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A Rendezvous Search on a Linear Graph

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1 Introducton

This note proposes a rendezvous-search model with examination cost and defines a simplified strategy for it. In a discrete form of the rendezvous search problem, the seekers move in discrete time from node to node on a finite and connected graph after they are placed randomly on nodes of the graph. The nodes can be marked if they have a name that both seekers can recognize when they go there; they are markable if each seeker knows whether it has visited a node or not or they are unmarkable if each seeker forgets whether it has visited a node or not. In a rendezvous search, when seekers can choose different strategies, the model is called asymmetric, while it is called symmetric if seekers must choose the same strategy (See Book II of Alpern and Gal (2003)). In a basic model of rendezvous search, the cost consists of only the time needed to find each other. One out of many results in Howard (1999) is that it analyzed an asymmetric model on a linear graph in which the nodes can be marked. He gave an optimal strategy when the probability distribution of initial placement is nondecreasing from the left to the right. Chester and Tütüncü (2004) considered the model on a linear graph in which the probability distribution is symmetric around the center.

In our model the cost consists of the examination cost and the cost for time, and the nodes can be marked. In section 2, we give a rendezvous search model on a finite graph. In Section 3 we propose a twice-examination strategy (TES). An optimal TES gives an upper bound for the value of the original search problem. In Section 4 we restrict our attention to TES on a linear graph. In Section 5 we consider a search problem on a linear graph, in which one of the two seekers does not move.

2 Search on a Finite Graph

Let (N, E) be a graph where N is the set of nodes with $|N| < +\infty$ and $E \subseteq N \times N$ is the set of edges. We assume that the graph (N, E) is connected and undirected. For each $i \in N$, $\delta(i)$ is the set of nodes adjacent to i,that is, $\delta(i) \equiv \{j \in N : (i,j) \in E\} \cup \{i\}$. Let Z^+ be the set of all positive integers. Two players (called Players I and II) are placed on nodes with probability $p(i,j)$ at the Nodes i and j, where

$$\sum_{i,j \in N} p(i,j) = 1, p(i,j) > 0, \quad \forall i,j \in N.$$

The nodes can be marked, that is, each player can distinguish where he/she is and also can distinguish the directions. At each step each player at the node $i \in N$ can choose one of alternatives :

(1) move to a node in $\delta(i)$ and examine ;

(2) move to a node in $\delta(i)$ and does not examine.

Here "move to $i \in \delta(i)$" means he stays at the node i. At the step 1, both players stay at the nodes where they are placed, and they decide whether they examine those nodes or not respectively. The examination at a node means that each player can check that node and adjacent nodes simultaneously. We denote by x_s and y_s the nodes where Players I and II are at the end of the step $s, s \geq 1$. So in the first step, Players I and II are placed at x_1 and y_1 respectively with probability $p(x_1, y_1)$. They can find each other only when (i) they are either at the same node or at adjacent nodes, and (ii) at least one of them examines. It costs c_i when a player examines at the node $i \in N$, while it costs 1 for each step. For simplicity, we have the next assumption. But it is not clear whether we could assume this without loss of generality.

Assumption 1. At every step Player II does not examine.

A path for Player I at $i \in N$ is a pair of a sequence x^i and a subset $S^i \subseteq Z^+$ where

$$x^i = \{x_1^i, \ldots, x_s^i, \ldots\}, x_s^i \in \delta(x_{s-1}^i) \text{ for } s \geq 2, \text{ and } x_1^i = i,$$

and S^i is the set of all steps where Player I examines. Let X^i be the set of all paths for Player I at $i \in N$. Let $\alpha(i)$ be a probability distribution on X^i such that

$$|\{(x^i, S^i) : \alpha(i, (x^i, S^i)) > 0\}| < +\infty.$$

A plan for Player I is defined by$\alpha = \{\alpha(i) : i \in N\}$. A path for Player II at $i \in N$ is a sequence y^j such that

$$y^j = \{y_1^j, \dots, y_s^j, \dots\}, y_s^j \in \delta(y_{s-1}^j) \text{ for } s \geq 2, \text{ and } y_1^j = j.$$

