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Estimating Age Replacement Policies with a Censored Small Sample Data

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Abstract— In this article, we consider the typical age replacement models so as to minimize the relevant expected costs, and formulate the statistical estimation problems with the censored sample of failure time data. Based on the concept of total time on test statistics, we show that the underlying optimization problems are translated to the graphical ones on the data space. Next, we utilize a kernel density estimator and improve the statistical estimation algorithms in terms of convergence speed. Throughout simulation experiments, the developed algorithms are useful especially for the small sample problem, and enable us to estimate the optimal age replacement times with higher accuracy.

Keywords— Age replacement, Total time on test, Non-parametric estimation, Kaplan-Meier estimation, Kernel density estimation, Statistical optimization

1. Introduction

Since Barlow and Proschan published their remarkable book [1], a number of optimal replacement models under uncertainty have been developed in the literature. For the ordinary age replacement problems, Bergman [2] and Bergman and Klefsjö [3, 4] developed the non-parametric estimation algorithms based on the total time on test statistics to obtain the optimal age replacement times from the complete sample of failure time data. If a lot of sample of failure time data can be obtained, then with probability 1, the estimate of the optimal age replacement time based on their algorithm asymptotically converges on a true optimal solution. Hence, the non-parametric estimation algorithms are useful to realize an adaptive maintenance control. When the adaptive control is carried out, it is important to obtain more accurate solution in the situation where only fewer failure data are obtained. Recently, Rinsaka and Dohi [5] proposed the non-parametric estimation algorithm based on the kernel density estimation [6–10] to improve the estimation accuracy of the optimal age replacement times for age replacement problems with a complete small sample data.

In many situations, however, it is difficult to collect all the failure time data, since it is necessary to continue the experiment until the last item on test or in service has failed. Under these circumstances, it is desirable to discontinue the study prior to failure of all items in the sample. Then, some observations may be censored or truncated from the right, referred to as right-censorship. Data of this type are called censored data. Reineke *et al.* [11, 12] proposed the non-parametric estimation algorithm based on the Kaplan-Meier estimation to obtain the optimal age replacement times from the censored sample of failure time data.

The aim of this paper is to improve the estimation accuracy of the optimal age replacement times for age replacement problems with a censored small sample data. More precisely, we propose the non-parametric estimation algorithm based on the kernel density estimation [13,14] for some typical age replacement problems. In Section **2**, we formulate the expected cost per unit time for the ordinary age replacement model and the imperfect maintenance model. In section **3** and **4**, we formulate the statistical estimation problems with the censored sample of failure time data. Based on the concept of total time on test statistics, we show that the underlying optimization problems are translated to the graphical ones on the data space. In section **5**, we utilize a kernel density estimator and improve the statistical estimation algorithms in terms of convergence speed. Throughout simulation experiments, the developed algorithms are useful especially for the censored small sample problem, and enable us to estimate the optimal age replacement times with higher accuracy.

2. Age Replacement Problems

Under the age replacement policy, a unit is replaced at failure or at age T (> 0) whichever occurs first. Let F and $R = 1 - F$ denote the cumulative distribution function and the survival function of the time to failure of a unit. It is assumed that F is continuous and strictly increasing and that the mean $\mu = \int_0^\infty R(t)dt$ is finite. We assume that the unit can be replaced at failure at a cost $c + K$ ($c > 0$, $K > 0$) and a preventive replacement at cost c. Here, K can be thought of as a consequence cost.

Model 1: The first model is the ordinary age replacement problem [1,2], which consists in finding an optimal age $T = T^*$ minimizing the expected cost per unit time

$$C(T) = \frac{(c+K)F(T) + cR(T)}{\int_0^T tdF(t) + TR(T)} = \frac{c + KF(T)}{\int_0^T R(t)dt}. \tag{1}$$

