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Optimal Portfolio Selection by CVaR-Based Sharpe Ratio — Sequential Linear Programming Approach —

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1 Introduction

We address an optimal portfolio selection problem of maximizing so we call CVaR (Conditional Value-at-Risk)-based Sharpe ratio of portfolio's return rate, which is defined as the ratio of the expected excess return to CVaR. The Sharpe ratio defined as the ratio of expected excess return to standard deviation, the most common traditional performance measure, takes standard deviation as a risk measure, however, its has been received a lot of criticisms. In our CVaR-based Sharpe ratio, the standard deviation is replaced with CVaR, which is a remarkable coherent risk measure which overcomes essential defects of standard deviation. Although our new performance measure is expected to enlarge the applicable area of practical investment problems for which the original Sharpe ratio is not suitable, however, we should device effective computational methods to solve optimal portfolio selection problems with very large number of investment opportunities.

In order to deal with rather complicated non-concave objective function, which comes from the introduction of CVaR, we propose the following SLP (Sequential Linear Programming) approach: By introducing a real parameter, we make the non-concave maximization problem to a parametric family of concave maximization problems. Then, for each of these problems, utilizing the results of Rockafellar and Uryasev (2000), we introduce an auxiliary decision variable to obtain a tractable concave maximization problem. Furthermore, if we estimate or approximate required expected values by sampling methods or historical data, we can reduce the parametric concave maximization problems to LP (Linear Programming) problems. Therefore, our problem could be finally reduced to a sequence of LP problems. Numerical experiments from real Japanese financial data are conducted to test our SLP approach.

The paper is organized as follows. In Section 2, we introduce downside risk measures: VaR and CVaR. in Section 3, we make a brief review of parametric approach to fractional programming. Sample approach will be presented in Section 4. Further, in Section 5, an empirical study is given.

2 VaR and CVaR

Let $\tilde{r}$ denote a random variable denoting a rate of return on an asset or a portfolio of assets. Value at Risk (VaR) of $\tilde{r}$ with confidence level $\beta \in [0, 1]$, denoted $\mathrm{VaR}_\beta[\tilde{r}]$, is defined as the negative of $(1-\beta)$–quantile of $\widetilde{r}$:

$$\begin{aligned}\mathrm{VaR}_\beta[\widetilde{r}] &:= -\inf\{r\in\mathbb{R} : \mathbb{P}(\widetilde{r}\le r)\ge 1-\beta\}\\ &= \sup\{u\in\mathbb{R} : \mathbb{P}(-\widetilde{r}\ge u)\ge 1-\beta\}.\end{aligned} \tag{1}$$

Conditional Value at Risk (CVaR) of $\widetilde{r}$ with confidence level $\beta\in[0,1]$, denoted $\mathrm{CVaR}_\beta[\widetilde{r}]$, is then defined as follows:

$$\mathrm{CVaR}_\beta[\widetilde{r}] := \frac{1}{1-\beta}\int_0^{1-\beta}\mathrm{VaR}_\alpha[\widetilde{r}]\mathrm{d}\alpha. \tag{2}$$

It can be shown that

$$\mathrm{CVaR}_\beta[\widetilde{r}] = \frac{1}{1-\beta}\mathbb{E}\left[-\widetilde{r}; -\widetilde{r}\ge \mathrm{VaR}_\beta[\widetilde{r}]\right] - \mathrm{VaR}_\beta[\widetilde{r}]\left\{\mathbb{P}\left(-\widetilde{r}\ge\mathrm{VaR}_\beta[\widetilde{r}]\right)-(1-\beta)\right\}, \tag{3}$$

where $\mathbb{E}[Y;A]$ denotes the partial expectation of a random variable Y on an event A; that is $\mathbb{E}[Y;a];=\mathbb{E}[Y1_A]$. Although this expression is somewhat complex, if

$$\mathbb{P}\left(-\widetilde{r}\ge\mathrm{VaR}_\beta[\widetilde{r}]\right) = 1-\beta, \tag{4}$$

then the second term vanishes and it becomes

$$\mathrm{CVaR}_\beta[\widetilde{r}] = \frac{1}{1-\beta}\mathbb{E}\left[-\widetilde{r}; -\widetilde{r}\ge\mathrm{VaR}_\beta[\widetilde{r}]\right] = \mathbb{E}\left[-\widetilde{r}\right| -\widetilde{r}\ge\mathrm{VaR}_\beta[\widetilde{r}]\right]. \tag{5}$$

Average Value at Risk (AVaR), Expected Shortfall (ES), Tail Conditional Expectation (TCE), and others are similar concepts, not few researchers prefer one of these terms to CVaR, but these become identical when the above condition holds (whose sufficient condition is the continuity of cumulative distribution function (cdf) of $\widetilde{r}$).

