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Renormalized Dissipative Solutions and Applications

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1 Introduction

We consider the following Cauchy problem:

$$\text{(CP)} \quad \begin{cases} u_t + \operatorname{div}\mathbf{F}(u) - \displaystyle\sum_{i,j=1}^{N} A_{ij}(u)_{x_i x_j} = f & \text{in } Q := (0,T)\times\mathbf{R}^N, \\ u(0,\cdot) = u_0 & \text{in } \mathbf{R}^N, \end{cases}$$

where $T > 0$ and $N \geq 1$. Here $f \in L^1(Q)$ and $u_0 \in L^1(\mathbf{R}^N)$ are given functions, the flux $\mathbf{F} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz-continuous function and the diffusion function $A(u)$ is a nonnegative symmetric $N \times N$ matrix.

In the case where the diffusion term degenerates, the equation becomes a conservation law $u_t + \operatorname{div}\mathbf{F}(u) = f$. As a typical example of appearance of a conservation law, we now consider the traffic flow on an expressway. Let $u(t,x)$ be the density on an expressway at time t and point x. We assume, for simplicity, that u is continuous in t and x, and the speed s of the cars depends only upon their density, which means that $s = s(u)$ and $s' < 0$. For any two points a, b on the expressway, the number of cars between a and b depends upon the inflow at $x = a$ and outflow at $x = b$, namely,

$$\begin{aligned} \frac{d}{dt}\int_a^b u(t,x)\,dx &= s(u(t,a))\,u(t,a) - s(u(t,b))\,u(t,b) \\ &= -\int_a^b \Big(s(u(t,x))\,u(t,x)\Big)_x\,dx \end{aligned}$$

holds for any a, b. Since u is continuous and a, b are arbitrary, we have the conservation law $u_t + F(u)_x = 0$ with the flux $F(u) := s(u)\, u$.

We are interested in finding all solutions of a partial differential equation and furthermore investigating the existence, uniqueness, asymptotic behavior and other properties of solutions for general data. Most partial differential equations, however, are not expected to have smooth solutions. For example, we consider the Cauchy problem of the inviscid Burgers' equation

$$u_t + uu_x = 0, \qquad u(0, x) = \frac{1}{1 + x^2}. \tag{1.1}$$

It is known that a smooth solution for (1.1) blows up in finite time, namely, the discontinuity of the smooth solution appears in finite time even if the initial datum is sufficiently smooth. This corresponds a shock wave in gas dynamics. In this problem, we see that the smooth solution for (1.1) blows up at $t = 8/\sqrt{27}$ by the implicit function theorem.

In order to deal with phenomena precisely, we need to extend the notion of solutions to nonsmooth solutions including discontinuous solutions interpreted in the sense of distributions, which is the so-called weak solution. Nevertheless, it is known that there exist many weak solutions in general for the Cauchy problem of nonlinear degenerate parabolic equations including conservation laws. Thereupon, Kružkov [10] introduced a new notion of an entropy solution which is a weak solution satisfying an entropy inequality, and proved the uniqueness of an entropy solution for a conservation law. This 'entropy' comes, roughly speaking, from the thermodynamic principle that physical entropy can not decrease as time goes forward. The entropy inequality is a suitable criterion to extract accurately the exact one weak solution according as physical demands, and ensure the uniqueness of weak solutions.

Chen and Perthame [4] extended the notion of entropy solutions to general degenerate parabolic equations with anisotropic nonlinearity, and obtained uniqueness of an entropy solution by utilizing a kinetic formulation and regularization by convolution. At the same time, Portilheiro [12] defined a dissipative solution of scalar conservation laws with globally Lipschitz-continuous flux

$\mathbf{F}$, which was established first by Evans, and showed the equivalence of such solutions and entropy solutions by accretive operator theory. Furthermore, the notion of dissipative solutions was extended by Perthame and Souganidis [11] to the second order degenerate parabolic balance laws and the equivalence result was obtained. The definition of dissipative solutions is more simple and flexible, and also suitable to study asymptotic problems handling relaxation systems than entropy solutions. Direct proofs of existence and uniqueness of dissipative solutions, however, have not been obtained yet.