Model 2: The second model considers a more general situation that the preventive maintenance at T is imperfect in some sense [3]. Let p ($0 \le p \le 1$) denote the probability that the preventive maintenance is imperfect. Then the expected cost per unit time $C_p(T)$ is given by

$$\begin{aligned} C_p(T) &= \frac{(c+K)F(T) + [c + p(c+K)]\,R(T)}{\int_0^T R(t)dt} \\ &= \frac{[K - p(c+K)] \times [F(T) + d(p)]}{\int_0^T R(t)dt}, \end{aligned} \tag{2}$$

where

$$d(p) = \frac{c/(c+K) + p}{K/(c+K) - p}. \tag{3}$$

3. The TTT Concept

To derive the optimal age replacement time on the graph, we define the total time on test (TTT) transform and the scaled TTT transform [15] for the lifetime distribution function $F(t)$ by

$$H^{-1}(u) = \int_0^{F^{-1}(u)} R(t)dt \tag{4}$$

and

$$\phi(u) = \frac{1}{\mu}\int_0^{F^{-1}(u)} R(t)dt, \tag{5}$$

respectively. Since $F(t)$ is a nondecreasing function, there always exists its inverse function

$$F^{-1}(u) = \inf\{t; F(t) \ge u\}, \quad 0 \le u \le 1. \tag{6}$$

If the expected costs per unit time given by Eq.(1) and Eq.(2) are rewritten in terms of the scaled TTT transform of $F(t)$, the following result is obtained [2,3].

Theorem 1: Obtaining the optimal age replacement time which minimizes the expected cost per unit time at Model i ($i = 1, 2$) can be reduced to the following maximization problem:

$$\max_{0 \le u \le 1} \frac{\phi(u)}{u + \eta_i}, \tag{7}$$

where

$$\eta_1 = c/K, \quad \eta_2 = d(p). \tag{8}$$

Theorem 1 can be obtained by transforming $C(T)$ and $C_p(T)$ to a function of u by means of $u = F(t)$. If the lifetime distribution $F(t)$ is known, then the optimal age replacement time can be obtained from Theorem 1 by $T^* = F^{-1}(u^*)$. Here, $u^*(0 \le u^* \le 1)$ is given by the x coordinate value u^* for the point of the curve with the largest slope among the line pieces drawn from the point $(-\eta_i, 0)$ $(-\infty < -\eta_i < 0)$ on a two-dimensional plane to the curve $(u, \phi(u)) \in [0,1] \times [0,1]$.

4. The Kaplan-Meier Estimator

Often in life testing, as well as in operational situations, it is not possible to observe the failure time of every unit. Failure time data often include some units that do not fail during their experiment period. The data on these units are said to be right-censored. Let $X_1, X_2, \cdots, X_n$ denote the true survival times of n units which are censored on the right by a sequence $U_1, U_2, \cdots, U_n$ which in general may be either constants or random variables.

The observed right-censored data are denoted by the pairs (Y_j, δ_j), $j = 1, \cdots, n$, where

$$Y_j = \min\{X_j, U_j\}, \quad \delta_j = \begin{cases} 1 & \text{if } X_j \le U_j, \\ 0 & \text{if } X_j > U_j. \end{cases} \tag{9}$$

Thus, it is known which observations are times of failure death and which ones are censored or loss times. In this paper, we assume that $U_1, \cdots, U_n$ constitute a random sample from a distribution G (which is usually unknown) and are independent of $X_1, \cdots, X_n$. That is, (Y_j, δ_j), $j = 1, 2, \cdots, n$, is called a randomly right-censored sample.

Based on the censored sample (Y_j, δ_j), $j = 1, \cdots, n$, a popular estimator of the survival probability is the Kaplan-Meier estimator [16] as the nonparametric maximum likelihood estimator of $R(t)$. Let $(Y_{(j)}, \delta_{(j)})$, $j = 1, \cdots, n$, denote the ordered Y_j's along with the corresponding δ_j's. The Kaplan-Meier estimator of R is defined by

$$\hat{R}_{\mathrm{KME}}(t) = \begin{cases} 1, & 0 \le t \le Y_{(1)}, \\ \displaystyle\prod_{j=1}^{k-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{(j)}}, & t \in (Y_{(k-1)}, Y_{(k)}],\ k = 2, \cdots, n, \\ 0, & t > Y_{(n)}. \end{cases} \tag{10}$$

Let s_j denote the jump of $\hat{R}_{\mathrm{KME}}$ at $Y_{(j)}$, that is,

$$s_j = \begin{cases} 1 - \hat{R}_{\mathrm{KME}}(Y_{(2)}), & j = 1, \\ \hat{R}_{\mathrm{KME}}(Y_{(j)}) - \hat{R}_{\mathrm{KME}}(Y_{(j+1)}), & j = 2, \cdots, n-1, \\ \hat{R}_{\mathrm{KME}}(Y_{(n)}), & j = n. \end{cases} \tag{11}$$

Note that $s_j = 0$ if and only if $\delta_j = 0$, $j < n$, that is, if $Y_{(j)}$ is a censored observation.