Let Y^j be the set of all paths for Player II at $i \in N$. Let $\beta(j)$ be a probability distribution on Y^j such that

$$|\{y^j : \beta(j, y^j) > 0\}| < +\infty.$$

A plan for Player II is defined by $\beta = \{\beta(j) : j \in N\}$. A strategy is a pair (α, β). We denote by $f((x^i, S^i), y^j; i, j)$ the cost when Players follow paths $(x^i, S^i), y^j$ after they are placed at the nodes i, j initially. We denote by $f(\alpha, \beta; i, j)$ the expected cost when Players adopt a strategy (α, β) and they are placed at the nodes i, j initially and by $f(\alpha, \beta)$ the expected cost when Players adopt a strategy (α, β):

$$f(\alpha, \beta; i, j) = \sum_{X^i, Y^k} \alpha(i, (x^i, S^i))\beta(j, y^j) f((x^i, S^i), y^j; i, j),$$

$$f(\alpha, \beta) = \sum_{i \in N} \sum_{j \in N} p(i, j) f(\alpha, \beta; i, j).$$

The problem is to find a strategy (α, β) which minimizes the expected cost $f(\alpha, \beta)$.

We let

$$v \equiv \inf f(\alpha, \beta).$$

For $i, j \in N$, let $d(i, j)$ be the minimum of the numbers of edges of paths connecting the node i with the node j. It is clear that both players can meet by coming to a specified node and staying there until the other player reaches there.

Proposition 2.1.

$$v \leq \min_{i \in N} \{ \max_{j \in N \setminus \{i\}} d(i, j) + c_i \}.$$

Proof: From every node, each player can reach the node i by at most $\max_{j\in N\setminus\{i\}} d(i,j)$ steps and Player I can examine the node i where the cost is $\max_{j\in N\setminus\{i\}} d(i,j) + c_i$. A plan for Player I is $\alpha = \{\alpha(j) : j \in N\}$ where $\alpha(j,(x^j,S^j)) = 1$ and $x^j_{d(i,j)} = i, S^j = \{d(i,j)\}$ for every $j \in N$. ∎

3 Twice-examination Strategy

In this section we define a special strategy and state a relation to a cooperative version of an ambush game.

Definition. A strategy (α,β) is said to be a *twice-examination* strategy (TES) if for every pair (x^i,S^i) and y^j such that $\alpha(i,(x^i,S^i)) > 0$ and $\beta(j,y^j) > 0$,

$$|S^i| \le 2 \text{ and } \quad \exists s \in S^i \text{ s.t. } y^j_s \in \delta(x^i_s).$$

In other words, a strategy is a TES if Player I examines at most twice and if they can meet certainly.

Under a TES, to meet certainly, both players must be at the same place or adjacent places at some step when Player I examines. If the cost $c_i, i \in N$ is relatively small, in comparison to the cost for each step, Player I will examine twice so that they meet certainly at the second examination and he will do the first examination so that the expected cost becomes as small as possible. We are interested in the step when and where Player I does the first examination. Suppose $S^i = \{s_1, s_2\}$ where $s_1 < s_2$. Then

$$f((x^i,S^i),y^j;i,j) = \begin{cases} s_1 + c_{x^i_{s_1}}, & \text{if } y^j_{s_1} \in \delta(x^i_{s_1}) \text{ ;} \\ s_2 + c_{x^i_{s_1}} + c_{x^i_{s_2}}, & \text{if } y^j_{s_1} \notin \delta(x^i_{s_1}), \text{ and } y^j_{s_2} \in \delta(x^i_{s_2}); \\ +\infty, & \text{if } y^j_{s_1} \notin \delta(x^i_{s_1}), \text{ and } y^j_{s_2} \notin \delta(x^i_{s_2}). \end{cases}$$

Remark. When we restrict our attention to TES, the model is a cooperative version of the following non-cooperative game. There are two players (Players I and II). A strategy for Player II is to choose a plan β. A strategy for player I is to choose a plan α such that for every $i \in N$ and every (x^i,S^i) with $\alpha(i,(x^i,S^i)) > 0$, $1 \le |S^i| \le 2$.

Suppose that Players I and II have chosen strategies α and β respectively. The payoff for player II, $g(\alpha, \beta)$, is defined as follows. First the nature chooses two points $i, j \in N$ with probability $p(i,j)$. Then both players choose (x^i, S^i) and y^j with probabilities $\alpha(i, (x^i, S^i))$ and $\beta(j, y^j)$ respectively. Suppose $S^i = \{s_1, s_2\}$ where $s_1 < s_2$. Then $g((x^i, S^i), y^j; i, j) = f((x^i, S^i), y^j; i, j)$. Furthermore,

$$g(\alpha, \beta) = \sum_{i,j \in N} p(i,j) \sum_{X^i, Y^j} \alpha(i, (x^i, S^i)) \beta(j, y^j) g((x^i, S^i), y^j; i, j).$$

Player I is the minimizer of g and Player II is the maximizer. If $|N|$ is large then player II will win certainly, i.e., $g = +\infty$ by using $y^j_s = j$ for all $s \in Z^+$ and all $j \in N$.