We here introduce an idea of dissipative solutions. We say u is a dissipative solution of the equation $Au = f$ with possibly multivalued accretive operator $A : D(A) \to 2^X$ defined as a subset of some Banach space X if

$$[u - \phi, f - A\phi]_+ \geq 0$$

holds for every 'nice' function ϕ, where $[\cdot,\cdot]_+$ denotes the Kato bracket defined as

$$[u, v]_+ := \lim_{\lambda \downarrow 0} \frac{\|u + \lambda v\| - \|u\|}{\lambda}.$$

We note that if $X = L^1(Q)$ particularly, then the Kato bracket is given by

$$[f, g]_+ := \iint_{f \neq 0} S_0(f)\, g \, dxdt + \iint_{f=0} |g| \, dxdt$$

for any $f, g \in L^1(Q)$, where $S_0(s)$ takes 1 if $s > 0$ or 0 otherwise for $s \in \mathbf{R}$.

On the other hand, it is known that if $u_0 \in L^1(\mathbf{R}^N)$ and $f \in L^1(Q)$, then the mild solution u of (CP) constructed by nonlinear semigroup theory is a unique entropy solution, which is unbounded in general. In the case where $\mathbf{F}$ is only locally Lipschitz-continuous, the flux function $\mathbf{F}(u)$ may fail to be locally integrable since no growth condition is assumed on the flux $\mathbf{F}$, and hence (CP) does not possess a solution even in the sense of distributions. To overcome this, the notion of renormalized entropy solutions has been introduced by Bénilan et al. [3] for scalar conservation laws and by Bendahmane and Karlsen [2] for second order degenerate parabolic equations. Furthermore, the existence and uniqueness of a renormalized entropy solution of these equations have been established and the semigroup solutions of (CP) in L^1 spaces are characterized. The arguments in [11] and [12], however, do not work well in the case where $\mathbf{F}$ is only locally Lipschitz-continuous and the solution u is unbounded. The notion of renormalized solutions has been introduced by DiPerna and Lions [5] for dealing with the existence of a solution of the Boltzmann equation and utilized for degenerate elliptic and degenerate parabolic problems in the L^1-setting in the last decade.

A new concept of renormalized dissipative solutions for a conservation law with L^1 data has been established in [9] and the equivalence of such solutions and renormalized entropy solutions in the sense of [3] was proved. Existence of renormalized dissipative solutions for a contractive relaxation system describing discrete velocity models and chemical reaction models has been also shown in general L^1-settings in [9] and solutions of the system were characterized. In this paper, we extend this notion to quasilinear anisotropic degenerate parabolic equations including conservation laws and apply this notion to some relaxation systems.

2 Equivalence

We begin with some notations and definitions. Let $s \in \mathbf{R}$ and $j \in [-1, 1]$. We set $s^+ := \max\{s, 0\}$ and $s^- := -\min\{s, 0\}$. Note that $s^- \geq 0$ and $s = s^+ - s^-$. Define a sign function S_j by $S_j(s) = 1$ if $s > 0$, $S_j(s) = -1$ if $s < 0$ or $S_j(0) = j$, and set $S_j^+(s) := \max\{S_j(s), 0\}$ and $S_j^-(s) := \min\{S_j(s), 0\}$.

For $s \in \mathbf{R}$, the diffusion function $A(s) = (a_{ij}(s))$ is a nonnegative symmetric $N \times N$ matrix of the form

$$a_{ij}(s) = \sum_{m=1}^{M} \sigma_{im}(s)\, \sigma_{jm}(s), \qquad \sigma_{im} \in L^\infty_{loc}(\mathbf{R})$$

for $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, where $M \leq N$ can be thought to be the maximal rank of the matrix. Let $T_\ell : \mathbf{R} \to [-\ell, \ell]$ denote the truncation function with height $\ell > 0$, that is, $T_\ell(s) := \min\{\max\{s, -\ell\}, \ell\}$ for any

$s \in \mathbf{R}$. For $1 \leq m \leq M$, $1 \leq i \leq N$ and $s \in \mathbf{R}$, we set

$$\beta_{im}(s) := \int_0^s \sigma_{im}(r)\,dr, \qquad \boldsymbol{\beta}_m(s) = (\beta_{1m}(s), \cdots, \beta_{Nm}(s)),$$

and for any $\psi \in C(\mathbf{R})$

$$\beta_{im}^{\psi}(s) := \int_0^s \psi(r)\,\sigma_{im}(r)\,dr, \qquad \boldsymbol{\beta}_m^{\psi}(s) = (\beta_{1m}^{\psi}(s), \cdots, \beta_{Nm}^{\psi}(s)).$$

Following [2] we define an entropy-entropy flux triple and a renormalized entropy solution of (CP).

