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On dense ideals in commutative Banach algebras

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Abstract Segal algebras are dense ideals of group algebras of locally compact groups, which constitute Banach algebras with respect to some norms and have some homogeneous structures. Since H. Reiter introdued this notion in 1965, many interesting and important results on Segal algebras have been accumulated. It is interesting that some properties of group algebras are hereditary in Segal algebras, but other's are not. Segal algebras may be regarded as generalizations of group algebras.

On the other hand, a generalization of the notion of Segal algebras to a notion on more general Banach algebras are attempted by J. T. Burnham and otheres.

In this note we fix a class of commutative semisimple Banach algebras, denoted by A, and define Segal algebras in A, which are generalizaitions of the classical Segal algebras. Then we define a new class of Segal algebras in A, and study some properties of them.

§1. Introduction In this note G stands for a non-discrete locally compact abelian group (LCA group) with character group $\hat{G}$. We denote by A a commutative semisimple Banach algebra which satisfies the following properties;

(α) A has bounded approximate identities,

(β) $(\hat{A}, \| \|_{\hat{A}})$ forms a Wiener algebra,

where $(\hat{A}, \ \| \|_{\hat{A}})$ denotes the Banach function algebra on Φ_A(the maximal ideal space of A) of Gelfand transforms of A with norm $\| \|_{\hat{A}}$ carried over from A. For the definition of a Wiener algebra, we refer to [5, chapter 2].

As examples of these A, we quote these algebras; group algebras $L^1(G)$ of LCA groups G, the Lipschtz algebra $Lip_0^1(R)$(cf. [3]), C^*-algebras $C_0(X)$ of non-compact locally compact Hausdorff spaces X, and some of thier ideals and quotient algebras.

In §2, definitions and results concernig normed ideals and Segal algebras are introduced briefly. In §3, notions of normed ideals and Segal algebras in $L^1(G)$ are generarized to notions of normed ideals and Segal algebras in A, and the results stated in §2 are generarized to the results on the normed ideals and Segal algebras in A. In §4, we introduce a class of Segal algebras in A and study thier properties.

§2. Classical Segal algebras

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In this section, we state the definitions and results concerning the theory of Segal algebras in $L^1(G)$, which are nessesary to state our results later.

Remark 1. Segal algebras are defined in a group algebra on a locally compact group.([5]) But in this note, we restrict ourselves to the commutative case. By "classical Segal algebras", we refer to Segal algebras in $L^1(G)$ on a non-discrete LCA group G.

Definition 1. (cf. [5]) A subalgebra $\mathcal{S}$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions.

(S_0) $\mathcal{S}$ is dense in $L^1(G)$.

(S_1) $\mathcal{S}$ is a Banach space under some norm $\|\cdot\|_S$, and

$$\|f\|_S \geq \|f\|_1 \quad (f \in \mathcal{S}).$$

(S_2) $\mathcal{S}$ is translation invariant ;

$$f \in \mathcal{S} \Rightarrow f_y \in \mathcal{S} \quad (y \in G)$$

and for each $f \in \mathcal{S}$ the mapping $y \to f_y$ of G into $\mathcal{S}$ is continuous.

Here we review typical examples of Segal algebras from [6].

Example 1. Let $\mathcal{S} := \{f \in C(R) : M(f) < \infty\}$, where $M(f) := \sum_{n\in Z} \sup_{0\leq x\leq 1} | f(x+n) |$. Then $\mathcal{S}$ is an ideal of $L^1(R)$ and $M(\cdot)$ is a complete algebra norm on $\mathcal{S}$, but not translation invariant. So, if we renorm $M(\cdot)$ by $\|\cdot\|_S$, where $\|f\|_S := \sup\{M(f_y) : y \in R\}$, then $\| \|_S$ is a translation invariant norm on $\mathcal{S}$ which is equivalent to $M(\cdot)$, and $(\mathcal{S}, \| \|_S)$ becomes a Segal algebra in $L^1(R)$.

Example 2. $S_p(G)$. For each $p(1 < p < \infty)$, put

$$S_p(G) := \{f \in L^1(G) : \|f\|_p < \infty\}, \; \|f\|_{S_p} := \|f\|_1 + \|f\|_p,$$

then $(S_p(G), \| \|_{S^p})$ is a Segal algebra in $L^1(G)$.

Example 3. $A_{\mu,p}(G), A_p(G)$. Let μ be an unbounded positive Radon measure on $\hat{G}$. For each $p(1 \leq p < \infty)$, put

$$A_{\mu,p}(G) := \{f \in L^1(G) : \hat{f} \in L^p(\mu)\}, \;\; \|f\|_{\mu,p} := \|f\|_1 + \|\hat{f}\|_{L^p(\mu)},$$

then $(A_{\mu,p}(G), \| \|_{\mu,p})$ is a Segal algebra in $L^1(G)$. Especially, in case μ is a Haar measure $m_{\hat{G}}$ of $\hat{G}$, we use, for this Segal algebra, an expression $(A_p(G), \| \|_{A_p})$, instead of the expression $(A_{m_{\hat{G}},p}(G), \| \|_{m_{\hat{G}},p})$.

J. Ciglar [2] Intoroduced a notion of normed ideals in $L^1(G)$, which is a generalization of the notion of Segal algebras, and gave a necessary and sufficient condition for a normed ideal to be a Segal algebra. Also, M. Riemersma [7] gave another nesessary and sufficient conditions for a normed ideal to be a Segal algebra.