Example. Let $N = \{1, \ldots, 9\}$ and

$$E = \{(1,3), (2,3), (3,4), (4,5), (5,6), (5,7), (5,8), (8,9)\}.$$

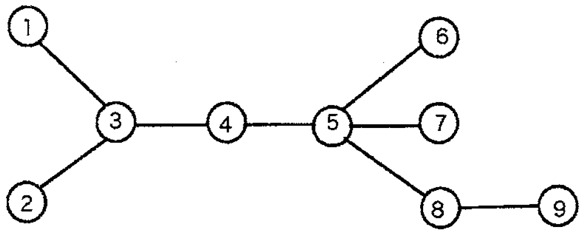

Figure 1 : A Finite Graph

Define a path x^i for every $i \in N$ by $x^i_s = 5$ for $s \geq d(i,5) + 1$ and let $y^i = x^i$ for every $i \in N$. Let

$$S^i = \begin{cases} \{2,4\}, & \text{if } i = 1,2; \\ \{1,3\}, & \text{if } 3 \leq i \leq 9. \end{cases}$$

Let $\alpha(i, (x^i, S^i)) = 1$ and $\beta(j, y^j) = 1$ for every $i \in N$. Then (α, β) is a TES. $\delta(x^1_4) = \delta(x^2_4) = \{4,5,6,7,8\}$ and $\delta(x^i_3) = \{4,5,6,7,8\}$ for $3 \leq i \leq 9$.

4 Search on a Linear Graph

In this section we restrict our attention to TES on a linear graph. We let $N = \{1, \ldots, n\}$ and $E = \{(i, i+1) : 1 \leq i \leq n-1\}$. For avoiding unnecessary complexity, we assume that n is an odd number. For the simplicity of the analysis we put

Assumption 2. $c_i = c > 0$ for all $i \in N$.
Assumption 3. $p(i,j) = \frac{1}{n^2}$ for all $1 \leq i \leq n$.

Proposition 4.1. Let α be a plan for Player I. For every (x^i, S^i) such that $\alpha(i,(x^i,S^i)) > 0$, let $S^i = \{s_1^i, s_2^i\}$ for $i \in N$. Then for every $i \in N$, $s_1^i \geq \max\{x^i_{s_1^i}, n+1-x^i_{s_1^i}\} - 1$ or $s_2^i \geq \max\{x^i_{s_2^i}, n+1-x^i_{s_2^i}\} - 1$.
Proof: Suppose there exists $i \in N$ such that $s_1^i < \max\{x^i_{s_1^i}, n+1-x^i_{s_1^i}\} - 1$ and $s_2^i < \max\{x^i_{s_2^i}, n+1-x^i_{s_2^i}\} - 1$. By the definition of a path, it must hold either $x^i_{s_1^i} \geq s_1^i - 1$ and $x^i_{s_2^i} \geq s_2^i - 1$ or $n+1-x^i_{s_1^i} \geq s_1^i - 1$ and $n+1-x^i_{s_2^i} \geq s_2^i - 1$. Then Player I could not find Player II when Player II starts at the node 1 and n. So the expected cost is not finite.∎

Example. Let $n = 9$. A TES related to Howard (1999) is as follows.

$$x^1 = \{1,2,3,4,5,5,\ldots\}, \quad x^2 = \{2,3,4,5,5,\ldots\}, \quad x^3 = \{3,4,5,6,5,5,\ldots\}$$
$$x^4 = \{4,5,6,5,5,\ldots\}, \quad x^5 = \{5,6,7,6,5,5,\ldots\}, \quad x^6 = \{6,7,6,5,5,\ldots\}$$
$$x^7 = \{7,8,7,6,5,5,\ldots\}, \quad x^8 = \{8,7,6,5,5,\ldots\}, \quad x^9 = \{9,8,7,6,5,5,\ldots\}$$

Let $y^i = x^i$ for every i. Let $s_2^i = 4$ if i is even, and $= 5$ if i is odd. Let $s_1^i \leq 3$ for every i. Let $\alpha(i,(x^i,S^i)) = \beta(i,y^i) = 1$ for every i. Then both players meet at most in 5 steps.