A very useful characterization is obtained by Pflug (2000), Uryasev (2000), and Rockafellar and Uryasev (2000, 2001). Let us introduce a function:

$$F_\beta(a;\widetilde{r}) := a + \frac{1}{1-\beta}\mathbb{E}\left[(-\widetilde{r}-a)^+\right], \quad a\in\mathbb{R}, \tag{6}$$

then the following theorem holds (for a real number $c\in\mathbb{R}$, $(c)^+ := \max\{c,0\}$ is the positive part of c).

Theorem 1.

(1) $\mathrm{CVaR}_\beta[\widetilde{r}]$ coincides with the minimum of function $F_\beta(\cdot;\widetilde{r})$:

$$\mathrm{CVaR}_\beta[\widetilde{r}] = \min\{F_\beta(a;\widetilde{r}) : a\in\mathbb{R}\}. \tag{7}$$

(2) The minimum of function $F_\beta(\cdot;\widetilde{r})$ is attained at when the variable is equal to $\mathrm{VaR}_\beta[\widetilde{r}]$:

$$\min\{F_\beta(a;\widetilde{r}) : a\in\mathbb{R}\} = F_\beta\left(\mathrm{VaR}_\beta[\widetilde{r}(\boldsymbol{x})];\widetilde{r}\right). \tag{8}$$

(3) $F_\beta(a;\widetilde{r})$ is convex both in $a\in\mathbb{R}$ and $\widetilde{r}$. □

Now let us consider a portfolio optimization of investments in financial assets numbered $i=1,\cdots,n$. We use the following notations:

- $\widetilde{r}_i$, $i = 1, \cdots, n$: the random rate of return on financial asset i;
- $\overline{r}_i := \mathbb{E}[\widetilde{r}_i]$, $i = 1, \cdots, n$: the mean (or expected) rate of return on financial asset i;
- x_i ($\in \mathbb{R}$), $i = 1, \cdots, n$: a portfolio weight, that is, a proportion of investment in financial asset i;
- $\widetilde{\boldsymbol{r}} := (\widetilde{r}_1, \cdots, \widetilde{r}_n)^\top$: the random vector of return rates on financial assets $i = 1, \cdots, n$;
- $\overline{\boldsymbol{r}} := (\widetilde{r}_1, \cdots, \widetilde{r}_n)^\top$: the vector of mean return rates on financial assets $i = 1, \cdots, n$;
- $\boldsymbol{x} := (x_1, \cdots, x_n)^\top$: the portfolio of investment proportions in financial assets $i = 1, \cdots, n$.

Further, we let

$$\widetilde{r}(\boldsymbol{x}) := \widetilde{\boldsymbol{r}}^\top \boldsymbol{x} = \sum_{i=1}^{n} \widetilde{r}_i x_i; \tag{9}$$

$$\overline{r}(\boldsymbol{x}) := \mathbb{E}[\widetilde{r}(\boldsymbol{x})] = \mathbb{E}[\widetilde{\boldsymbol{r}}^\top \boldsymbol{x}] = \sum_{i=1}^{n} \mathbb{E}[\widetilde{r}_i] x_i = \overline{\boldsymbol{r}}^\top \boldsymbol{x} = \sum_{i=1}^{n} \overline{r}_i x_i. \tag{10}$$

The above theorem is particularly useful when we must consider the minimization of $\mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})]$ of return rate $\widetilde{r}(\boldsymbol{x})$ on portfolio $\boldsymbol{x}$. According to the definition of $\mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})]$, for every evaluation of the objective function at $\boldsymbol{x} \in X$, we must evaluate the values in the order:

$$(1) \quad \mathrm{VaR}_\beta[\widetilde{r}(\boldsymbol{x})] \quad \Longrightarrow \quad (2) \quad \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})], \tag{11}$$

but these are tremendous tasks. The following theorem implies that the evaluation and minimization of $\mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})]$ can be done by the simultaneous minimization of function $F(a; \widetilde{r}(\boldsymbol{x}))$ with respect to the original decision variable $\boldsymbol{x} \in X$ and an auxiliary variable $a \in \mathbb{R}$.