Definition 2.1. *For any convex C^2 entropy function $\eta : \mathbf{R} \to \mathbf{R}$, the corresponding entropy fluxes*

$$\mathbf{q} = (q_1, \cdots, q_N) : \mathbf{R} \to \mathbf{R}^N \quad \text{and} \quad R = (r_{ij}) : \mathbf{R} \to \mathbf{R}^{N\times N}$$

are defined by $q_i'(s) = \eta'(s)\,F_i'(s)$ and $r_{ij}'(s) = \eta'(s)\,a_{ij}(s)$ for $i, j = 1, \cdots, N$ and $s \in \mathbf{R}$. Then, we define $(\eta, \mathbf{q}, R)$ as an entropy-entropy flux triple.

Definition 2.2. *We say $u \in L^\infty(0, T; L^1(\mathbf{R}^N))$ is a renormalized entropy solution of* (CP) *if a measurable function $u : Q \to \mathbf{R}^N$ satisfies the following conditions:*

(E1) *For any $m = 1, \cdots, M$,*

$$\boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q)^N \quad \text{and} \quad \operatorname{div}\boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q) \quad \text{for all} \quad \ell > 0.$$

(E2) *For any $m = 1, \cdots, M$ and $\psi \in C(\mathbf{R})$,*

$$\operatorname{div}\boldsymbol{\beta}_m^{\psi}(T_\ell(u)) = \psi(T_\ell(u))\,\operatorname{div}\boldsymbol{\beta}_m(T_\ell(u))$$

a.e. in Q and in $L^2(Q)$ for all $\ell > 0$.

(E3) *For any $\ell > 0$ and any entropy-entropy flux triple $(\eta, \mathbf{q}, R)$ with $|\eta'| \leq K$ for some given $K > 0$, there exists for any $\ell > 0$ a nonnegative bounded Radon measure μ_ℓ^K on Q with $\mu_\ell^K(Q) \to 0$ as $\ell \to \infty$ such that*

$$\eta(T_\ell(u))_t + \operatorname{div}\mathbf{q}(T_\ell(u)) - \sum_{i,j=1}^N r_{ij}(T_\ell(u))_{x_i x_j} - \eta'(T_\ell(u))\,f$$
$$\leq\; -\eta''(T_\ell(u)) \sum_{m=1}^M \left(\operatorname{div}\boldsymbol{\beta}_m(T_\ell(u))\right)^2 + \mu_\ell^K \qquad \text{in } \mathcal{D}'(Q).$$

(E4) $u(t,\cdot) \to u_0$ *in* $L^1(\mathbf{R}^N)$ *as* $t \downarrow 0$ *essentially.*

Next, we introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in the sense of [11].

Definition 2.3. *We say* $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ *is a renormalized dissipative solution of* (CP) *if a measurable function* $u : Q \to \mathbf{R}^N$ *satisfies the following conditions:*

(D1) *For any* $m = 1, \cdots, M$,

$$\boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q)^N \quad \text{and} \quad \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u)) \in L^2(Q) \quad \text{for all } \ell > 0.$$

(D2) *For any* $m = 1, \cdots, M$ *and* $\psi \in C(\mathbf{R})$,

$$\operatorname{div} \boldsymbol{\beta}_m^\psi(T_\ell(u)) = \psi(T_\ell(u)) \operatorname{div} \boldsymbol{\beta}_m(T_\ell(u))$$

a.e. in Q *and in* $L^2(Q)$ *for all* $\ell > 0$.