Let $\chi_1, \chi_2, \cdots, \chi_m$ denote the observed failure times and let $\chi_{(1)} \le \chi_{(2)} \le \cdots \le \chi_{(m)}$ denote the order statistics of the χ_j, where m $(\le n)$ is the number of observed (uncensored) failures. For randomly censored data, the TTT-plot can be can be constructed using the Kaplan-Meier estimator by letting $u_{(j)} = 1 - \hat{R}_{\mathrm{KME}}(\chi_{(j)})$, $j = 1, 2, \cdots, m$, for the ordered failure time j and by estimating the TTT-transform with

$$\begin{aligned} H_{\mathrm{KME}}^{-1}(u_{(j)}) &= \int_0^{\chi_{(j)}} \hat{R}_{\mathrm{KME}}(t)dt \\ &= \sum_{k=1}^{j} \left(\chi_{(k)} - \chi_{(k-1)}\right) \hat{R}_{\mathrm{KME}}(\chi_{(k-1)}), \quad j = 1, 2, \cdots, m; \quad \chi_{(0)} = 0. \end{aligned} \tag{12}$$

The TTT-plot is obtained by plotting the coordinates

$$\left\{ u_{(j)}, \frac{\hat{H}_{\mathrm{KME}}^{-1}(u_{(j)})}{\hat{H}_{\mathrm{KME}}^{-1}(u_{(n)})} \right\}, \quad j = 1, 2, \cdots, m. \tag{13}$$

By connecting the points in a staircase pattern, the scaled TTT plot is obtained. Since the estimate in Eq.(13) is a nonparametric estimate of $(u, \phi(u))$, $u \in [0,1]$, the following theorem on the optimal age replacement time is obtained by direct application of the result in Theorem 1 [11, 12].

Theorem 2: It is assumed that the randomly censored failure time data (Y_j, δ_j), $j = 1, \cdots, n$ are observed in Model i $(i = 1, 2)$. The nonparametric estimate $\hat{T}$ of an optimal age replacement time minimizing the expected cost per unit time is given by $\chi_{(j^*)}$ satisfying the following:

$$j^* = \left\{ j \,\middle|\, \max_{0 \le j \le n} \frac{\hat{H}_{\mathrm{KME}}^{-1}(u_{(j)})/\hat{H}_{\mathrm{KME}}^{-1}(u_n)}{u_{(j)} + \eta_i} \right\}. \tag{14}$$

5. The Kernel Density Estimation

In this section, we propose the kernel density estimation to obtain the optimal age replacement time from the censored small sample data. Suppose that the true lifetimes $X_1, \cdots, X_n$ are the nonnegative independent identically distributed random variables with common unknown distribution function F and the density function f. Again, we assume that the right-censored data can be observed. Then we define the kernel density estimator [13, 14] by

$$\hat{f}_{\mathrm{KDE}}(\tau) = h^{-1} \sum_{j=1}^{n} s_j \Phi\left(\frac{\tau - Y_j}{h}\right) \tag{15}$$

where, s_j is given by Eq.(11). The parameter h (> 0) is the window width, also called the smoothing parameter or bandwidth. The function Φ is called the kernel function which satisfies the condition

$$\int \Phi(t)dt = 1, \quad \int t\Phi(t)dt = 0 \quad \text{and} \quad \int t^2\Phi(t)dt = r^2 \neq 0. \tag{16}$$

Usually, but not always, Φ will be a symmetric probability density function. In this paper, the Epanechnikov kernel function [10]

$$\Phi(t) = \begin{cases} \frac{3}{4}\left(1 - \frac{1}{5}t^2\right)/\sqrt{5} & \text{for } |t| < \sqrt{5} \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

is utilized to estimate the density function of lifetime.