Definition 2. (cf. [2]) Let $\mathcal{N}$ be a linear subspace of $L^1(G)$. $\mathcal{N}$ is called a normed ideal in $L^1(G)$ if $\mathcal{N}$ satisfies the following condtions;
(a) $\mathcal{N}$ is a dense ideal in A,
(b) $\mathcal{N}$ is a Banach space for some norm $\|\ \|_{\mathcal{N}}$ such that

$$\begin{aligned} \|f\|_1 &\leq \|f\|_{\mathcal{N}} \quad (f \in \mathcal{N}), \\ \|fg\|_{\mathcal{N}} &\leq \|f\|_1 \|g\|_{\mathcal{N}} \quad (f \in L^1(G), g \in \mathcal{N}). \end{aligned}$$

Next we state remarkable properties of Segal algebras or normed ideals.

Theorem A. If $\mathcal{N}$ is a normed ideal in $L^1(G)$, we have;
(i) If U is a neighbourhood of $\gamma_0 \in \hat{G}$, there is an $f \in \mathcal{N}$ such that supp $\hat{f} \subset U$ and $\hat{f}(\gamma) = 1$ for every γ in a neighbourhood of γ_0.
(ii) If $K, U \subset \hat{G}$ such that K is compact and U is open with $K \subset U$, then there is an $e \in \mathcal{N}$ such that $\hat{e}(\gamma) = 1$ $(\gamma \in K)$ and supp $\hat{e} \subset U$.
(iii) $L^1_c(G)$ is contained in $\mathcal{N}$, where $L^1_c(G) := \{f \in L^1(G) : \text{supp } \hat{f} \text{ is compact}\}$.

Theorem B. ([2], [5]) For a normed ideal $\mathcal{N}$, the following (a), (b), and (c) are equivalent each other.
(a) $\mathcal{N}$ is a Segal algebra.
(b) $\mathcal{N}$ has approximate units, that is;

$$\forall f \in \mathcal{N}, \forall \varepsilon > 0, \exists e \in \mathcal{N}; \text{ s. t. } \|f - f * e\| < \varepsilon.$$

(c) $\mathcal{N} = \mathcal{N}_0$, where $\mathcal{N}_0$ is the norm closure of $L^1_c(G)$ in $\mathcal{N}$.

Theorem C. (H. Reiter) Let $\mathcal{S}$ be a Segal algebr in $L^1(G)$.
(i) The ideal theory of $\mathcal{S}$ is the same as that of $L^1(G)$. More precisely, if $\mathcal{I}$ is a closed ideal of $L^1(G)$ then $\mathcal{I} \cap \mathcal{S}$ is a closed ideal of $\mathcal{S}$, and conversely each closed ideal of $\mathcal{S}$ is of this form for a unique closed ideal $\mathcal{I}$ of $L^1(G)$.
(ii) The maximal ideal spaces of $\mathcal{S}$ and $L^1(G)$ are homeomorphic. We can naturally identify $\Phi_{\mathcal{S}}$ with $\hat{G}$, that is, the Gelfand transform of $\mathcal{S}$ is equal the Fourier transform restricted to $\mathcal{S}$.

Theorem D. (i) Let $\mathcal{S}$ be a Segal algebra, and let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity of $L^1(G)$ composed of elements in $L^1_c(G)$. Then $\{e_\lambda\}_{\lambda \in \Lambda}$ is a bounded approximate identity of $\mathcal{S}$ which is bounded with respect to the multiplication oeprator norm;

$$\|T_f\|_{op} := \sup\{\|fg\|_S : g \in \mathcal{S}, \|g\|_S \leq 1\} \quad (f \in \mathcal{S})$$

(ii) If a Segal algebra $\mathcal{S}$ has a bounded approximate identity, then we have $\mathcal{S} = L^1(G)$.

Theorem E. If $(\mathcal{S}_1, \| \, \|_{S_1})$ and $(\mathcal{S}_2, \| \, \|_{S_2})$ are Segal algebras, then $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ becomes a Segal algebra with respect to the norm $\| \, \|_S = \| \, \|_{S_1} + \| \, \|_{S_2}$.

It is known that Segal algebras are normed ideals ([2]), and with the virtue of Theorem B, we can define Segal algebras in A, which are generalizations of classical Segal algebras

In the next section, we will give precise definitons of normed ideals and Segal algebras in A, and then we will show that Theorem A, B, C , D and E above are also valid for all nomed ideals or all Segal algebras in A.

§3. Definitions and fundamental propreties of normed idesls and Segal algebras in A

Recall that A stands for a semisimple commutative Banach algebras with the properties;

(α) A has bonded approximate identities; here we fix one, say,

$\{e_\lambda\}_{\lambda \in \Lambda}$ with $\mathrm{supp}_{\lambda \in \Lambda} \|e_\lambda\|_A = M < \infty$.

(β) $(\hat{A}, \| \, \|_{\hat{A}})$ forms a Wiener algebra.

Φ_A denotes the maximal ideal space of A. For $x \in A$, $\hat{x}$ is the Gelfand transform of x. A_c is the set of all $x \in A$ such that supp $\hat{x}$(the support of $\hat{x}$) is compact.

Since A_c is dense in A by (β), we can assume without loss of generality that $\{e_\lambda\}_{\lambda \in \Lambda}$ is contained in A_c.

In [1] Burnham defined abstract Segal algebras (ASA) in general Banach algebras, which is a generalization of the Cigler's normed ideals [2].

In this section, we will define 'Segal algebra in A', which is a generalization of classical Segal algebras.