Theorem 2.

(1)

$$\min\{\mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] : \boldsymbol{x} \in X\} = \min\{F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R};\ \boldsymbol{x} \in X\}. \tag{12}$$

(2) For $\boldsymbol{x}^* \in X$,

$$\min\{\mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] : \boldsymbol{x} \in X\} = \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x}^*)] \tag{13}$$

if and only if

$$\min\{F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R};\ \boldsymbol{x} \in X\} = F_\beta\left(\mathrm{VaR}_\beta[\widetilde{r}(\boldsymbol{x}^*)]; \widetilde{r}(\boldsymbol{x}^*)\right). \tag{14}$$

(3) $F_\beta(a; \widetilde{r}(\boldsymbol{x}))$ is convex both in $a \in \mathbb{R}$ and $\boldsymbol{x} \in X$. □

Accordingly, the original convex programming problem with $n+1$ decision variables;

$$\left|\begin{array}{ll} \text{Minimize} & \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] \\ \text{subject to} & \boldsymbol{x} \in X, \end{array}\right. \tag{15}$$

could be reduced to the following convex programming problem with $n+1$ decision variables;

$$\left|\begin{array}{ll} \text{Minimize} & F_\beta(a; \widetilde{r}(\boldsymbol{x})) := a + \dfrac{1}{1-\beta}\mathbb{E}\left[(-\widetilde{r}(\boldsymbol{x}) - a)^+\right] \\ \text{subject to} & a \in \mathbb{R}; \\ & \boldsymbol{x} \in X, \end{array}\right. \tag{16}$$

which is more tractable than the original problem.

3 Fractional Programming

Let us consider a fractional programming problem formulated as follows:

$$[\mathrm{P}] \quad \left| \begin{array}{ll} \text{Maximize} & h(\boldsymbol{x}) := \dfrac{f(\boldsymbol{x})}{g(\boldsymbol{x})} \\ \text{subject to} & \boldsymbol{x} \in X, \end{array} \right. \tag{1}$$

where

- $\boldsymbol{x} = (x_1, \cdots, x_n)^\top$;
- $X \subset \mathbb{R}^n$: a convex and compact constraint set;
- $f : X \longrightarrow \mathbb{R}$: a continuous function on X;
- $g : X \longrightarrow \mathbb{R}_{++}$: a continuous positive–valued function on X.
- $h : X \longrightarrow \mathbb{R}$: a continuous function on X.

If

(A1) f: a linear (or, more generally, concave) function on X;

(A2) g: a convex function on X

then

$$h := f/g : X \longrightarrow \mathbb{R} : \text{a(n essentially) quasi–concave function on } X \tag{2}$$

because, for a (nonnegative) level $z \in \mathbb{R}_+$, the level set

$$\begin{aligned} L_h(z) &:= \{\boldsymbol{x} \in X : h(\boldsymbol{x}) \geq z\} \\ &= \{\boldsymbol{x} \in X : f(\boldsymbol{x}) - zg(\boldsymbol{x}) \geq 0\} : \text{a convex set in } \mathbb{R}^n \end{aligned} \tag{3}$$

which is due to

$$f - zg : X \longrightarrow \mathbb{R} : \text{a concave function on } X. \tag{4}$$

Now, by introducing a real parameter $z \in \mathbb{R}$, let us consider

$$[\mathrm{Q}(z)] \quad \left| \begin{array}{ll} \text{Maximize} & u(\boldsymbol{x}; z) \\ \text{subject to} & \boldsymbol{x} \in X, \end{array} \right. \tag{5}$$

where, for each $z \in \mathbb{R}$, we define

$$u(\boldsymbol{x}; z) := f(\boldsymbol{x}) - zg(\boldsymbol{x}) : X \longrightarrow \mathbb{R} \tag{6}$$

Since, for any $z \in \mathbb{R}$, the function $u(\boldsymbol{x}; z)$ is a continuous function of $\boldsymbol{x} \in X$, and $X \in \mathbb{R}^n$ is a compact set, by Weierstrass Theorem, the problem $\mathrm{Q}(z)$ has an optimal solution, say $\boldsymbol{x}(z) \in X$, Further Let us define the optimal value function by

$$\begin{aligned} v(z) &:= \max\{u(\boldsymbol{x}; z) : \boldsymbol{x} \in X\} \\ &= \max\{f(\boldsymbol{x}) - zg(\boldsymbol{x}) : \boldsymbol{x} \in X\} \\ &= \{f(\boldsymbol{x}(z)) - zg(\boldsymbol{x}(z)), \quad z \in \mathbb{R}. \end{aligned}$$