(D3) *For any* $\ell > 0$, $\xi \in C_0^2(\mathbf{R}^N)$ *and* $\theta \in C_0^2(\mathbf{R})^+$ *with* $\operatorname{spt}\theta \subset (-\ell, \ell)$, *there exists a nonnegative bounded Radon measure* ν_ℓ *on* Q *with* $\nu_\ell(Q) \to 0$ *as* $\ell \to \infty$ *such that*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\left(T_\ell(u) - k - \xi\right)^+ dk dx \\
&\leq \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(u) - k - \xi) \\
&\qquad\qquad \times \left(f - \operatorname{div}\mathbf{F}(k+\xi) + \sum_{i,j=1}^N A_{ij}(k+\xi)_{x_i x_j}\right) dk dx \\
&\quad - \int_{\mathbf{R}^N} \theta(T_\ell(u) - \xi) \sum_{m=1}^M \left(\operatorname{div}\boldsymbol{\beta}_m(T_\ell(u)) - \boldsymbol{\sigma}_m(T_\ell(u))\cdot\nabla\xi\right)^2 dx \\
&\quad + \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\, S_0^+(\ell - k - \xi)\, dk d\nu_\ell \qquad \text{in } \mathcal{D}'(0,T),
\end{aligned}$$

where $A_{ij}'(\cdot) := a_{ij}(\cdot)$, $\boldsymbol{\sigma}_m(\cdot) = (\sigma_{1m}(\cdot), \cdots, \sigma_{Nm}(\cdot))$ *and* $C_0^2(\mathbf{R})^+$ *denotes the space of all nonnegative functions in* $C_0^2(\mathbf{R})$ *as usual.*

(D4) $u(t,\cdot) \to u_0$ *in* $L^1(\mathbf{R}^N)$ *as* $t \downarrow 0$ *essentially.*

Then we obtain the following main result.

Theorem 2.4. *Suppose that $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$. Then, u is a renormalized entropy solution of* (CP) *if and only if u is a renormalized dissipative solution of* (CP).

Sketch of the proof. Let $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ be a renormalized dissipative solution of (CP). Then, we consider a function $\alpha \in C_0^2(\mathbf{R}^N)^+$ and for each ε, $\lambda > 0$ a nondecreasing smooth function $\xi_{\varepsilon,\lambda}$ defined by $\xi_{\varepsilon,\lambda}(s) = 0$ if $|s| \le \lambda$, $\xi_{\varepsilon,\lambda}(s) = 1/\varepsilon$ if $|s| \ge \lambda + \varepsilon$ and strictly increasing otherwise. Let $V(N)$ denote the volume of the unit ball in $\mathbf{R}^N$. Using the test function $\xi_{\varepsilon,\lambda}(x-y)$, multiplying by $\alpha_\lambda(y) := V(N)^{-1}\lambda^{-N}\alpha(y)$, integrating with respect to y and applying the Lebesgue differentiation theorem yield for any $\phi \in C_0^1(0,T)^+$,

$$
\begin{aligned}
&\iint_Q\int_{\mathbf{R}} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha\,dkdxdt + \iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla\alpha(x)\,\phi\,dxdt \\
&\quad + \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\,f\,\theta(k)\,\phi\,\alpha\,dkdxdt \\
&\quad + \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\sum_{i,j=1}^N \big(A_{ij}(k)-A_{ij}(T_\ell(u))\big)\,\alpha_{x_ix_j}\,\theta(k)\,\phi\,dkdxdt \\
&\quad - \iint_Q \theta(T_\ell(u))\sum_{m=1}^M \Big(\operatorname{div}\beta_m(T_\ell(u))\Big)^2 \phi\,\alpha\,dxdt \\
&\quad - 2\iint_Q \Theta(T_\ell(u))\sum_{m=1}^M \Big(\operatorname{div}\beta_m(T_\ell(u))\Big)\Big(\sigma_m(T_\ell(u))\cdot\nabla\alpha\Big)\phi\,dxdt \\
&\quad + \iint_Q\int_{\mathbf{R}} S_0^+(\ell-k)\,\theta(k)\,\phi\,\alpha\,dkd\nu_\ell \;\ge\; 0.
\end{aligned}
$$

Following the definition of an entropy-entropy flux triple, we see that

$$
\begin{aligned}
\eta(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\,(T_\ell(u)-k)^+\,dk, \\
\eta'(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\,S_0^+(T_\ell(u)-k)\,dk, \\
q_i(T_\ell(u))_{x_i} &= \eta'(T_\ell(u))\,F_i(T_\ell(u))_{x_i}, \\
r_{ij}(T_\ell(u))_{x_ix_j} &= \eta'(T_\ell(u))_{x_j}\,A_{ij}(T_\ell(u))_{x_i} + \eta'(T_\ell(u))\,A_{ij}(T_\ell(u))_{x_ix_j}.
\end{aligned}
$$