Definition 3. (cf. [2]) An ideal $\mathcal{N}$ in A is called a normed ideal in A if $\mathcal{N}$ satisfies the following condtions;

(a) $\mathcal{N}$ is dense in A,

(b) $\mathcal{N}$ is a Banach space for some norm $\| \ \|_{\mathcal{N}}$ such that

$$\begin{aligned} \|a\|_A &\leq \|a\|_{\mathcal{N}} \ \ (a \in \mathcal{N}) \\ \|ax\|_{\mathcal{N}} &\leq \|a\|_A \|x\|_{\mathcal{N}} \ \ (a \in A, x \in \mathcal{N}). \end{aligned}$$

Definition 4. (cf.[7]) A normed ideal $(\mathcal{N}, \| \ \|_{\mathcal{N}})$ in A is called a Segal algebra in A if $\mathcal{N}$ has approximate units, that is, $\mathcal{N}$ satisfies;

$$\forall x \in \mathcal{N},\ \forall \varepsilon > 0,\ \exists e \in \mathcal{N} \text{ such that } \|x - xe\|_{\mathcal{N}} < \varepsilon$$

Under the above definitions of normed ideals and Segal algebras in A, all the therems(Theorem A, B, C, D and E of the previous section)are also valid. Although the proofs of them are the same as that in the case of classical ones, we will give thier proofs for the sake of completeness.

Theorem A'. *If $\mathcal{N}$ is a normed ideal in A, we have;*

(i) *If U is a neighbourhood of $\varphi_0 \in \Phi_A$, there is an $x \in \mathcal{N}$ such that $\hat{x}(\varphi) = 1$ for all φ in a neibourhood of φ_0, and that* supp $\hat{x} \subset U$.

(ii) *If $K, U \subset \Phi_A$ such that K is compact and U is open with $K \subset U$, then there is an $e \in \mathcal{N}$ such that $\hat{e}(\varphi) = 1$ $(\varphi \in K)$ and* supp $\hat{e} \subset U$.

(iii) $A_c \subset \mathcal{N}$.

Proof. (i) Since $\mathcal{N}$ is dense in A, there exists $x \in \mathcal{N}$ such that $\hat{x}(\varphi_0) \neq 0$. Choose $y \in A$ such that $\hat{y}(\varphi_0) \neq 0$ with supp $\hat{y} \subset U$, and choose $z \in A$ such that $\hat{z}(\varphi) = 1/(\hat{x}(\varphi)\hat{y}(\varphi))$ for all φ in a neighbourhood of φ_0. Then if we put $e = xyz \in \mathcal{N}$, it is easy to see that e satisfies the desired properties.

(ii) For each $\varphi \in K$, there exists(by (i)) an $a_\varphi \in \mathcal{N}$ and a neighbourhood V_φ of φ such that supp $\widehat{a_\varphi} \subset U$ and $\widehat{a_\varphi} = 1$ on V_φ. We can choose a finite number of elements $\varphi_1, ..., \varphi_n \in K$ such that $\cup_{i=1}^n V_{\varphi_i} \supset K$. Then if we define $e \in \mathcal{N}$ by $1 - e = (1 - a_{\varphi_1}) \cdots (1 - a_{\varphi_n})$, then it is easy to see that e satisfies the desired properties.

(iii) Let $x \in A_c$ be arbitrary, and put $K :=$ supp $\hat{x}$. Then, by (ii), there is an $e \in \mathcal{N}$ such that $\hat{e} = 1$ on K, and hence . $x = xe \in \mathcal{N}$. Thus A_c is contained in $\mathcal{N}$. Q.E.D.

Theorem B'. (cf. [2]) *Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity of A contained in A_c such that* $\sup_{\lambda \in \Lambda} \|e_\lambda\|_A = M < \infty$. *If $\mathcal{N}$ is a normed ideal in A, the following* (a), (b), and (c) *are equivalent each other.*

(a) *$\mathcal{N}$ is a Segal algebra in A.*

(b) $\{e_\lambda\}_{\lambda\in\Lambda}$ *is an approximate identity of* $\mathcal{N}$.

(c) $\mathcal{N} = \mathcal{N}_0$, *where* $\mathcal{N}_0$ *is the norm closure of* A_c *in* $\mathcal{N}$.

Proof. (b) implies (a) is trivial by Definition 4. To prove (a) implies (c), let $x \in \mathcal{N}$ and $\varepsilon > 0$ be arbitraly. Choose $e \in \mathcal{N}$ and $\lambda_0 \in \Lambda$ such that $\|x - xe\|_{\mathcal{N}} < \varepsilon/2$ and $\|ee_{\lambda_0} - e\|_A < \varepsilon/(2\|x\|_{\mathcal{N}})$. Then $ee_{\lambda_0}x \in A_c$, and

$$\begin{aligned}\|x - ee_{\lambda_0}x\|_{\mathcal{N}} &\leq \|x - ex\|_{\mathcal{N}} + \|ex - ee_{\lambda_0}x\|_{\mathcal{N}}\\ &\leq \varepsilon/2 + \|x\|_{\mathcal{N}}\|e - ee_{\lambda_0}\|_A\\ &\leq \varepsilon/2 + \|x\|_{\mathcal{N}}\Big(\varepsilon/(2\|x\|_{\mathcal{N}})\Big) = \varepsilon.\end{aligned}$$

Thus $\mathcal{N} = \mathcal{N}_0$.