Since, for each $\boldsymbol{x} \in X$, $u(\boldsymbol{x}; z) = f(\boldsymbol{x}) - zg(\boldsymbol{x})$ is a monotone decreasing linear function of $z \in \mathbb{R}$, and $v(z)$ is a function composed of pointwise maximum of such linear functions, we conclude $v(z)$ is a (possibly non–smooth) monotone decreasing convex function $z \in \mathbb{R}$. Furthermore, it is noted that a sub–differential (a slope) of the function $v(z)$ at $z \in \mathbb{R}$ is given by $-g(\boldsymbol{x}(z))$, that is, let $z' \in \mathbb{R}$ be another point, then

$$\begin{aligned} v(z') - v(z) &= \max\{f(\boldsymbol{x}) - z'g(\boldsymbol{x}) : \boldsymbol{x} \in X\} - \max\{f(\boldsymbol{x}) - zg(\boldsymbol{x}) : \boldsymbol{x} \in X\} \\ &= \max\{f(\boldsymbol{x}) - z'g(\boldsymbol{x}) : \boldsymbol{x} \in X\} - \{f(\boldsymbol{x}(z)) - zg(\boldsymbol{x}(z))\} \\ &\geq \{f(\boldsymbol{x}(z)) - z'g(\boldsymbol{x}(z))\} - \{f(\boldsymbol{x}(z) - zg(\boldsymbol{x}(z))\} \\ &= \{-g(\boldsymbol{x}(z))\}(z' - z), \quad z' \in \mathbb{R}. \end{aligned} \tag{7}$$

Therefore, a supporting line at $(z, v(z))$ is represented by

$$v' = \{-g(\boldsymbol{x}(z))\}(z' - z) + v(z), \quad (z', v') \in \mathbb{R}^2. \tag{8}$$

It is noted that the zero of the above linear function is

$$z' = \frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}. \tag{9}$$

In the theory of fractional programming, the following theorem is known.

Theorem 3. The following two statements are equivalent:

(S1) For the problem P, $z^* \in \mathbb{R}$ is the optimal value and $\boldsymbol{x}^* \in X$ is an optimal solution, that is,

$$z^* = \max\left\{\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})} : \boldsymbol{x} \in X\right\} = \frac{f(\boldsymbol{x}^*)}{g(\boldsymbol{x}^*)}. \tag{10}$$

(S2) For $z^* \in \mathbb{R}$, $\boldsymbol{x}^* \in X$ is an optimal solution of $\mathrm{Q}(z^*)$, and its optimal value is 0, that is,

$$v(z^*) = \max\{f(\boldsymbol{x}) - z^*g(\boldsymbol{x}) : \boldsymbol{x} \in X\} = f(\boldsymbol{x}^*) - z^*g(\boldsymbol{x}^*) = 0. \tag{11}$$

□

This theorem implies the fractional programming problem (P) is reduced to the nonlinear equation with one unknown variable: Find a zero point of (possibly) non–smooth optimal value function $v : \mathbb{R} \longrightarrow \mathbb{R}$ of the family of maximization problems of concave functions $u(\boldsymbol{x}; z)$ subject to $\boldsymbol{x} \in X$ with a real parameter $z \in \mathbb{R}$:

$$[\text{NLE}] \qquad |\text{Find } z^* \in \mathbb{R} \text{ such that } v(z^*) = 0. \tag{12}$$

And it also suggests a numerical procedure for finding the optimal solution of the fractional programming problem (P).

Dinkelbach (1962) reduces the solution of a linear fractional programming problem to the solution ‹ ˆ a sequence of linear programming problems. The method is general in as much as it can be pplied even when we have a ratio of functions that not necessarily linear.

The Ɲ ewton algorithm for solving NLE becomes as follows:

Algorithm 1 (Newton Method).

Step 0: (Initialization) Set $k \longleftarrow 0$ and $z^0 \in \mathbb{R}_+$ arbitrarily.