Putting $\theta = \eta''$ and $\Theta = \eta'$, we obtain that

$$\begin{aligned} 0 \;\leq\; & \iint_Q \eta(T_\ell(u))\,\phi'\,\alpha\,dxdt + \iint_Q \eta'(T_\ell(u))\,f\,\phi\,\alpha\,dxdt \\ & + \iint_Q \mathbf{q}(T_\ell(u))\cdot\nabla\alpha\,\phi\,dxdt + \iint_Q \sum_{i,j=1}^{N} r_{ij}(T_\ell(u))\,\alpha_{x_ix_j}\,\phi\,dxdt \\ & - \iint_Q \eta''(T_\ell(u)) \sum_{m=1}^{M} \Big(\operatorname{div}\beta_m(T_\ell(u))\Big)^2 \phi\,\alpha\,dxdt + \iint_Q \eta'(\ell)\,\phi\,\alpha\,d\nu_\ell, \end{aligned}$$

which is exactly (E3).

We next assume that $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ is a renormalized entropy solution of (CP). For given $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell,\ell)$, we observe that $\eta(T_\ell(u)) = \int_{\mathbf{R}} \big(T_\ell(u)-k-\xi(y)\big)^+\,\theta(k)\,dk$ is a smooth entropy. Let ϕ and ρ be standard mollifiers on $(0,T)$ and $\mathbf{R}^N$, respectively. Define ρ_ε by $\rho_\varepsilon(x-y) := \rho((x-y)/\varepsilon)\varepsilon^{-N}$, and let ψ_n be a nonnegative smooth function taking 1 if $|x| \leq n$ or 0 if $|x| \geq 2n$, and $|\nabla\psi_n| \leq C/n$ for some $C > 0$. Putting $\zeta = \rho_\varepsilon(x-y)\,\phi(t)\,\psi_n(t,x)$, integrating with respect to y over $\mathbf{R}^N$ and using the properties of the Dirac mass and the Lebesgue convergence theorem, we obtain that

$$\begin{aligned} 0 \leq\; & \iint_Q\int_{\mathbf{R}} (T_\ell(u)-k-\xi)^+\,\theta\,\phi'\,dkdxdt + \iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi)\,\theta\,\phi\,dkd\mu_\ell \\ & - \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\,dkdxdt \\ & + \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi) \sum_{i,j=1}^{N} A_{ij}(k+\xi)_{x_ix_j}\,\theta\,\phi\,dkdxdt \\ & - \iint_Q \theta(T_\ell(u)-\xi) \sum_{m=1}^{M} \Big(\operatorname{div}\beta_m(T_\ell(u)) - \sigma_m(T_\ell(u))\cdot\nabla\xi\Big)^2 \phi\,dxdt \\ & + \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,f\,\theta\,\phi\,dkdxdt. \end{aligned}$$

This is exactly (D3). Thus we complete the proof of the equivalence result. $\square$

3 Applications

We now apply the notion of renormalized dissipative solutions to two relaxation systems.

Example 1: Let a degenerate parabolic equation

$$u_t + \operatorname{div} \mathbf{F}(u) - \sum_{i,j=1}^{N} A_{ij}(u)_{x_i x_j} = f$$

be given. We assume that the initial datum $u_0(x)$ takes values in some interval and $\mathbf{F}(0) = \mathbf{0}$. Let $\omega_i > 0$ and suppose that $V_{n,i}$ satisfy the conditions $\sum_{i=1}^{N} V_{n,i}^{-1} \inf_{|u|\le n} F_i'(u) > -1$ and

$$\left(1 + \sum_{j=1}^{N} \omega_j\right) V_{n,i}^{-1} \sup_{|u|\le n} F_i'(u) < \omega_i \left(1 + \sum_{j=1}^{N} V_{n,j}^{-1} \inf_{|u|\le n} F_j'(u)\right)$$