To complete the proof, suppose (c) and let $x \in \mathcal{N}$ and $0 < \varepsilon(\leq 1)$ be arbitrary. Then there is an $x_\varepsilon \in A_c$ such that $\|x - x_\varepsilon\|_{\mathcal{N}} < \dfrac{\varepsilon}{2(M+1)}$. Choose $e \in \mathcal{N}$ and $\lambda_0 \in \Lambda$ such that $x_\varepsilon e = x_\varepsilon$ and $\|e_\lambda e - e\|_A < \dfrac{\varepsilon}{2(\|x_\varepsilon\|_{\mathcal{N}}+1)}$ $(\lambda \geq \lambda_0)$. Then we have

$$\begin{aligned}\|e_\lambda x_\varepsilon - x_\varepsilon\|_{\mathcal{N}} &= \|e_\lambda x_\varepsilon e - x_\varepsilon e\|_{\mathcal{N}} \leq \|e_\lambda e - e\|_A\|x_\varepsilon\|_{\mathcal{N}}\\ &\leq \frac{\varepsilon}{2(\|x_\varepsilon\|_{\mathcal{N}}+1)}\|x_\varepsilon\|_{\mathcal{N}} \leq \frac{\varepsilon}{2}\ (\lambda \geq \lambda_0),\end{aligned}$$

and hence we have

$$\begin{aligned}\|e_\lambda x - x\|_{\mathcal{N}} &= \|e_\lambda(x - x_\varepsilon) + (e_\lambda x_\varepsilon - x_\varepsilon) + (x_\varepsilon - x)\|_{\mathcal{N}}\\ &\leq \|e_\lambda\|_A\|x - x_\varepsilon\|_{\mathcal{N}} + \|e_\lambda x_\varepsilon - x_\varepsilon\|_{\mathcal{N}} + \|x_\varepsilon - x\|_{\mathcal{N}}\\ &\leq (M+1)\frac{\varepsilon}{2(M+1)} + \varepsilon/2 = \varepsilon\ \ (\lambda \geq \lambda_0).\end{aligned}$$

Thus we get that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{N}$, and (b) follows. Q.E.D.

Theorm C'. Let $\mathcal{S}$ be a Segal algebra in A.

(i) The ideal theory of $\mathcal{S}$ is the same as that of A. More precisely, if $\mathcal{I}$ is a closed ideal of A then $\mathcal{I} \cap \mathcal{S}$ is a closed ideal of $\mathcal{S}$, and conversely each closed ideal of $\mathcal{S}$ is of this form for a unique closed ideal $\mathcal{I}$ of A.

(ii) The maximal ideal spaces of $\mathcal{S}$ and A are homeomorphic. We can naturally identify $\Phi_{\mathcal{S}}$ with Φ_A, that is, the Gelfand transform of $\mathcal{S}$ is equal to the Gelfand transform of A restricted to $\mathcal{S}$.

Proof. (i) We denote by $\overline{Ideal}(A)$(resp.$\overline{Ideal}(\mathcal{S})$) the set of all the closed ideals of A(resp.$\mathcal{S}$). For each $\mathcal{I} \in \overline{Ideal}(A)$, we have $\pi(\mathcal{I}) := \mathcal{I} \cap \mathcal{S} \in \overline{Ideal}(\mathcal{S})$ by the continuity of the identity map of $\mathcal{S}$ into A. We will show that the map π is a bijection of $\overline{Ideal}(A)$ and $\overline{Ideal}(\mathcal{S})$.

(a) Let $\mathcal{J} \in \overline{Ideal}(\mathcal{S})$ be arbitrary. Denote by $\overline{\mathcal{J}}$ the closure of $\mathcal{J}$ in A. One can show easily that $\overline{\mathcal{J}} \in \overline{Ideal}(A)$, and we omit its proof. For each $x \in \overline{\mathcal{J}} \cap \mathcal{S}$

and $\varepsilon > 0$, there exists $e \in \mathcal{S}$ such that $\|x - xe\|_S < \varepsilon/2$. Choose $y \in \mathcal{J}$ such that $\|x - y\|_A \leq \varepsilon/(2\|e\|_S)$. Then we have

$$\begin{aligned} \|x - ye\|_S &\leq \|x - xe\|_S + \|xe - ye\|_S \leq \varepsilon/2 + \|x - y\|_A\|e\|_S \\ &\leq \varepsilon/2 + \frac{\varepsilon}{2\|e\|_S}\|e\|_S = \varepsilon. \end{aligned}$$

Since $ye \in \mathcal{J}$ and $\mathcal{J}$ is closed, we have $x \in \mathcal{J}$. Thus $\overline{\mathcal{J}} \cap \mathcal{S} = \mathcal{J}$ for each $\mathcal{J} \in \overline{Ideal}(\mathcal{S})$, which implies that π is onto.

(b) Let $\mathcal{I} \in \overline{Ideal}(A)$ be arbitrary. For each $x \in \mathcal{I}$ and $\varepsilon > 0$, there exists $e \in A$ such that $\|x - xe\|_A < \varepsilon/2$. Since A_c is dense in A, we can choose $e' \in A_c$ such that $\|e - e'\|_A < \varepsilon/(2\|x\|_A)$. Then we have

$$\begin{aligned} \|x - xe'\|_A &\leq \|x - xe\|_A + \|xe - xe'\|_A \\ &\leq \varepsilon/2 + \|x\|_A\|e - e'\|_A \leq \varepsilon. \end{aligned}$$

Since $xe' \in \mathcal{I} \cap A_c$ and A_c is contained in $\mathcal{S}$ by (iii) of Theorem A', we have $\mathcal{I} = \overline{\mathcal{I} \cap \mathcal{S}}$. This proves that π is one to one.

(ii) For the proof of (ii), we refer to [1, Theorem 2.1]. Q.E.D.