Step 1: For $z^k \in \mathbb{R}_+$, solve

$$[\mathrm{Q}(z^k)] \quad \left| \begin{array}{ll} \text{Maximize} & u(\boldsymbol{x}; z^k) := f(\boldsymbol{x}) - z^k g(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X, \end{array} \right. \tag{13}$$

and set the optimal solution as $\boldsymbol{x}^k \in X$, and the optimal value as $v(z^k)$.

Step 2: For a pre-specified accuracy bound $\varepsilon \in \mathbb{R}_{++}$, if

$$|v(z^k)| = |u(\boldsymbol{x}^k; z^k)| = |f(\boldsymbol{x}^k) - z^k g(\boldsymbol{x}^k)| < \varepsilon \tag{14}$$

then stop; else set

$$z^{k+1} \longleftarrow \left(\frac{f(\boldsymbol{x}^k)}{g(\boldsymbol{x}^k)} \right)^+ ; \tag{15}$$

$$k \longleftarrow k + 1 \tag{16}$$

and go to Step 1. □

3.1 Implementation for Maximization of CVaR–based Sharpe Ratio

For our maximization problem of CVaR–based Sharpe ratio, for $\boldsymbol{x} \in X$, let us define

$$\begin{aligned} f(\boldsymbol{x}) &:= \overline{r}(\boldsymbol{x}) - r_f = \mathbb{E}[\widetilde{r}(\boldsymbol{x})] - r_f = \mathbb{E}[\widetilde{\boldsymbol{r}}^\top \boldsymbol{x}] - r_f = \overline{\boldsymbol{r}}^\top \boldsymbol{x} - r_f \\ &= \sum_{i=1}^n \overline{r}_i x_i - r_f = \sum_{i=1}^n (\overline{r}_i - r_f)\, x_i; \end{aligned} \tag{17}$$

$$\begin{aligned} g(\boldsymbol{x}) &:= \mathrm{CVaR}_\beta(\boldsymbol{x}) := \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] \\ &= \min\{F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R}\}, \end{aligned} \tag{18}$$

where we assume that

$$g(\boldsymbol{x}) = \mathrm{CVaR}_\beta(\boldsymbol{x}) = \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] > 0, \quad \forall \boldsymbol{x} \in X. \tag{19}$$

Then, the objective function in $\mathrm{Q}(z)$ ($z \in \mathbb{R}_+$) to be maximized, becomes

$$\begin{aligned} u(\boldsymbol{x}; z) &= f(\boldsymbol{x}) - z g(\boldsymbol{x}) \\ &= (\overline{r}(\boldsymbol{x}) - r_f) - z \mathrm{CVaR}_\beta(\boldsymbol{x}) \\ &= (\overline{r}(\boldsymbol{x}) - r_f) - z \min\{F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R}\} \\ &= \max\{(\overline{r}(\boldsymbol{x}) - r_f) - z F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R}\}. \end{aligned} \tag{20}$$

Therefore, the problem $\mathrm{Q}(z)$ ($z \in \mathbb{R}_+$) is reduced to the following concave maximization problem with $n+1$ decision variables $\boldsymbol{x} \in X$, $a \in \mathbb{R}$:

$$[\mathrm{Q}(z)] \quad \left| \begin{array}{ll} \text{Maximize} & (\overline{r}(\boldsymbol{x}) - r_f) - z F_\beta(a; \widetilde{r}(\boldsymbol{x})) \\ \text{subject to} & a \in \mathbb{R}, \\ & \boldsymbol{x} \in X. \end{array} \right. \tag{21}$$

Here,

$$(\overline{r}(\boldsymbol{x}) - r_f) - z F_\beta(a; \widetilde{r}(\boldsymbol{x})) = (\mathbb{E}[\widetilde{r}(\boldsymbol{x})] - r_f) - z \left(a + \frac{1}{1-\beta} \mathbb{E}\left[(-\widetilde{r}(\boldsymbol{x}) - a)^+\right] \right). \tag{22}$$