for $n = 1, 2, \cdots$ and $i = 1, 2, \cdots, N$. Following [7, Lemma 4.1], we see that there exist a strictly increasing function $r_n : [-n, n] \to \mathbf{R}$ defined by

$$w = r_n(u) := \left(1 + \sum_{i=1}^{N} \omega_i\right)^{-1} \left(u + \sum_{i=1}^{N} V_{n,i}^{-1} F_i(u)\right)$$

and strictly decreasing functions $h_{n,i} : [r_n(-n), r_n(n)] \to \mathbf{R}$ with $h_{n,i}(0) = 0$ such that $w - \sum_{i=1}^{N} h_{n,i}(w) = u$ and $\omega_i V_{n,i} w + V_{n,i} h_{n,i}(w) = F_i(u)$ for $u \in [-n, n]$.

Now we consider a relaxation system for w^ε and $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \cdots, z_N^\varepsilon)$ with relaxation parameter $\varepsilon > 0$:

$$\text{(RS1)} \begin{cases} w_t^\varepsilon + \sum_{i=1}^{N} \omega_i V_{n,i} w_{x_i}^\varepsilon = \frac{1}{\varepsilon} \sum_{i=1}^{N} (h_{n,i}(w^\varepsilon) - z_i^\varepsilon) & \text{in } Q, \\ (z_i^\varepsilon)_t - V_{n,i} (z_i^\varepsilon)_{x_i} = \frac{1}{\varepsilon} (h_{n,i}(w^\varepsilon) - z_i^\varepsilon) & \text{in } Q, \quad i = 1, \cdots, N, \\ w^\varepsilon(0, \cdot) = w_0 & \text{in } \mathbf{R}^N, \\ z_i^\varepsilon(0, \cdot) = z_{i0} & \text{in } \mathbf{R}^N, \ i = 1, \cdots, N, \end{cases}$$

with $a \le w_0 \le b$ and $h_{n,i}(b) \le z_{i0} \le h_{n,i}(a)$. Here $a < 0$ and $b > 0$ are constants satisfying $-n \le a + \sum_{i=1}^{N} h_{n,i}(b) \le b + \sum_{i=1}^{N} h_{n,i}(a) \le n$.

Set $u^\varepsilon = w^\varepsilon - \sum_{i=1}^N z_i^\varepsilon$ and $u_0 = w_0 - \sum_{i=1}^N z_{i0} \in L^1(\mathbf{R}^N)$. Then, from [7], we see that $\overline{u}_n = \lim_{\varepsilon\downarrow 0} u^\varepsilon$ exists in $L^1(Q)$ and $\overline{u}_n$ is an entropy solution of (CP) with $A \equiv O$ and $f = 0$ satisfying $-n \le \overline{u}_n \le n$. Let $u_0 \in L^1(\mathbf{R}^N)$ and choose sequences of functions $\{w_{0,n}\}_{n\ge 1}$ and $\{z_{i0,n}\}_{n\ge 1}$ with the previous conditions for $i = 1, \cdots, N$. Moreover, we assume that $u_{0,n} = w_{0,n} - \sum_{i=1}^N z_{i0,n}$ converges to u_0 in $L^1(\mathbf{R}^N)$ as $n \to \infty$. Then we obtain that

Theorem 3.1. *The limit function $\overline{u} = \lim_{n\to\infty} \overline{u}_n$ in $L^1(Q)$ is a unique renormalized dissipative solution of* (CP) *with $A \equiv O$ and $f = 0$.*

Example 2: We next consider the following system for w^ε and z^ε with relaxation parameter $\varepsilon > 0$:

$$
\text{(RS2)} \qquad \begin{cases} w_t^\varepsilon + \operatorname{div} \mathbf{G}(w^\varepsilon) - \displaystyle\sum_{i,j=1}^N B_{ij}(w^\varepsilon)_{x_i x_j} = -\frac{1}{\varepsilon} w^\varepsilon z^\varepsilon & \text{in } Q, \\ z_t^\varepsilon = -\dfrac{1}{\varepsilon} w^\varepsilon z^\varepsilon & \text{in } Q, \\ w^\varepsilon(0,\cdot) = w_0 & \text{in } \mathbf{R}^N, \\ z^\varepsilon(0,\cdot) = z_0 & \text{in } \mathbf{R}^N, \end{cases}
$$

with $0 \le z_0 \le a$ and $0 \le w_0 \le g_n(a)$ almost everywhere in $\mathbf{R}^N$, where $g_n : [0, n] \to \mathbf{R}^+$ is a strictly increasing function and a is a nonnegative constant such that $-n \le -a \le g_n(a) \le n$ for $n = 1, 2, \cdots$. In addition, we assume on the data as follows:

(H1) $B_{ij} = B_{ji} \in C^2(\mathbf{R})$ and $B = (b_{ij}) \ge 0$ with $b_{ij}(\cdot) := B'_{ij}(\cdot)$ and $b_{ij}(0) = 0$ for $i, j = 1, \cdots, N$.

(H2) $\mathbf{G} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz-continuous flux with $\mathbf{G}(0) = \mathbf{0}$.

(H3) $w_0, z_0 \in \left(L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)\right)^+$ with $\int_{\mathbf{R}^N} |x|^2 w_0 \, dx < \infty$.

(H4) For $i, j, m = 1, \cdots, N$, $\sum_{m=1}^N \tau_{im}(s)\, \tau_{jm}(s) = b_{ij}(s)$, $\gamma'_{im}(s) = \tau_{im}(s)$ and $\gamma_m(w^\varepsilon) \in L^2(Q)^N$ with $\gamma_m(s) := (\gamma_{1m}(s), \cdots, \gamma_{Nm}(s))$ for $s \in \mathbf{R}$.

This system describes the evolution of a chemical or a biological species which is called a tracer in a porous medium. This tracer is supposed to be stuck on the surface of the solid frame. Belhadj et al. [1] studied this system and obtained the existence of entropy solutions with continuously differentiable flux $\mathbf{G}$. In case of locally Lipschitz-continuous $\mathbf{G}$, as in the analogous argument we obtain the following results:

Proposition 3.2. *Suppose that* (H1)-(H4). *Then, the problem* (RS2) *has a unique entropy solution* $(w^\varepsilon, z^\varepsilon) \in C((0,T); L^1(\mathbf{R}^N))^2$ *satisfying the following properties:*

(P1) $0 \le w^\varepsilon(t,x) \le \|w_0\|_{L^\infty(\mathbf{R}^N)}$ *and* $0 \le z^\varepsilon(t,x) \le \|z_0\|_{L^\infty(\mathbf{R}^N)}$ *almost every* $(t,x) \in Q$.

(P2) *If* $(w^\varepsilon, z^\varepsilon)$ *and* $(\overline{w}^\varepsilon, \overline{z}^\varepsilon)$ *are two solutions corresponding to the initial data* (w_0, z_0) *and* $(\overline{w}_0, \overline{z}_0)$, *respectively, then we have*

$$\begin{aligned} &\|w^\varepsilon(t) - \overline{w}^\varepsilon(t)\|_{L^1(\mathbf{R}^N)} + \|z^\varepsilon(t) - \overline{z}^\varepsilon(t)\|_{L^1(\mathbf{R}^N)} \\ \le\ & \|w_0 - \overline{w}_0\|_{L^1(\mathbf{R}^N)} + \|z_0 - \overline{z}_0\|_{L^1(\mathbf{R}^N)} \qquad \text{for all } \ t \ge 0. \end{aligned}$$

(P3) *Let* $(w^\varepsilon, z^\varepsilon)$ *and* $(\overline{w}^\varepsilon, \overline{z}^\varepsilon)$ *be two solutions corresponding to the initial data* (w_0, z_0) *and* $(\overline{w}_0, \overline{z}_0)$, *respectively. If* $w_0 \le \overline{w}_0$ *and* $z_0 \le \overline{z}_0$, *then we have* $w^\varepsilon(t) \le \overline{w}^\varepsilon(t)$ *and* $z^\varepsilon(t) \le \overline{z}^\varepsilon(t)$ *almost everywhere in* $\mathbf{R}^N$.

(P4) $\operatorname{div} \gamma_m(w^\varepsilon) \in L^2(Q)$ *for* $m = 1, \cdots, N$.