Theorem D'. (i) *Let $\mathcal{S}$ be a Segal algebra in A, and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity of A composed of elements in A_c. Then $\{e_\lambda\}_{\lambda\in\Lambda}$ is a bounded approximate identity of $\mathcal{S}$ which is bounded with respect to the multiplication oeprator norm;*

$$\|T_x\|_{op} := \sup\{\|xy\|_S : y \in \mathcal{S}, \|y\|_S \leq 1\} \quad (x \in \mathcal{S}).$$

(ii) *If a Segal algebra $\mathcal{S}$ in A has a bounded approximate identity, then we have $\mathcal{S} = A$.*

Proof. (i) $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $\mathcal{S}$ by Theorem B'. It is multiplication operator bounded since for each $\lambda_0 \in \Lambda$,

$$\|T_{e_{\lambda_0}}\|_{op} = \sup_{x\in\mathcal{S}, \|x\|_S\leq 1} \|e_{\lambda_0}x\|_S \leq \sup_{x\in\mathcal{S}, \|x\|_S\leq 1} \|e_{\lambda_0}\|_A\|x\|_S \leq \sup_{\lambda\in\Lambda}\|e_\lambda\|_A < \infty.$$

(ii) Suppose that $\mathcal{S}$ has a bounded approximate identity $\{u_\omega\}_{\omega\in\Omega}$ such that $\sup_{\omega\in\Omega}\|u_\omega\|_S = M_1 < \infty$.

Let x be arbitrary in A_c. Choose $\omega_0 \in \Omega$ such that $\|u_{\omega_0}x - x\|_S < \|x\|_A$. Then $\|x\|_S \leq \|x\|_A + \|u_{\omega_0}\|_S\|x\|_A \leq (1 + M_1)\|x\|_A$. Since A_c is dense in $\mathcal{S}$ we get that $\|x\|_S \leq (1 + M_1)\|x\|_A \quad (x \in \mathcal{S})$. Therefore $\|\ \|_S$ and $\|\ \|_A$ are equvalent norm on, which implies that $A = \mathcal{S}$. Q.E.D.

Theorem E.' *If* $(\mathcal{S}_1, \| \|_{\mathcal{S}_1})$ *and* $(\mathcal{S}_2, \| \|_{\mathcal{S}_2})$ *are Segal algebras in A, then* $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ *becomes a Segal algebra in A with respect to the norm* $\| \|_{\mathcal{S}} = \| \|_{\mathcal{S}_1} + \| \|_{\mathcal{S}_2}$.

Proof. It is easy to see that $(\mathcal{S}, \| \|_{\mathcal{S}})$ is a normed ideal in A, and we omit its proof. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a bounded approximate identity of A contained in A_c, then by Theorem B', $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $(\mathcal{S}_i, \| \|_{\mathcal{S}})$ $i = 1, 2$. Let $x \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary, and choose λ_i(i=1,2) such that $\|x - e_\lambda x\|_{\mathcal{S}_i} \leq \varepsilon/2$ $(\lambda \geq \lambda_i)$ for $i = 1, 2$. Therefore if we take $\lambda_3 \in \Lambda$ such that $\lambda_3 \geq \lambda_i$ $(i = 1, 2)$, then

$$\|x - xe_\lambda\|_{\mathcal{S}} = \|x - xe_\lambda\|_{\mathcal{S}_1} + \|x - xe_\lambda\|_{\mathcal{S}_2} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (\lambda \geq \lambda_3).$$

Thus $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of $(\mathcal{S}, \| \|_{\mathcal{S}})$, and hence the assertion of the theorem follows from Theorem B'. Q.E.D.

§4. Segal algebras induced by local multipliers of A

Definition 5. Let τ be a complex continuous function on Φ_A. We call τ a local multiplier of A if we have $\hat{x}\tau \in \hat{A}$ $(x \in A_c)$. The set of local multipliers of A is denoted by $\hat{\mathcal{M}}_{loc}(A)$.

Definition 6. If $\tau \in \hat{\mathcal{M}}_{loc}(A)$, we put $A_\tau := \{x \in A : \hat{x}\tau \in \hat{A}\}$. Obviously, A_τ is a linear subspace of A which contains A_c. For each $x \in A_\tau$, there is a unique $a_x \in A$ such that $\widehat{a_x} = \hat{x}\tau$, and set

$$\|x\|_\tau := \|x\|_A + \|a_x\|_A \quad (x \in A_\tau).$$

It turns out that $\| \|_\tau$ is a complete algebra norm on A_τ as the following proposition shows.

Proposition 1. *For each* $\tau \in \hat{\mathcal{M}}_{loc}(A)$, $(\hat{A}, \| \|_\tau)$ *is a Segal algebra in A. Moreover, if* $\sup\{| \tau(\varphi) | : \varphi \in \Phi_A\} = \infty$, *we have* $A \neq A_\tau$.

Proof. It is easy to see that A_τ is a linear subspace of A. For each $a \in A$ and $x \in A_\tau$, $(ax)^\wedge\tau = \hat{a}(\hat{x}\tau) \in \hat{A}$, and hence A_τ is an ideal of A. That A_τ is dense in A follows from the fact that $A_c \subset A_\tau$.

