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Dynamic Programming creates The Golden Ratio, too

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Abstract

Since ancient times of Greek as the Parthenon at Athens, the Golden Ratio has been keeping to give a profound influence in many various fields. Mathematicians love the number to explain the nature of the universe and of human life. It comes up even with a formula for the human decision making process; aesthetics, etc. Also in the typical sequential or multi-stage decision processes, the rule is not exceptional. We will show a few explicit famous problems which cooperate with Dynamic Programming: Allocation problem and Linear-Quadratic control problem. For both problems, there appears the Golden Ratio(ϕ) in the solution of Bellman equation.

Keywords: *Dynamic Programming; The Golden Ratio; Allocation problem; Optimal Control.*

1 Introduction

The Golden Ratio($\phi = 1.161803\cdots$) has been a profound influence since ancient times such as the Parthenon at Athens. The shape of the Golden Ratio is supposed to be interesting in a graphic forms for their sculptures and paintings. The beauty appears even in the ingredient of nature creatures. The most influential mathematics textbook by Euclid of Alexandria defines the proportion. These information presents a broad sampling of ϕ−related topics in an engaging and easy-to-understand format.

The Fibonacci sequence(1,1,2,3,5,8,13,$\cdots$) occurs closely the Golden Ratio between two successive numbers. It is known also that the diagonal summation produces the Fibonacci sequence in the Pascal's triangle. These repeated procedure or iteration have something in common.

The principle of Dynamic Programming is said to 'divide and conquer'. In fact, if it is not possible to work out directly, divide up a problem into smaller ones. The basic idea of Dynamic Programming is aiming to provide effective sub-problem to the original problem. The Bellman's curse of dimensionality conquers the computational explosion with the problem dimension through the use of parametric representations. The more it's in a complex, the more it is divided. When a problem were needed in a multi-stage or sequential decision, we should consider the problem in repeatedly. If the procedure of this reduction gives a self-similar one, the methodology shows its effectiveness. The Golden Ratio is created repeatedly by its own in a quite same form. Recurrence relations are ubiquitous including that a beautiful continued fraction represents the Golden number. It is interesting that this quite introductory problem of Dynamic programming produce the basic mathematical aspects.

In the following section, we treat a few exact problems which is typical of Dynamic Programming; Allocation problem and Linear-Quadratic Control problem. However problems are in a simple fashion, it figure out the essence of Dynamic Programming.

2 Dynamic Programming

The conceptual cluster of Dynamic Programming are profound ed throughout the mathematics. Not only the analytical aspect of optimization method, but also the repeated structure of investigate problem. To give a useful explanation and an interesting implication, we show some problems which are quit easy and explicitly solved.

First the general setting of Dynamic Programming problem are illustrated. It is composed as (S, A, r, T). Let S be a state space in the Euclidean space R and $A = (A_x), A_x \subset R, x \in S$ means a feasible action space depending on a current state $x \in S$. The immediate reward is a function of $r = r(x, a, t), x \in S, a \in A_x, t = 0, 1, 2, \cdots$. And the terminal reward $K = K(x), x \in S$ is given. The transition law from the current x to the new $y = m(x, a, t)$ by the action or decision $a \in A_x$ at a time t. If the transition law $m(x, a, t)$ does not depend t, it is called a stationary $m(x, a, t)$ and we treat it in this paper. Here we consider additive costs and the optimal value of a_t will be depend on the decision history. Assume its value at time t denoted by x_t, which enjoy the following properties:

(a) The value of x_t is observable at time t.

(b) The sequence $\{x_t\}$ follow a recurrence in time:

$$x_{t+1} = m(x_t, a_t, t). \tag{2.1}$$

It is termed that the function $y = m(x, \pi(x, t), t)$ means a move from the current x to the next y at t so the law of motion or the plant equation by adapting a policy $a = \pi(x, t)$.

(c) The set of a_t may adopt depends on x_t and t.

(d) The cost function $C_\pi(x, t)$ starting a state x at time-to-go $T_t = T - t$ to optimize over all policies π has the additive form;

$$C_\pi(x_0, t_0) = \sum_{t=t_0}^{T-1} r(x_t, a_t, t) + K(x_T) \tag{2.2}$$

with $x_0 = x_{t_0}$ and

$$x_{t+1} = m(x_t, \pi(x_t, t), t), \quad a_t = \pi(x_t, t)$$

where T is a given finite planning horizon.

Let

$$F(x, t) = \inf_\pi C_\pi(x, t).$$

It is well known that the optimization with state structure of recursively satisfies the optimality equation(DP equation):

$$F(x,t+1) = \inf_{a \in A_x} [r(x,a,t) + F(m(x,a,T_t),t)] \tag{2.3}$$

with the boundary $F(x,T) = K(x)$ where $T_t = T-t$ for $x \in S$, $0 \le t < T$.

All of these are referred from text books by Bertsekas[1], Whittle[4], Sniedovich[5], etc.

