,d\}$ such that $\dim(E_i^*W)=0$ for any $i<j$ and $\dim(E_j^*W)\ne 0$, we say that $W$ is *of endpoint* $j$ (ref. [4].) P. Terwilliger conjectured the following: If a commutative association scheme $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ is Q-polynomial, then one of the following holds (1) $\mathcal{X}$ is formally self-dual or (2) for any $x\in X$, the Terwilliger algebra $T(x)$ is thin.

It is well-known that the association scheme obtained from the Grassmann graph is Q-polynomial. The association scheme is not formally self-dual but for any $x\in X$, the Terwilliger algebra $T(x)$ is thin, that is, the above conjecture holds. Since conditions of Q-polynomial and self-dual depends only on the parameters $b_i$ and $c_i$, the association scheme obtained from the twisted Grassmann graph is also Q-polynomial but not formally self-dual.

For a graph $\Gamma$, let $A=A_\Gamma$ be the adjacency matrix of $\Gamma$. We call the eigenvalues and multiplicities of $A$ the *eigenvalues* and *multiplicties* of $\Gamma$ respectively. Let $\theta_1,\theta_2,\cdots,\theta_t$ and $m_1,m_2,\cdots,m_t$ are respectively the eigenvalues and corresponding multiplicities of $\Gamma$. Then for any non-negative integer $i$,

$$\sum_{p=1}^{t} m_p\theta_p^i = \text{Trace } A^i = \#\{\text{ closed path of length } i \text{ in } \Gamma\}$$

where closed path of length $i$ means that a sequence $x_1,x_2,\cdots,x_i$ of vertices satisfying any consective two vertices are adjacent and $x_i$ and $x_1$ is also adjacent. Define that a closed path of length 0 is a vertex. For a distance-regular graph $\Gamma$ with parameters $b_i$ and $c_i$, let $A$ be a adjacency matrix of the local graph with respect to a vertex $x$. We can easily see that

$$\text{Trace } A^0 = b_0, \text{ Trace } A^1 = 0, \text{ and Trace } A^2 = b_0(b_0 - b_1 - 1).$$

In particular, for the Grassmann graph $J_q(2e+1,e)$ and the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$, we have that Trace $A^0 = q\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$, Trace $A^1 = 0$ and Trace $A^2 = q\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e+1 \\ 1 \end{bmatrix}\left(q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1\right)$.

A distance-regular graph $\Gamma$ has *classic parameter* $(d,q,\alpha,\beta)$ if

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \end{aligned}$$

The Grassmann graph $J_q(n,e)$ has classic parameter $\left(e,q,q,q\begin{bmatrix} n-e \\ 1 \end{bmatrix}\right)$. Similarly the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ also has calssic parameter $\left(e,q,q,q\begin{bmatrix} e+1 \\ 1 \end{bmatrix}\right)$.

## 2 Computing the eigenvalues of graphs

In this section, we show the method to compute the eigenvalues of graphs. For a graph $\Gamma$ on the vertex set $V$ and for an automorphism group $G$, not necessarily Aut$(G)$, consider actions of $G$ on $V$ and $V \times V$. Let $O_1, O_2, \cdots O_p$ be the orbits on $V$ and $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_{p'}$ be the orbits on $V \times V$. Suppose that $\mathcal{O}_1, \cdots, \mathcal{O}_s$ satisfies that $\cup_{i=1}^s \mathcal{O}_i = O_1 \times V$. For $1 \le i,j \le s$ and for $(x,y) \in \mathcal{O}_i$,

$$P(i,j) = \#\{z: \text{ vertex } \mid (x,z) \in \mathcal{O}_j \text{ and } y,z \text{ are adjacent}\}$$

is independent of the choice of $(x,y)$ and only depends on $i$ and $j$. Let $P_1 = (P(i,j))_{1 \le i,j \le s}$. Similarly we can construct matrices $P_2, \cdots, P_p$.