Next we will show that $\| \|_\tau$ is a complete norm on A_τ. Note that the map: $x \to a_x$ is a linear transformation from A_τ to A, and hence it is easy to see that $\| \|_\tau$ is a norm on A_τ. Thus we have only to show that $\| \|_\tau$ is a comple norm. To see this, let $\{x_n\}$ be a Cauchy sequence in A_τ. Then $\lim_{m,n\to\infty} \|x_m - x_n\|_A = \lim_{m,n\to\infty} \|a_{x_m} - a_{x_n}\|_A = 0$ and hence there exist $a, x \in A$ such that $\lim_{n\to\infty} \|x - x_n\|_A = 0$ and

$\lim_{n\to\infty} \|a - a_{x_n}\|_A = 0$ because $\| \ \|_A$ is a complete norm on A. Since

$$\hat{x}(\varphi)\tau(\varphi) = \lim_{n\to\infty} \widehat{x_n}(\varphi)\tau(\varphi) = \lim_{n\to\infty} \widehat{a_{x_n}}(\varphi) = \hat{a}(\varphi)$$

for all $\varphi \in \Phi_A$, it follows that $\hat{x}\tau = \hat{a} \in \hat{A}$, and hence $x \in A_\tau$ and $a = a_x$. Therefore

$$\lim_{n\to\infty} \|x_n - x\|_\tau = \lim_{n\to\infty} \Big(\|x_n - x\|_A + \|a_{x_n} - a_x\|_A\Big) = 0.$$

Consequently, $\| \ \|_\tau$ is complete.

Next let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity of A. As stated above, $\{e_\lambda\}_{\lambda\in\Lambda}$ can be chosen in A_c and hence in A_τ. We show that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate identity of A_τ.

Let $x \in A_\tau$ and $\varepsilon > 0$ be arbitrary. Since

$$(e_\lambda x - x)^\wedge\tau = (e_\lambda x)^\wedge\tau - \hat{x}\tau = \hat{e}_\lambda\hat{x}\tau - \hat{x}\tau = \hat{e}_\lambda\hat{a}_x - \hat{a}_x = (e_\lambda a_x - a_x)^\wedge,$$

it follows that $\|e_\lambda x - x\|_\tau = \|e_\lambda x - x\|_A + \|e_\lambda a_x - a_x\|_A$. Then we obtain the desired result by taking the limit with respect to $\lambda \in \Lambda$.

We further see that $\|ax\|_\tau \le \|a\|_A\|x\|_\tau$ for all $a \in A$ and $x \in A_\tau$. In fact, let $a \in A$ and $x \in A_\tau$. Since $(ax)^\wedge\tau = \hat{a}\hat{x}\tau = \hat{a}\widehat{a_x} = (aa_x)^\wedge$, it follows that

$$\|ax\|_\tau = \|ax\|_A + \|aa_x\|_A \le \|a\|_A\|x\|_A + \|a\|_A\|a_x\|_A = \|a\|_A\|x\|_\tau,$$

and hence we obtain the desired result. Q.E.D.

Definition 7. For $\tau \in \hat{M}_{loc}(A)$, we call $(A_\tau, \| \ \|_\tau)$ the Segal algebra in A induced by τ.

Proposition 2. *If $x \in A$ such that* supp $\hat{x}$ *is σ-compact but not compact, the we have $x \notin \cap\{A_\tau : \tau \in \hat{M}_{loc}(A)\}$.*

Proof. Let x be an element in A such that supp $\hat{x}$ is σ-compact but not compact, and denote the open set $\{\varphi \in \Phi_A : \hat{x}(\varphi) \neq 0\}$ of Φ_A by Ω. We argue in the topological space $X := \overline{\Omega}(=$ supp $\hat{x})$, and for $E \subset X, E^\circ$ means the interier of E in X. Since X is σ-compact, there exists a sequence $\{K_n\}$ of compact seubsets of X which satisfies; (i) $K_j \subset K_{j+1}^\circ$ for $j = 1, 2, \ldots$, (ii) $X = \cup_{j=1}^\infty K_j$. In fact, since X is σ- compact, it is easy to see that we have an expression $X = \cup_{j=1}^\infty U_j$, where $\{U_j : j = 1, 2, \ldots\}$ is an increasing sequence of relatively compact open subsets of X. Put $K_1 = \overline{U_1}$. Next choose a positive integer n such that U_n properly contains K_1, and put $K_2 = \overline{U_n}$. When K_1, K_2,...,K_m have defined, choose n such that U_n properly contains K_m. Put $K_{m+1} := \overline{U_n}$. In the way, we can get a sequence $\{K_n\}_{n=1}^\infty$ which satisfies (i) and (ii).

Since $x \notin A_c$, Ω is not contained in any $K_j, j = 1, 2, ...$, and there exists an infinite strictly increasing sequence of positive integers $n_1, n_2, ...$ such that

$$\exists \varphi_j \in K_{n_{j+1}} \setminus K_{n_j} \text{ such that } \hat{x}(\varphi_j) \neq 0 \text{ for } j = 1, 2, ...$$

For each positive integer j we can choose $x_j \in A_c$ such that $\hat{x}_j(\varphi_j) = 1/\hat{x}(\varphi_j)$ and supp $\hat{x}_j \subset ((K^\circ_{n_{j+1}} \cap \Omega) \setminus K_{n_j})$. If we define a complex function τ on Φ_A by

$$\tau(\varphi) := \begin{cases} \hat{x}_j(\varphi) & \text{if } \varphi \in (K_{n_{j+1}} \setminus K_{n_j}) \text{ for some } j, \\ 0 & \text{if } \varphi \in (\Phi_A \setminus X) \cup K_{n_1}. \end{cases}$$

Note that supp $\tau \cap K^\circ_{n_{j+1}} = \cup^j_{k=1}$supp $\widehat{x_k} \subset \Omega, j = 1, 2, ...$ and hence supp $\tau \subset \Omega$. We claim here that τ is continuous in Φ_A. To prove the claim, we may only show that τ is continuous at each point in supp τ. Let $\varphi \in$ supp τ be arbitrary. Then there exists j such that $\varphi \in K^\circ_{n_{j+1}}$ Since $\tau = \sum^j_{i=1} \widehat{x_i}$ on $K^\circ_{n_{j+1}}$ and $K^\circ_{n_{j+1}} \cap \Omega$ is a open set of Φ_A which contains φ, it follows that τ is continuou at φ. Thus the claim is proved.