When we utilize the kernel method, the problem of choosing how much to smooth is of crucial importance. The maximum likelihood criterion for selecting the ideal value of h for a given censored sample is feasible for $\hat{f}_{\mathrm{KDE}}$ but does not seem to be tractable, even using numerical method. Scott and Factor [17] considered choosing ideal value of h which maximizes the likelihood

$$L(h) = \prod_{i=1}^{n} \left[\hat{f}_{\mathrm{KDE}}(y_i)\right]^{\delta_i} \left[\int_{y_i}^{\infty} \hat{f}_{\mathrm{KDE}}(x)dx\right]^{1-\delta_i}. \tag{18}$$

Obviously, by definition of $\hat{f}_{\mathrm{KDE}}$, the maximum of Eq.(18) is $+\infty$ at $h = 0$. Padgett [14] considered the following modified likelihood criterion:

$$\max_{h \ge 0} \quad L_1(h) = \prod_{k=1}^{n} \left[\hat{f}_{nk}(y_k)\right]^{\delta_k} \left[\int_{y_k}^{\infty} \hat{f}_{nk}(x)dx\right]^{1-\delta_k}, \tag{19}$$

where,

$$\hat{f}_{nk}(y_k) = h^{-1} \sum_{\substack{j=1 \\ j \neq k}}^{n} s_j \Phi\left(\frac{y_k - y_j}{h}\right). \tag{20}$$

Table 1: Censoring parameters and s-expected proportion of censoring

q	0.1	0.2	0.3	0.4	0.5
ν	82.90	38.50	23.63	16.12	11.55
q	0.6	0.7	0.8	0.9	
ν	8.43	6.12	4.25	2.59	

Now, define the scaled total time on test transform of the estimator $\hat{F}_{\mathrm{KDE}}(t) = 1-\hat{R}_{\mathrm{KDE}}(t) = \int_0^t \hat{f}_{\mathrm{KDE}}(s)ds$ of lifetime distribution by

$$\phi_{\mathrm{KDE}}(u) = \frac{1}{\hat{\mu}_n}\int_0^{\hat{F}_{\mathrm{KDE}}^{-1}(u)} \hat{R}_{\mathrm{KDE}}(t)dt, \tag{21}$$

where, $\hat{\mu}_n$ is the estimate of mean time to failure (MTTF) and can be estimated as

$$\hat{\mu}_n = \int_0^{\infty} \hat{R}_{\mathrm{KDE}}(t)dt. \tag{22}$$

The following theorem on the optimal age replacement time is obtained by direct application of the result in Theorem 1.

Theorem 3: It is assumed that the randomly censored failure time data (Y_j, δ_j), $j = 1, \cdots, n$, are observed in Model i $(i = 1, 2)$. The nonparametric estimate $\hat{T}$ of an optimal age replacement time minimizing the expected cost per unit time is given by $T^* = \hat{F}_{\mathrm{KDE}}^{-1}(u^*)$ satisfying the following:

$$\max_{0\leq u\leq 1} \frac{\phi_{\mathrm{KDE}}(u)}{u+\eta_i}. \tag{23}$$

6. Simulation Experiments

Of our interest in this section is the investigation of asymptotic properties and convergence speed of estimators proposed in previous sections. Suppose that the lifetime of the unit obeys the Weibull distribution:

$$F(t) = 1 - R(t) = 1 - \exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right] \qquad t \geq 0. \tag{24}$$

We assume that the unit is tested and subject to random censoring generated by the exponential distribution with mean ν, and defined by the cumulative distribution function

$$G(t) = 1 - \exp\left(-\frac{t}{\nu}\right) \qquad t \geq 0. \tag{25}$$

The s-expected proportion of censoring, q, is given by [18]

$$q = \int_0^{\infty} R(t)dG(t). \tag{26}$$

In the following simulation experiments, the Weibull shape and scale parameters are fixed $\gamma = 2.0$ and $\theta = 10.0$. Table 1 shows the censoring parameters and corresponding s-expected proportion of censoring. The other parameters are fixed $c = 1$, $K = 9$, $p = 0.2$. Through the TTT transform, the optimal age replacement times for Models 1 and 2 can be derived as $T^* = 3.365$ and $T^* = 6.790$, respectively.