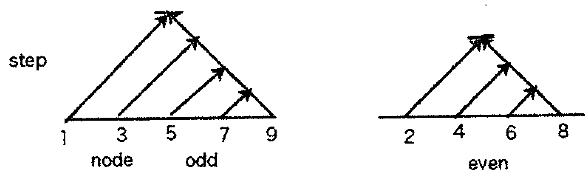

Figure 2 : A Strategy

5 A Search Problem on a Linear Graph

In this section we analyze a search problem on a linear graph and compare the cost with it in the rendezvous search. Assumptions 2 and 3 are still valid in this section. Let $N = \{1,\ldots,n\}$ and $E = \{(i,i+1) : 1 \leq i \leq n-1\}$. Players I and II are placed with probability $p(i,j) = \frac{1}{n^2}$ at the nodes i and j. Player II stays until Player I finds him/her. The nodes can be marked, so Player I can distinguish where he/she is and also can distinguish the directions. At each step Player I at the node $i \in N$ can choose one of alternatives : (1) move to a node in $\delta(i)$ and examine ; (2) move to a node in $\delta(i)$ and does not examine. A path and a plan for Player I are defined in the same way as in Section 2. A unique plan for Player II is β^w where $\beta^w(j, y^{*j}) = 1$ for every $j \in N$ and y^{*j} is defined by $y^{*j}_s = j$ for all $s \geq 1$. The problem here is to find a plan for Player I which minimizes the expected cost $f(\alpha, \beta^w)$. Define a path (x^{+i}, S^{+i}) for Player I by

$$x^{+i} = \{i, i{+}1, \ldots, n, n{-}1, \ldots, i, \ldots, 1\}, \text{ and } S^{+i} = \{1, \ldots, n{-}i{+}1, 2n{-}2i{+}2, \ldots, 2n{-}i\}.$$

In the same way, define a path (x^{-i}, S^{-i}) for Player I by

$$x^{-i} = \{i, i-1, \ldots, 1, 2, \ldots, i, \ldots, n\}, \text{ and } S^{-i} = \{1, \ldots, i, 2i, \ldots, n+i-1\}.$$

Proposition 5.1. For the search problem, the following plan $\alpha = \{\alpha(i) : i \in N\}$ for Player I minimizes the expected cost:

$$\alpha(i, (x^{+i}, S^{+i})) = \alpha(i, (x^{-i}, S^{-i})) = \frac{1}{2}, \text{ for every } i \in N.$$

Proof: Suppose that Player I is placed at $i \in N$. Since Player II does not move, Player I must behave, following an optimal strategy in the search problem with traveling cost on a tree. So Player I must take the plan α in the statement (Kikuta(1995)). ∎

It is possible to calculate the expected cost $f(\alpha, \beta^w)$ for α in the statement of Proposition 5.1.

$$f((x^{+i}, S^{+i}), y^{*j}; i, j) = \begin{cases} 1 + c, & \text{if } j = i, i+1, i-1; \\ (j-i)(1+c), & \text{if } j > i+1; \\ (n-j)c + 2n - i - j, & \text{if } j < i-1. \end{cases}$$

$$f((x^{-i}, S^{-i}), y^{*j}; i, j) = \begin{cases} 1 + c, & \text{if } j = i, i+1, i-1; \\ (i-j)(1+c), & \text{if } j < i-1; \\ (j-1)(1+c) + i - 1, & \text{if } j > i+1. \end{cases}$$

Then

$$f(\alpha, \beta^w) = \frac{1}{2n^2} \sum_{i,j \in N} \{f((x^{+i}, S^{+i}), y^{*j}; i, j) + f((x^{-i}, S^{-i}), y^{*j}; i, j)\}.$$

It is interesting to see that $\frac{f(\alpha,\beta^w)}{n}$ may converge as n becomes large.

Remark. A study on a search game with examination cost suggests that if we remove Assumption 2 then an optimal plan would become very complex.

6 Final Remark

In this report we proposed a rendezvous-search model with examination cost on a finite graph. Furthermore, we proposed a simplified strategy (called TES in this report). It is known that it is very difficult to solve a search game with examination cost when the underlying graph has a cycle. This suggests us that we must first study a rendezvous-search problem on a tree. So to find an optimal strategy, or more simply, to find an optimal TES is a problem to be solved when the graph is a linear graph.

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