If $y \in A_c$, we can choose j such that supp $\hat{y} \cap X \subset K^\circ_{n_{j+1}}$. Then it is easy to see that $\hat{y}\tau = \sum^j_{k=1} \hat{y}\widehat{x_k} \in \hat{A}$, and hence $\tau \in \hat{\mathcal{M}}_{loc}(A)$. Moreover $x \notin A_\tau$ since $\varphi_j \to \infty$ and $\hat{x}(\varphi_j)\tau(\varphi_j) = 1 \;\; j = 1, 2, ...$ This complete the proof. Q.E.D.

Corollary 3. *Suppose that Φ_A is σ-compact, or discrete. Then we have $\cap\{A_\tau : \tau \in \hat{\mathcal{M}}_{loc}(A))\} = A_c$.*

Proof. For each $x \in A$, supp $\hat{f}$ is σ-compact, and hence the assetion follows from Proposition 2. Q.E.D.

Corollary 4. *Let G be a non-discrete locally compact abelian group. Then we have $\cap\{L^1(G)_\tau : \tau \in \hat{\mathcal{M}}_{loc}(L^1(G))\} = L^1(G)_c$.*

Proof. For each $f \in L^1(G)$, supp $\hat{f}$ is σ-compact, and hence the assetion follows from Proposition 2. Q.E.D.

Let $\mathcal{S}$ be a Segal algebra in A. A multiplier of $\mathcal{S}$ to A is a bounded linear operator of $\mathcal{S}$ into A such that $(Tx)y = x(Ty)$ for each $x, y \in \mathcal{S}$. The set of all multiplies of $\mathcal{S}$ to A is denoted by $\mathcal{M}(\mathcal{S}, A)$.

Proposition 5. *If $\mathcal{S}$ is a Segal algebra in A and if T is a linear operator of $\mathcal{S}$ into A, the following* (a) *and* (b) *are equivalent each other.*

(a) *$T \in \mathcal{M}(\mathcal{S}, A)$.*

(b) *There exists an unique continuous function τ on Φ_A such that $\widehat{Tx} = \hat{x}\tau \;\; (x \in \mathcal{S})$.*

Proof. By Theorem C', we can naturally identify $\Phi_{\mathcal{S}}$ with Φ_A.

(a)$\Rightarrow$ (b). For each $\varphi \in \Phi_A$, choose x and $y \in \mathcal{S}$ such that $\hat{x}(\varphi) \neq 0$, and $\hat{y}(\varphi) \neq 0$. Then we have $(Tx)^\wedge(\varphi)\hat{y}(\varphi) = \hat{x}(\varphi)(Ty)^\wedge(\varphi)$. Hence we have

$$(Tx)^\wedge(\varphi)/\hat{x}(\varphi) = (Ty)^\wedge(\varphi)/\hat{y}(\varphi)..........(1).$$

Define a complex function τ on Φ_A by

$$\tau(\varphi) := (Tx)^\wedge(\varphi)/\hat{x}(\varphi)........(2).$$

The definition is well defined by (1), and by (2) we get

$$(Tx)^\wedge(\varphi) = \hat{x}(\varphi)\tau(\varphi) \quad (x \in \mathcal{S}, \varphi \in \Phi_A).........(3).$$

Note that (3) is true even if $\hat{x}(\varphi) = 0$. For, in this case, choosing $y \in \mathcal{S}$ with $\hat{y}(\varphi) \neq 0$, we have $(Tx)^\wedge(\varphi)\hat{y}(\varphi) = \hat{x}(\varphi)(Ty)^\wedge(\varphi) = 0$, and hence $(Tx)^\wedge(\varphi) = 0$.

By (2), it follows that τ is continuous on Φ_A. The uniqueness of τ easily verified by the routine procedure.

(b)$\Rightarrow$ (a). For each $x \in \mathcal{S}$, there exists a unique $a_x \in A$ such that

$$(a_x)^\wedge(\varphi) = \hat{x}(\varphi)\tau(\varphi) \quad (\varphi \in \Phi_A).$$

We define $Tx = a_x$ $(x \in \mathcal{S})$. Then that T satisfies the property $(Tx)y = x(Ty)$ for $x, y \in \mathcal{S}$ is obvious, and the boundedness of T is easily shown by using the closed graph theorem. Q.E.D.

Definition 8. Suppose $\mathcal{S}$ is a Segal algebra in A. If $T \in \mathcal{M}(\mathcal{S}, \mathcal{A})$ there is , by Proposition 5, a unique $\tau \in C(\Phi_A)$ such that $(Tx)^\wedge = \hat{x}\tau$. We denote this τ by $\hat{T}$, and call the Gelfand trasnsform of T. We set $\hat{\mathcal{M}}(\mathcal{S}, A) := \{\hat{T} : T \in \mathcal{M}(\mathcal{S}, A)\}$.

Remark 2. (1) If τ is a local multiplier of A, we have, by Propositin 5, $\tau \in \hat{\mathcal{M}}(A_\tau, A)$.