**Proposition 3** *The union of eigenvalues of $P_i$'s is just the eigenvalues of $\Gamma$.*

From now on, we put $\Gamma$ as the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$. We consider an action of the stabilizer of $U \in \mathcal{B}_1 \cup \mathcal{B}_2$ in $G = P\Gamma L(2e+1,q)_H$ as automorphism group. Then we need to separate computation of eigenvalues in each case $U \in \mathcal{B}_1$ or $U \in \mathcal{B}_2$.

### 2.1

Fix $U \in \mathcal{B}_1$. Then the neighbors of $U$ in $\Gamma$ consists of the following two sets:

$$A := \{W \in \mathcal{B}_1 \mid W \text{ is adjacent to } U\}, \ B := \{W \in \mathcal{B}_2 \mid W \text{ is adjacent to } U\}.$$

The $A$ and $B$ forms the $G_U$-orbitals on the neighbors of $U$ and $G_U$-orbitals on $A \cup B$ are following:

$$\begin{aligned}
A_0 &:= \{(W_1, W_1) \mid W_1 \in A\}, \\
A_1 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap U = W_2 \cap U, \langle W_1, U\rangle = \langle W_2, U\rangle, W_1 \neq W_2\}, \\
A_2 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap U = W_2 \cap U, \langle W_1, U\rangle \neq \langle W_2, U\rangle\}, \\
&\quad \text{(In the cases of } A_1 \text{ and } A_2,\ W_1 \cap W_2 \text{ is a } (e-2)\text{-dimensional subspace in } U.) \\
A_3 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap W_2 : (e-2)\text{-dimensional subspace not in } U\}, \\
A_4 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap W_2 : (e-3)\text{-dimensional subspace}\}, \\
&\quad \text{(In this case, } W_1 \cap W_2 \text{ is in } U.) \\
AB_1 &:= \{(W_1, W_2) \in A \times B \mid W_1 \subset W_2\}, \\
AB_2 &:= \{(W_1, W_2) \in A \times B \mid W_1 \not\subset W_2\}, \\
BA_1 &:= \{(W_1, W_2) \in B \times A \mid W_2 \subset W_1\} (= \text{transpose of } AB_1), \\
BA_2 &:= \{(W_1, W_2) \in B \times A \mid W_2 \not\subset W_1\} (= \text{transpose of } AB_2), \\
B_0 &:= \{(W_1, W_1) \mid W_1 \in B\}, \\
B_1 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 :\ e\text{-dimensional subspace in } H\} \\
B_2 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 :\ e\text{-dimensional subspace not in } H\} \\
B_3 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 = U\}
\end{aligned}$$

For two $G_U$-orbitals $K$ and $K'$ and for $(W_1, W_2) \in K$, put

$$p(K, K') := \{W \in A \cup B \mid (W_1, W) \in K',\ W \text{ is adjacent to } W_2\}.$$

Then the following holds:

| $p(K,K')$ | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $AB_1$ | $AB_2$ |
|---|---|---|---|---|---|---|---|
| $A_0$ | $0$ | $q-1$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$ | $0$ | $q^e$ | $0$ |
| $A_1$ | $1$ | $q-2$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$ | $0$ | $q^e$ | $0$ |
| $A_2$ | $1$ | $q-1$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}-1$ | $0$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$ | $0$ | $q^e$ |
| $A_3$ | $1$ | $q-1$ | $0$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}-1$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $q^e$ | $0$ |
| $A_4$ | $0$ | $0$ | $q$ | $q$ | $x$ | $0$ | $q^e$ |
| $AB_1$ | $1$ | $q-1$ | $0$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$ | $0$ | $q^e-1$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ |
| $AB_2$ | $0$ | $0$ | $q$ | $0$ | $q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$ | $q$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}+q^e-q-1$ |

where $x = q^2\left(\begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} e-2 \\ 1 \end{bmatrix}\right) - q - 1$. Considering the above array as a $7 \times 7$ matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and $-1$. Similarly, we have the following results:

| | $B_0$ | $B_1$ | $B_2$ | $B_3$ | $BA_1$ | $BA_2$ |
|---|---|---|---|---|---|---|
| $B_0$ | $0$ | $q^e-1$ | $q^2\begin{bmatrix}e\\1\end{bmatrix}$ | $0$ | $q^2\begin{bmatrix}e-1\\1\end{bmatrix}$ | $0$ |
| $B_1$ | $1$ | $q^e-2$ | $q^2$ | $q^3\begin{bmatrix}e-1\\1\end{bmatrix}$ | $q^2\begin{bmatrix}e-1\\1\end{bmatrix}$ | $0$ |
| $B_2$ | $1$ | $q-1$ | $q^2\begin{bmatrix}e-1\\1\end{bmatrix}+q^2-1$ | $q(q^2-1)\begin{bmatrix}e-1\\1\end{bmatrix}$ | $0$ | $q\begin{bmatrix}e-1\\1\end{bmatrix}$ |
| $B_3$ | $0$ | $q$ | $q(q+1)$ | $y$ | $0$ | $q\begin{bmatrix}e-1\\1\end{bmatrix}$ |
| $BA_1$ | $1$ | $q^e-1$ | $0$ | $0$ | $q\begin{bmatrix}e-1\\1\end{bmatrix}-1$ | $q^2\begin{bmatrix}e\\1\end{bmatrix}$ |
| $BA_2$ | $0$ | $0$ | $q$ | $q^e-q$ | $q$ | $z$ |

where $y=q^3\begin{bmatrix}e\\1\end{bmatrix}+q^e-2q-1$, $z=q^2\left(\begin{bmatrix}e\\1\end{bmatrix}+\begin{bmatrix}e-2\\1\end{bmatrix}\right)-1$. Considering the above array as a $6\times 6$ matrix, the eigenvalues are $q(q+1)\begin{bmatrix}e\\1\end{bmatrix}-1$, $q^2\begin{bmatrix}e\\1\end{bmatrix}-1$, $q^2\begin{bmatrix}e-1\\1\end{bmatrix}-1$ and $-q-1$. Therefore we have the following conclusion.

**Proposition 4** *For the local graph of $\tilde{J}_q(2e+1,e)$ around $W\in\mathcal{B}_1$, the eigenvalues are $q(q+1)\begin{bmatrix}e\\1\end{bmatrix}-1$, $q^2\begin{bmatrix}e\\1\end{bmatrix}-1$, $q^2\begin{bmatrix}e-1\\1\end{bmatrix}-1$, $-q-1$ and $-1$.*

The number of 3-cycles in $\Gamma(U)$ is equal to $qx(qx+1)(2q^3x^2+(q^4-q^3-q^2-3q)x+q^3-q^2+2)$. Let $m_1,m_2,m_3$ and $m_4$ be the multiplicities of $q^2\begin{bmatrix}e\\1\end{bmatrix}-1$, $q^2\begin{bmatrix}e-1\\1\end{bmatrix}-1$, $-q-1$ and $-1$ respectively. From them, we conclude that $m_1=\begin{bmatrix}e-1\\1\end{bmatrix}, m_2=q\begin{bmatrix}e+1\\1\end{bmatrix}-1, m_3=q^2\left(\begin{bmatrix}e+1\\1\end{bmatrix}-q^{e-1}\right)\begin{bmatrix}e-1\\1\end{bmatrix}$, and $m_4=\begin{bmatrix}e-1\\1\end{bmatrix}(q^{e+1}-1)$.

## 2.2

Fix $U\in\mathcal{B}_2$. Then the neighbors of $U$ in $\Gamma$ consists of the following three $G$-invariant sets:

$$\begin{aligned} C &:= \{W\in\mathcal{B}_2 \mid W\cap U=U\cap H\},\\ D &:= \{W\in\mathcal{B}_2 \mid W\cap U\neq U\cap H,\ W \text{ is adjacent to } U\},\\ E &:= \{W\in\mathcal{B}_1 \mid W\subset U\}. \end{aligned}$$

These three set forms the $G_U$-orbits on the neighbors of $U$. The $G_U$-orbitals on $C$ are following:

$$\begin{aligned} C_0 &:= \{(W_1,W_1)\mid W_1\in C\},\\ C_1 &:= \{(W_1,W_2)\in C\times C \mid \langle W_1,U\rangle=\langle W_2,U\rangle, W_1\neq W_2\},\\ C_2 &:= \{(W_1,W_2)\in C\times C \mid \langle W_1,U\rangle\neq\langle W_2,U\rangle\}. \end{aligned}$$