4 Sampling Approach

Let

$$\boldsymbol{d}^1 = (d_1^1, \cdots, d_n^1)^\top, \cdots, \boldsymbol{d}^m = (d_1^m, \cdots, d_n^m)^\top \in \mathbb{R}^n \tag{1}$$

be a sample of data with size $m \in \mathbb{Z}_{++}$, which are drawn from the population of random vector $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \cdots, \widetilde{r}_n)^\top$. Then, a natural unbiased estimator of $\overline{\boldsymbol{r}} = (\overline{r}_1, \cdots, \overline{r}_n)^\top$ ($\in \mathbb{R}^n$) is given by

$$\widehat{\overline{\boldsymbol{r}}} := \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{d}^j, \qquad \text{or,} \qquad \widehat{\overline{r}}_i := \frac{1}{m}\sum_{j=1}^{m} d_i^j, \quad i = 1, \cdots, n. \tag{2}$$

Furthermore, the corresponding natural unbiased estimator of mean $\overline{r}(\boldsymbol{x}) = \overline{\boldsymbol{r}}^\top \boldsymbol{x}$ of random return rate $\widetilde{r}(\boldsymbol{x}) = \widetilde{\boldsymbol{r}}^\top \boldsymbol{x}$ of portfolio $\boldsymbol{x} \in X$ is given by

$$\widehat{\overline{r}(\boldsymbol{x})} := \left\{\frac{1}{m}\sum_{j=1}^{m} \boldsymbol{d}^j\right\}^\top \boldsymbol{x} = \frac{1}{m}\sum_{j=1}^{m} {\boldsymbol{d}^j}^\top \boldsymbol{x}, \quad \boldsymbol{x} \in X. \tag{3}$$

On the other hand, in order to estimate

$$\mathrm{CVaR}_\beta(\boldsymbol{x}) = \mathrm{CVaR}_\beta[\widetilde{r}(\boldsymbol{x})] = \min\{F_\beta(a; \widetilde{r}(\boldsymbol{x})) : a \in \mathbb{R}\}, \quad \boldsymbol{x} \in X, \tag{4}$$

we use the final representation to obtain

$$\widehat{\mathrm{CVaR}_\beta(\boldsymbol{x})} := \min\left\{F_\beta\widehat{(a; \widetilde{r}(\boldsymbol{x}))} : a \in \mathbb{R}\right\}, \quad \boldsymbol{x} \in X, \tag{5}$$

where

$$F_\beta\widehat{(a; \widetilde{r}(\boldsymbol{x}))} := a + \frac{1}{1-\beta}\left[\frac{1}{m}\sum_{i=1}^{m}\left(-{\boldsymbol{d}^j}^\top \boldsymbol{x} - a\right)^+\right] \quad a \in \mathbb{R};\ \boldsymbol{x} \in X. \tag{6}$$

Accordingly, the objective function of Q(z) to be maximized, is now estimated as

$$\begin{aligned}
(\widehat{\overline{r}(\boldsymbol{x}) - r_f}) - zF_\beta\widehat{(a; \widetilde{r}(\boldsymbol{x}))} &= \left(\widehat{\overline{r}(\boldsymbol{x})} - r_f\right) - zF_\beta\widehat{(a; \widetilde{r}(\boldsymbol{x}))}, \\
&= \left(\frac{1}{m}\sum_{j=1}^{m} {\boldsymbol{d}^j}^\top \boldsymbol{x} - r_f\right) - z\left(a + \frac{1}{1-\beta}\left[\frac{1}{m}\sum_{i=1}^{m}\left(-{\boldsymbol{d}^j}^\top \boldsymbol{x} - a\right)^+\right]\right) \\
&= \left(\frac{1}{m}\sum_{j=1}^{m} {\boldsymbol{d}^j}^\top \boldsymbol{x} - r_f\right) - z\left(a + \frac{1}{(1-\beta)m}\sum_{i=1}^{m} u_j\right), \\
&\quad a \in \mathbb{R};\ \boldsymbol{x} \in X,
\end{aligned} \tag{7}$$

where we introduced auxiliary variables $\boldsymbol{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$:

$$u_j := \left(-{\boldsymbol{d}^j}^\top \boldsymbol{x} - a\right)^+, \quad j = 1, \cdots, m. \tag{8}$$