(2) If $\mathcal{S}$ is a Segal algebra in A, and if $T \in \mathcal{M}(\mathcal{S}, A)$, it is easy to see that $\hat{T}$ is a local multiplier of A which satisfies $\mathcal{S} \subset A_{\hat{T}}$.

(3) If $\tau \in \hat{\mathcal{M}}_{loc}(L^1(G))$ such that $\tau = \hat{\mu}$ for some $\mu \in M(G)$, we have $L^1(G)_\tau = L^1(G)$ and $\| \|_\tau$ is equivalent to $\| \|_1$.

Proposition 6. *If* $1 < p < \infty$ *and if* $\tau \in \hat{\mathcal{M}}_{loc}(L^1(G))$ *such that* $\mathcal{S}_p(G) \subseteq L^1(G)_\tau$, *then we have* $\tau = \hat{\mu}$ *for some* $\mu \in M(G)$.

Proof. It follows that $\tau \in \hat{\mathcal{M}}(\mathcal{S}_p(G), L^1(G))$ by the condition on τ. Since it is known that $\hat{\mathcal{M}}(\mathcal{S}_p(G), L^1(G)) = \{\hat{\mu} : \mu \in M(G)\}$ ([4, p. 79 Theorem 3.5.1]), we have $\tau = \hat{\mu}$ for some $\tau = \hat{\mu}$ for some $\mu \in M(G)$). Q.E.D.

Lemma 7. *If G is non-compct, and if p $(1 \leq p < \infty)$, then we have* $\hat{\mathcal{M}}(A_p(G), L^1(G)) = \{\hat{\mu} :\in M(G)\}$.

Proof. Let $T \in \mathcal{M}(A_p(G), L^1(G))$ be arbitrary. Observe that $\hat{T}$ is a bounded function on $\hat{G}$. In fact, if we choose $e \in A_p(G)$ such taht $\hat{e}(0) = 1$, then $\{e_\gamma(x) := e(x)(x,\gamma); \gamma \in \hat{G}\}$ is a set of bounded functions in $A_p(G)$, and hence there is a positive number M such that $| \widehat{e_\gamma}(\gamma)\hat{T}(\gamma) | \leq M \quad (\gamma \in \hat{G})$. This with the relation $\widehat{e_\gamma}(\gamma) = 1$ $(\gamma \in \hat{G})$ implies $\|\hat{T}\|_\infty \leq M$.

From the relations; $\|Tf\|_1 \leq \|T\| \|f\|_{A_p}$ and

$$\|(Tf)^\wedge\|_p = \left(\int_{\hat{G}} \left|\hat{f}(\gamma)\hat{T}(\gamma)\right|^p d\gamma\right)^{1/p} \leq \|\hat{T}\|_\infty \|\hat{f}\|_p,$$

we have

$$\|Tf\|_{A_p(G)} \leq \|T\|\|f\|_{A_p} + \|\hat{T}\|_\infty \|\hat{f}\|_p \leq (\|T\| + \|\hat{T}\|_\infty)\|f\|_{A_p} \quad (f \in A_p(G)).$$

Therefore $T \in \mathcal{M}(A_p(G))$. Since it is known that $\mathcal{M}(A_p(G)) = \{\hat{\mu} : \ \mu \in M(G)\}$ (cf. [4, p.204 Theorem 6.3.1]), we get the desired result. Q.E.D.

Proposition 8. *Suppose G is non-comact and $1 \leq p < \infty$. If $\tau \in \hat{M}_{loc}(L^1(G))$ and $A_p(G) \subseteq L^1(G)_\tau$, then we have $\tau = \hat{\mu}$ for some $\mu \in M(G)$.*

Proof. The assumptions on τ implies that $\tau \in \hat{\mathcal{M}}(A_p(G), L^1(G))$. As the same way as the proof of Proposition 6, with the aid of Lemma 7, we get that $\tau = \hat{\mu}$ for some $\mu \in M(G)$. Q.E.D.

Remark 3. Proposition 6 (resp. Propsition 8) shows that there are no proper Segal algebras in the type $L^1(G)_\tau, \tau \in \hat{\mathcal{M}}(L^1(G))$ which contain $S_p(G)$(resp. $A_p(G)$). Bu next proposition shows that this is not the case for the Segal algebras of type $A_{\nu,1}(G)$ of an infinite compact abelian group G.

Proposition 9. *Let G be an infinite compact abelian group. Suppose that $\tau \in \hat{\mathcal{M}}_{loc}(L^1(G))$ satisfies $0 < \tau$ and $\sup_{\gamma \in \hat{G}} \tau(\gamma) = \infty$, and define an unbounded Radon measure ν on $\hat{G}$ by $\nu := \tau m_{\hat{G}}$, where $m_{\hat{G}}$ is a Haar measure of $\hat{G}$. Then we have $A_{\nu,1}(G) \subseteq L^1(G)_\tau \neq L^1(G)$.*

Proof. For $f \in L^1(G)$, we have

$$\begin{aligned} f \in A_{\nu,1}(G) \quad & \Leftrightarrow \int_{\hat{G}} | \hat{f}(\gamma) | \, d\tau(\gamma) m_{\hat{G}}(\gamma) < \infty \\ & \Rightarrow \hat{f}\tau \in L^1(\hat{G}) \subseteq L^2(\hat{G}) \subseteq L^1(G)^\wedge \\ & \Rightarrow f \in L^1(G)_\tau, \end{aligned}$$

and the result follows. Q.E.D.

Remark 4. For an infinite compact abelian group G, any complex function on $\hat{G}$ belongs to $\hat{\mathcal{M}}_{loc}(L^1(G))$.

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