The $G_U$-orbitals on $C\times D$ are following:

$$\begin{aligned} CD_1 &:= \{(W_1, W_2) \in C \times D \mid \dim W_1 \cap W_2 = e\}, \\ CD_2 &:= \{(W_1, W_2) \in C \times D \mid \dim W_1 \cap W_2 = e-1\}. \end{aligned}$$

The $G_U$-orbitals on $D \times C$ are $DC_1 := (CD_1)^t$ and $DC_2 := (CD_2)^t$. The sets $C \times E$ and $E \times C$ form $G_U$-orbitals. Let $W_1 \in D$. Then since $W \cap U$ is an $e$-dimensional subspace distinct from $U \cap H$, $U_1 := W_1 \cap U \cap H$ is an $(e-1)$-dimensional subspace and for some vectors $u_0, u_0' \in H$ and $u_1 \notin H$, $U = \langle U_1, u_0, u_1 \rangle$ and $W_1 = \langle U_1, u_0', u_1 \rangle$. The $(G_U)_{W_1}$-orbitals on $D$ are following:

$$\begin{aligned} D_0 &:= \{W_1\}, \\ D_1 &:= \{\langle U_1, u_1, u\rangle \mid u \in \langle u_0, u_0'\rangle\}, \\ D_2 &:= \{\langle U_1, u_1, u\rangle \mid u \in H \setminus \langle u_0, u_0'\rangle\}, \\ D_3 &:= \{\langle U_1, au_0 + u_1, u_0'\rangle \mid a \in \mathbf{F}_q^\times\}, \\ D_4 &:= \{\langle U_1, au_0 + u_1, u\rangle \mid a \in \mathbf{F}_q^\times, u \in \langle u_0, u_0'\rangle \setminus \{\langle u_0\rangle, \langle u_0'\rangle\}\}, \\ D_5 &:= \{\langle U_1, au_0 + u_1, u\rangle \mid a \in \mathbf{F}_q^\times, u \in H \setminus \langle U_1, u_0, u_0'\rangle\}, \\ D_6 &:= \{\langle U_1', au + u_1, bu + u_0', cu + u_0\rangle \mid a, b, c \in \mathbf{F}_q, U_1 = \langle U_1', u\rangle, \dim U_1' = e-2\}, \\ D_7 &:= \{\langle U_1', au + u_1, bu + u_0', v\rangle \mid a, b \in \mathbf{F}_q, U_1 = \langle U_1', u\rangle, \dim U_1' = e-2, \\ &\qquad\qquad v \in H \setminus \langle U_1, u_0, u_0'\rangle\}. \end{aligned}$$

The $G_U$-orbitals on $D \times E$ are $DE_1 := \{(W_1, W_1 \cap U \cap H) \mid W_1 \in D\}$ and its complement $DE_2$. $ED_1 := (DE_1)^t$ and $ED_2 := (DE_2)^t$ form the $G_U$-orbitals on $E \times D$. The $G_U$-orbitals on $E \times E$ are $E_0 := \{(W_1, W_1) \mid W_1 \in E\}$ and its complement $E_1$. Consider matrices whose entries are $p(K, K')$. First we can obtain the following table:

| | $C_0$ | $C_1$ | $C_2$ | $CD_1$ | $CD_2$ | $C \times E$ |
|---|---|---|---|---|---|---|
| $C_0$ | $0$ | $q-2$ | $q^e - q$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $0$ | $\begin{bmatrix} e \\ 1 \end{bmatrix}$ |
| $C_1$ | $1$ | $q-3$ | $q^e - q$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $0$ | $\begin{bmatrix} e \\ 1 \end{bmatrix}$ |
| $C_2$ | $1$ | $q-2$ | $q^e - q - 1$ | $0$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $\begin{bmatrix} e \\ 1 \end{bmatrix}$ |
| $CD_1$ | $1$ | $q-2$ | $0$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $1$ |
| $CD_2$ | $0$ | $0$ | $q-1$ | $q$ | $\alpha$ | $1$ |
| $C \times E$ | $1$ | $q-2$ | $q^e - q$ | $q^2$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |

where $\alpha = q^2\left(\begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} e-1 \\ 1 \end{bmatrix}\right) - q - 1$. Considering the above array as a $6 \times 6$ matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-1$ and the roots of $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$. The eigenvalue $-1$ has multiplicity 2