The relation between The Golden ratio formula and Fibonacci sequence is known as [7] etc. To produce the Fibonacci sequence, it is a good example in a recursive programming[6]. Also the Fibonacci sequence are related with continued fraction. For the notation of continued fraction, we adopt ourself to the following notations:

$$b_0 + \cfrac{c_1}{b_1 + \cfrac{c_2}{b_2 + \cfrac{c_3}{b_3 + \cfrac{c_4}{b_4 + \cdots}}}} = b_0 + \frac{c_1}{b_1+}\frac{c_2}{b_2+}\frac{c_3}{b_3+}\frac{c_4}{b_4+}\cdots.$$

Note that the Golden number satisfy the quadratic equation: $\phi^2 = 1 + \phi$. By using this relation repeatedly, $\phi = 1 + \frac{1}{\phi} = 1 + \cfrac{1}{1 + \cfrac{1}{\phi}} = 1 + \frac{1}{1+}\frac{1}{\phi} = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{\phi} = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$. Similarly the reciprocal (or sometimes called as a dual Golden number) is denoted $\phi^{-1} = 0.618\cdots = \phi - 1 = \frac{1}{\phi} = \frac{1}{1+}\frac{1}{\phi} = \frac{1}{1+}\frac{1}{1+}\frac{1}{\phi} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$.

This reproductive property suggests our fundamental claim for the following a typical example of Dynamic Programming. Before we solve the problem, let us induce sequences $\{\phi_n\}$ as

$$\phi_{n+1} = 1 + \frac{1}{1 + 1/\phi_n} = 1 + \frac{1}{1+}\frac{1}{\phi_n} \quad (n \ge 1),\ \phi_1 = 1. \tag{2.4}$$

Also let $\{\hat{\phi}_n\}$ as

$$\begin{aligned} &\hat{\phi}_{n+1} = \frac{1}{1+}\frac{1}{1+}\hat{\phi}_n \quad (n \ge 1),\ \hat{\phi}_1 = 1, \\ &i.e. \\ &\hat{\phi}_{n+1}^{-1} = \frac{1}{\hat{\phi}_{n+1}} = 1 + \frac{1}{1 + \hat{\phi}_n} = 1 + \frac{1}{1+}\hat{\phi}_n \quad (n \ge 1). \end{aligned} \tag{2.5}$$

The sequence $\{\phi_n\}$ of (2.4) satisfies

$$\begin{aligned} \phi_{n+1} &= 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\phi_{n-1}} \\ &= 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\phi_{n-2}} \end{aligned}$$

Similarly $\{\hat{\phi}_n\}$ of (2.5) satisfies

$$\begin{aligned} \frac{1}{\hat{\phi}_{n+1}} &= 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\hat{\phi}_{n-1} \\ &= 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\hat{\phi}_{n-2} \end{aligned}$$

From this definition, it is seen easily that

$$\lim_{n\to\infty} \phi_n = \phi = (\sqrt{5}+1)/2 \tag{2.6}$$

$$\lim_{n\to\infty} \hat{\phi}_n = 1/\phi = (\sqrt{5}-1)/2 \tag{2.7}$$

3 Linear-Quadratic Control problem

The Linear Quadratic(LQ) control problem is to consider minimizing a control for the linear system over the quadratic cost function. If the state of system $\{x_t\}$ moves on

$$x_{t+1} = x_t + a_t, \ t = 0, 1, 2, \cdots \tag{3.1}$$

with $x_0 = 1$ by an input control $\{a_t, -\infty < a_t < \infty\}$. The cost incured is

$$\sum_{t=0,1,2,\cdots,T-1} (x_t^2 + a_t^2) + x_T^2. \tag{3.2}$$

LQ of the DP equation

$$v_{t+1}(x) = \min_{a \in A_x} \{r(a, x) + v_t(a + x)\} \tag{3.3}$$

where

$$r(a, x) = a^2 + x^2,$$
$$a \in A_x = (-\infty, \infty), x \in (-\infty, \infty)$$

Theorem 3.1
The solution of (3.3) is given by

$$\begin{cases} v_0(x) = \phi_T x^2 \\ v_t(x) = \phi_{T-t} x^2, t = 1, 2, \cdots \end{cases}$$

using the Golden number related sequence $\{\phi_n\}$ by (2.4).

(Proof) The proof is immediately obtained by the elementary quadratic minimization and then the mathematical induction. □

4 Allocation problem

Allocation problem or sometime called as partition problem is the problem of the form

$$v_{t+1}(x) = \min_{a \in A_x} \{r(a, x) + v_t(a)\} \tag{4.1}$$

for $t = 0, 1, 2, \cdots, T$, where

$$r(a, x) = a^2 + (x - a)^2,$$
$$a \in A_x = [0, x], x \in (-\infty, \infty).$$

Theorem 4.1
The solution of (4.1) is given by using the dual golden number as

$$\begin{cases} v_0(x) = \hat{\phi}_T x^2 \\ v_t(x) = \hat{\phi}_{T-t} x^2, t = 1, 2, \cdots \end{cases}$$

(Proof) Using the Schwartz inequality, the following holds immediately: For given positive constants A and B with a fixed x,

$$\min_{0 \le a \le 1} \{Aa^2 + B(x - a)^2\} = \frac{x^2}{1/A + 1/B}.$$

So the proof could be done inductively. □

Remark 1 : We note here that the number $\phi^{-1} = 0.618\cdots$ of reciprocal of the Golden number is called sometimes *The Dual Golden number*. The above two problems are closely related.

Remark 2 : It is seen that the same quadratic function of the form; $v(x) = Cx^2$ where c is a constant, becomes a solution if the DP equation is, for Allocation and LQ,

$$v_{t+1}(x) = \min_{a \in A_x} \left\{ r(a,x) + 2\int_0^a v_t(y)/y\, dy \right\} \tag{4.2}$$

$$v_{t+1}(x) = \min_{a \in A_x} \left\{ r(a,x) + 2\int_0^{a+x} v_t(y)/y\, dy \right\} \tag{4.3}$$

respectively. Refer to [3].

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