Let us consider the estimation of an optimal age replacement time minimizing the expected cost per unit time when the random right-censored failure time data are already observed. It is assumed that the observed data consist of 30 pseudorandom numbers generated from the Weibull failure time distribution in Eq.(24) and the exponential censoring time distribution in Eq.(25), where the s-expected proportion of censoring is fixed

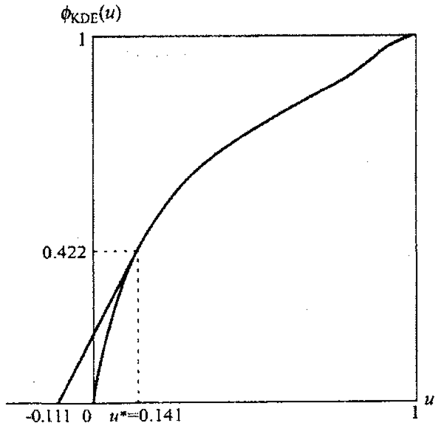

Figure 1: Estimation of the optimal age replacement time based on the kernel density estimation (Model 1).

$q = 0.2$. In Fig.1, we present an estimation example of the optimal age replacement time in Model 1 based on the kernel density estimation from 30 observed right-censored data, where the ideal smoothing parameter h which maximizes the modified likelihood criterion in Eq.(19) is used. The point providing the steepest slope among the line segments drawn from $(-\eta_1, 0) = (-0.111, 0)$ to the scaled TTT transform $\phi_{KDE}(u)$ is $u^* = 0.141$. Hence, the optimal age replacement time can be estimated as $\hat{T}^* = 4.025$.

Next, let us study the asymptotic behavior of two nonparametric estimation algorithms, namely, the TTT plot using the Kaplan-Meier estimation and the kernel density estimation. Monte Carlo simulations are carried out with pseudorandom numbers based on the Weibull failure time distribution and the exponential censoring time distribution, in order to investigate the convergence toward the real optimal solution. Figures 2 to 5 show the asymptotic behavior of the optimal age replacement time for Model 1 and 2. From these figures, it is found that the results converge to the real optimal solutions when the number of failure time data is close to 30.

Figures 6 to 9 show the mean square error of estimate of the optimal age replacement time, which are obtained by carrying out the above Monte Carlo simulations 100 times. When the sample size is extremely small, the estimation accuracy of the kernel density estimation is not always high. Once 20 or more sample data can be obtained, we can observe that the estimation accuracy of the optimal age replacement time can be improved by introducing the kernel density estimation. From these results, we conclude that the statistical algorithm based on the kernel density estimation can be recommended to estimate the optimal age replacement time, especially for the censored small sample problem.

7. Concluding Remarks

In this paper, we have considered the typical age replacement models so as to minimize the relevant expected costs, and formulated the statistical estimation problems with the censored sample of failure time data. Based on the concept of total time on test statistics, we have shown that the underlying optimization problems were translated to the graphical ones on the data space. Next, we have utilized a kernel density estimator and improve the statistical estimation algorithms in terms of estimation accuracy. Through the simulation experiments, it has been shown that the estimation accuracy of the kernel density estimation is higher than the Kaplan-Meier estimation, when the 20 or more sample data can be obtained. We have adopted the approach of maximizing the modified likelihood criterion for the method of determining the smoothing parameter. In the future effort, it would be interesting to improve the estimation accuracy by adopting the other criteria to determine the smoothing parameter, when the sample size is extremely small.

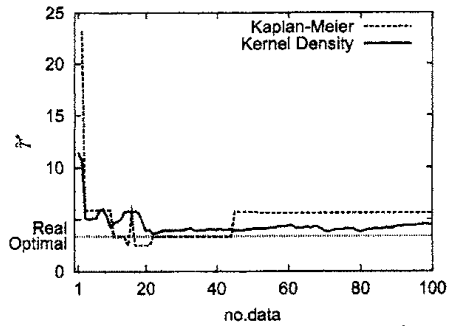

Figure 2: Asymptotic behavior of estimate of the optimal age replacement time (Model 1, $q = 0.2$).