We note that the equation $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$ has no roots in $\mathbf{Q}[q]$. Next we obtain the following table:

| | $E_0$ | $E_1$ | $ED_1$ | $ED_2$ | $E \times C$ |
|---|---|---|---|---|---|
| $E_0$ | 0 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | 0 | $q^e - 1$ |
| $E_1$ | 1 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$ | 0 | $q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$ | $q^e - 1$ |
| $ED_1$ | 1 | 0 | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} + q^2 - 1$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q - 1$ |
| $ED_1$ | 0 | 1 | $q^2$ | $q^2(q+1)\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$ | $q - 1$ |
| $E \times C$ | 1 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q^2$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q^e - 2$ |

Considering the above array as a $5 \times 5$ matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and $-1$. Finally we have the following tables:

(i):

| | $D_0$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ |
|---|---|---|---|---|---|---|---|---|
| $D_0$ | 0 | $q-1$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q-1$ | $(q-1)^2$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | 0 |
| $D_1$ | 1 | $q-2$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q-1$ | $(q-1)^2$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | 0 |
| $D_2$ | 1 | $q-1$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$ | 0 | 0 | $q(q-1)$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $D_3$ | 1 | $q-1$ | 0 | $q-2$ | $(q-1)^2$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | 0 |
| $D_4$ | 1 | $q-1$ | 0 | $q-1$ | $q^2-2q$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | 0 |
| $D_5$ | 0 | 0 | $q$ | 1 | $q-1$ | $\alpha$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $D_6$ | 1 | $q-1$ | 0 | $q-1$ | $(q-1)^2$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $D_7$ | 0 | 0 | $q$ | 0 | 0 | $q(q-1)$ | $q$ | $\beta$ |

where $\alpha = q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} + q^2 - 2q - 1$ and $\beta = q^2(q+1)\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - q - 1$. Put this table $D_{11}$.

(ii):

| | $DC_1$ | $DC_2$ | $DE_1$ | $DE_2$ |
|---|---|---|---|---|
| $D_0$ | $q-1$ | 0 | 1 | 0 |
| $D_1$ | $q-1$ | 0 | 1 | 0 |
| $D_2$ | 0 | $q-1$ | 1 | 0 |
| $D_3$ | $q-1$ | 0 | 1 | 0 |
| $D_4$ | $q-1$ | 0 | 1 | 0 |
| $D_5$ | 0 | $q-1$ | 1 | 0 |
| $D_6$ | $q-1$ | 0 | 0 | 1 |
| $D_7$ | 0 | $q-1$ | 0 | 1 |

Put this table $D_{12}$.

(iii):

| | $D_0$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ |
|---|---|---|---|---|---|---|---|---|
| $DC_1$ | 1 | $q-1$ | 0 | $q-1$ | $(q-1)^2$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | 0 |
| $DC_2$ | 0 | 0 | $q$ | 0 | 0 | $q(q-1)$ | 0 | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $DE_1$ | 1 | $q-1$ | $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ | $q-1$ | $(q-1)^2$ | $q^2(q^{e-1}-1)$ | 0 | 0 |
| $DE_2$ | 0 | 0 | 0 | 0 | 0 | 0 | $q^2$ | $q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |

Put this table $D_{21}$.

(iv):

| | $DC_1$ | $DC_2$ | $DE_1$ | $DE_2$ |
|---|---|---|---|---|
| $DC_1$ | $q-2$ | $q^e-q$ | 1 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $DC_2$ | $q-1$ | $q^e-q-1$ | 1 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $DE_1$ | $q-1$ | $q^e-q$ | 0 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$ |
| $DE_2$ | $q-1$ | $q^e-q$ | 1 | $q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1$ |

Put this table $D_{22}$.