Collecting the above results, through a sampling method, we can approximate the problem Q(z) by the following Linear Programming (LP) problem with $n + m + 1$ decision variables $a \in \mathbb{R}$; $\boldsymbol{x} = (x_1, \cdots, x_n)^\top \in \mathbb{R}^n$; $\boldsymbol{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$:

$$\widehat{[\mathrm{Q}(z)]} \quad \left|\begin{array}{ll}
\text{Maximize} & \left(\frac{1}{m}\sum_{j=1}^{m} {\boldsymbol{d}^j}^\top \boldsymbol{x} - r_f\right) - z\left(a + \frac{1}{(1-\beta)m}\sum_{i=1}^{m} u_j\right) \\
\text{subject to} & a \in \mathbb{R}; \\
& \boldsymbol{x} \in X; \\
& u_j \geq -{\boldsymbol{d}^j}^\top \boldsymbol{x} - a;\ u_j \geq 0, \quad j = 1, \cdots, m.
\end{array}\right. \tag{9}$$

5 Numerical Experiments: Historical Cases

In this section, we try to apply the new performance measure, CVaR–based Sharpe ratio, to a (virtual) risky investment on the NIKKEI stock indexes constructed from stocks traded at the Tokyo Stock Exchange. In the empirical study, we use the monthly stock return data of various types of industries from January 1995 to December 2003. We regard these 28 types of industry indexes as 28 kinds of risky securities. At first, we compute the 9–year average return rates by using 9–year monthly rate of returns to obtain $\hat{\bar{r}}_i$, $i = 1, \cdots, 28$ (see Table 1). For example, the estimated mean rate of return on building industry is $\hat{\bar{r}}_3$, and the weight on the building industry in the portfolio investment is x_3. The "aff" in Table 1 means the industry of agriculture, forestry, and fisheries.

We use the SLP (Sequential Linear Programming) approach to derive an optimal portfolio of maximizing the CVaR–based Sharpe ratio. The computation are carried out based on Algorithm 1 by utilizing the LP solver of MATLAB 6.5 on a 2.60 GHz Pentium 4 machine. The optimal weights x_i^*, $i = 1, \cdots, n$, and the optimal number K of risky securities included in the optimal investment are found out, and they are shown in Table 2. Then, for three different β–values 0.99, 0.95, 0.90, we also compute the β–VaR and β–CVaR of the optimal portfolio $\boldsymbol{x}^*$ as shown in Table 2. Further, in Table 2, N is the number of iterations required for the convergence to the optimal solution. We find that the algorithm converges after 4 iterations for all β values.

Table 1: Estimated Expected Rate of Return on Each Industry's from Jan. 1995 to Dec. 2003 (%)

type of industry	$\hat{\bar{r}}_i$	type of industry	$\hat{\bar{r}}_i$
aff (r_1)	−0.756	mining (r_2)	−0.519
building (r_3)	−0.537	grocery (r_4)	−0.14
fiber manufacture (r_5)	−0.27	valve.paper (r_6)	−0.19
medicament (r_7)	0.288	oil.coal (r_8)	−0.07
rubble (r_9)	0.22	glass.soil.stone (r_{10})	−0.08
steel (r_{11})	−0.20	nonferrous metal (r_{12})	−0.16
hardware (r_{13})	−0.21	machinery (r_{14})	−0.11
electric manufacture (r_{15})	0.35	transport application (r_{16})	0.51
precision instrument (r_{17})	0.675	others instrument (r_{18})	0.06
commerce (r_{19})	−0.05	finance.insurance (r_{20})	−0.56
real estate (r_{21})	0.17	transport (r_{22})	−0.06
shipping (r_{23})	0.38	airlift (r_{24})	−1.21
warehouse (r_{25})	−0.24	IT (r_{26})	0.26
electricity.gas (r_{27})	0.08	service (r_{28})	−0.21

References

[1] Andersson, F. and Uryasev, S. (1999), Credit Risk Optimization with Conditional Value–at–Risk Criterion, Research Report 99–9, ISE Department, University of Florida.

Table 2: Maximal Value of CVaR–based Sharpe ratio, $\boldsymbol{x}^*$, VaR_β, and CVaR_β Calculated by Sequential Linear Programming Algorithm for $\beta = 0.99, 0.95, 0.90$, and $r_f = 0.0005$, where relevant nonzero x_is are shown.

$\boldsymbol{x}^*$	$\beta = 0.99$	$\beta = 0.95$	$\beta = 0.90$
x_{16}	0	0.0846	0.1556
x_{17}	0.8768	0.8606	0.7905
x_{21}	0.1121	0.0207	0.0097
x_{23}	0.0111	0.0341	0.0441
max	0.02475	0.04	0.055
CVaR_β	24.8458	14.427	11.403
VaR_β	10.8443	8.9232	8.0086
N	4	4	4
K	3	4	4

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