Proposition 3.3. *Suppose that* (H1)-(H4). *Let* $n \ge 1$, $u^\varepsilon = w^\varepsilon - z^\varepsilon$ *and* $u_0 = w_0 - z_0 \in L^1(\mathbf{R}^N)$. *Then,* $\overline{u}_n = \lim_{\varepsilon \downarrow 0} u^\varepsilon$ *exists in* $L^1(Q)$ *and* $\overline{u}_n \in [-n, n]$ *is a unique entropy solution of the following generalized Stefan problem:*

$$\text{(GSP)} \qquad \begin{cases} u_t + \operatorname{div} \mathbf{G}(u^+) - \displaystyle\sum_{i,j=1}^{N} B_{ij}(u^+)_{x_i x_j} = 0 & \text{in } \ Q, \\ \qquad\qquad\qquad\qquad\qquad u(0,\cdot) = u_0 & \text{in } \ \mathbf{R}^N. \end{cases}$$

From these propositions, we finally obtain that

Theorem 3.4. *Suppose that* (H1)-(H4). *Then,* $\overline{u} = \lim_{n\to\infty} \overline{u}_n$ *in* $L^1(Q)$ *is a unique renormalized dissipative solution of the generalized Stefan problem* (GSP).

Sketch of the proof. Fix $\ell \geq 1$ and assume that $u_0 \geq -\ell$. Define t_0 by

$$t_0 = \begin{cases} 0 & \text{if } u(t) \in [-\ell, \ell] \text{ for all } t \geq 0, \\ \inf\{t > 0\,;\, u(t) = \ell\} & \text{otherwise.} \end{cases}$$

We set $Q_1 := (0, t_0] \times \mathbf{R}^N$ and $Q_2 := (t_0, T) \times \mathbf{R}^N$, and take any test functions $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell, \ell)$. Since constant functions $w \equiv \ell$ and $z \equiv 0$ satisfy the relaxation system (RS2) with appropriate test functions, we see that

$$\begin{aligned} 0 \leq & \iint_{Q_1}\int_{\mathbf{R}} \theta(k)\,(\ell - k - \xi)^+ \phi'\,dkdxdt \\ & + \iint_{Q_1}\int_{\mathbf{R}} \theta(k)\,S_0^+(\ell - k - \xi) \\ & \qquad \times \left(-\operatorname{div}\mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi\,dkdxdt \\ & - \iint_{Q_1} \theta(\ell - \xi) \sum_{m=1}^{N} \big(-\tau_m(\ell)\cdot\nabla\xi\big)^2 \phi\,dkdxdt. \end{aligned}$$

On the other hand, if $t \in [t_0, T]$, then by the comparison property for (RS2) we see that $u(t) \in [-\ell, \ell]$. From Proposition 3.3 and the equivalence result [11, Theorem 1.1], we obtain that

$$\iint_{Q_2}\int_{\mathbf{R}} S_0^+(\overline{u}_n - k - \xi)\,h(\overline{u}_n, k)\,dkdxdt \geq 0,$$

where

$$h(\overline{u}_n, k) := (\overline{u}_n - k - \xi)\,\theta\,\phi' + \left(-\operatorname{div}\mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}(\overline{u}_n^+)_{x_i x_j} \right) \theta\,\phi.$$

Passing to the limit, we finally obtain that

$$
\begin{aligned}
0 \le & \iint_Q \int_{\mathbf{R}} \theta(k)\left(T_\ell(\overline{u}) - k - \xi\right)^+ \phi' \, dk dx dt \\
& + \iint_Q \int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(\overline{u}) - k - \xi) \\
& \qquad \times \left(-\operatorname{div} \mathbf{G}((k+\xi)^+) + \sum_{i,j=1}^{N} B_{ij}((k+\xi)^+)_{x_i x_j} \right) \phi \, dk dx dt \\
& - \iint_Q \theta(T_\ell(\overline{u}) - \xi) \sum_{m=1}^{N} \left(\operatorname{div} \gamma_m(T_\ell(\overline{u})^+) - \tau_m(T_\ell(\overline{u})^+)\cdot\nabla\xi\right)^2 \phi \, dx dt
\end{aligned}
$$

for any $\phi \in C_0^1(0,T)^+$, which implies that $\overline{u}$ is a renormalized dissipative solution of (GSP). Moreover, by the uniqueness theorem in [3], we conclude that $\overline{u}$ is a unique solution. □

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