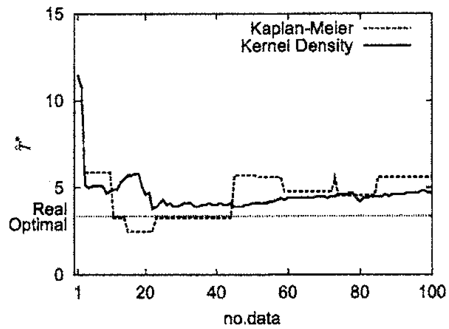

Figure 3: Asymptotic behavior of estimate of the optimal age replacement time (Model 1, $q = 0.4$).

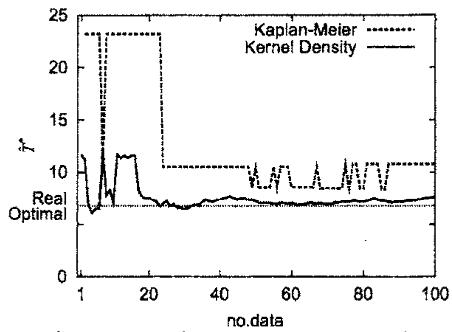

Figure 4: Asymptotic behavior of estimate of the optimal age replacement time (Model 2, $q = 0.2$).

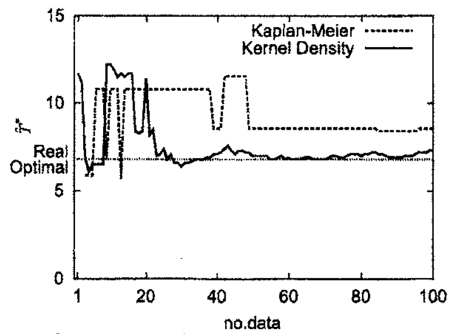

Figure 5: Asymptotic behavior of estimate of the optimal age replacement time (Model 2, $q = 0.4$).

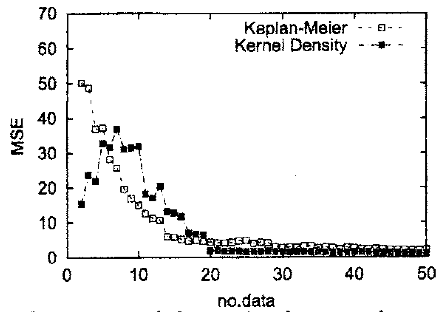

Figure 6: Mean square error of estimate of the optimal age replacement time (Model 1, $q = 0.2$).

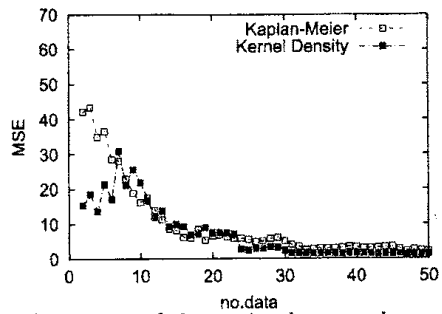

Figure 7: Mean square error of estimate of the optimal age replacement time (Model 1, $q = 0.4$).

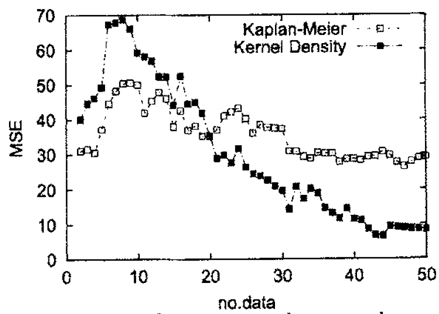

Figure 8: Mean square error of estimate of the optimal age replacement time (Model 2, $q = 0.2$).

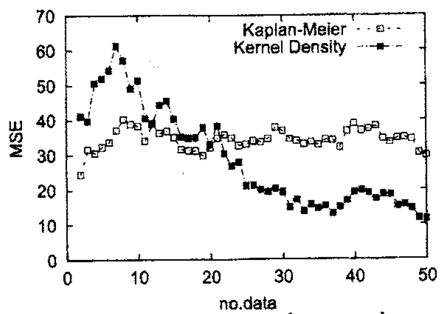

Figure 9: Mean square error of estimate of the optimal age replacement time (Model 2, $q = 0.4$).

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