Let $Z = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ be a $12 \times 12$ matrix. Then the eigenvalues and their multiplicities are as follows:

| Eigenvalue | Multiplicities |
|---|---|
| $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix}-1$ | 1 |
| $q\begin{bmatrix} e \\ 1 \end{bmatrix}-1$ | 1 |
| $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1$ | 2 |
| $-q-1$ | 3 |
| $-1$ | 3 |
| $\theta_1, \theta_2$ | 1 (for each root) |

where $\theta_1$ and $\theta_2$ are the roots of $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$.

**Proposition 5** *For the local graph of $\tilde{J}_q(2e+1, e)$ with respect to $W \in \mathcal{B}_2$, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$, $-1$ and the roots of $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$.*

The number of 3-cycles in $\Gamma(U)$ is equal to the sum of numbers obtained from (1) to (9), which is $qx(q^5x^3 + q^3x^3 + 3q^4x^2 - 6q^3x^2 + q^2x^2 + 5q^3x - 6q^2x^qx + 2)$. The multiplicities $m_1$, $m_2$, $m_3$, $m_4$ and $m_5$ satisfy that for any $i \geq 0$,

$$(q(q+1)x-1)^i + (qx-1)^i m_1 + (qx-q-1)^i m_2 + (-q-1)^i m_3 + (-1)^i m_4 + a_i m_5 = \mathrm{Tr}A^i \quad (1)$$

where $a_i$ is defined by as follows: $a_0 = 2$, $a_1 = q^2x - q - 2$, $a_i = (q^2x - q - 2)a_{i-1} + (q^3x - q - 1)a_{i-2}$ for $i \geq 2$, which means that $\theta_1^i + \theta_2^i = a_i$ for any $i$. From them, we can see that $m_1 = q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}, m_2 = q^e, m_3 = q^2\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e-1 \\ 1 \end{bmatrix}, m_4 = (q^{e+1}-1)\begin{bmatrix} e \\ 1 \end{bmatrix} - q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$, and $m_5 = q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$.

## 3 Thin and non-thin irreducible modules

Let $\Gamma$ be a distance-regular graph with classic parameter $(d, q, \alpha, \beta)$. For a local graph $\Gamma(x)$, if $\lambda \neq b_0 - b_1 - 1$ is an eigenvalue of the local graph, there exists an eigenvector $v$ of $E_1^* A_1 E_1^*$ whose eigenvalue is $\lambda$. Then $Tv$ forms an irreducible $T$-module of endpoint 1. Moreover any irreducible $T$-module of endpoint 1 is $Tv$ for some eigenvector $v$ of $E_1^* A_1 E_1^*$. P. Terwilliger proved the following [4]:

**Theorem 6** *In the above assumption, the irreducible module $Tv$ is thin if and only if*

$$\lambda \in \left\{ \alpha \begin{bmatrix} d-1 \\ 1 \end{bmatrix} - 1, \beta - \alpha - 1, -q - 1, -1 \right\}$$

As we noted, the Grassmann graph $J_q(2e+1, e)$ and the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ have classic parameter $(e, q, q, \begin{bmatrix} e+1 \\ 1 \end{bmatrix})$. In these cases,

$$\alpha \begin{bmatrix} d-1 \\ 1 \end{bmatrix} - 1 = q \begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1, \beta - \alpha - 1 = q \begin{bmatrix} e \\ 1 \end{bmatrix} - 1.$$

Hence the above set in the Theorem is just the eigenvalues of the local graph of Grassmann graph except $b_0 - b_1 - 1 = q(q+1) \begin{bmatrix} e \\ 1 \end{bmatrix} - 1$.

For the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$, let $U \in \mathcal{B}_1$. Then, from results in the previous section, we can see that the Terwilliger algebra $T(U)$ has 4 irreducible modules of endpoint 1. Moreover the above theorem, all such modules are thin. On the other hand, let $U \in \mathcal{B}_2$. Then there are 6 irreducible $T(U)$-modules of endpoint 1. From results in the previous section, we can see that three of them are thin and the other are non-thin. Therefore, the twisted Grassmann graph is a counterexample of the conjecture of Terwilliger.

## References


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