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ディラック作用素のスペクトルについての一注意

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Abstract

We have much information on the spectrum of the Laplace-Beltrami operators on non-compact simply connected complete Riemannian manifolds. In particular the spectrum depends on the sign of the Gaussian curvature when the dimension of the manifolds is two. In this paper we are concerned with Dirac operators on the manifolds and show that the spectrum does not depend on the sign of the Gaussian curvature.

1 Introduction

Let M be a simply connected complete Riemannian manifold with metric

$$ds^2 = g_{ij}dx^i dx^j.$$

As usual we use the following standard notation

$$(g^{ij}) = (g_{ij})^{-1}, \quad g = \det(g_{ij}).$$

Δ_M denotes the Laplace-Beltrami operator

$$\Delta_M = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}$$

We assume that $\dim M = 2$ and we can endow M with a global system of geodesic polar coordinates where the metric is given by

$$ds^2 = dr^2 + h^2(r,\theta)d\theta^2. \quad r \in [0,\infty), \ \theta \in [0,2\pi).$$

Thus, it is seen that

$$\Delta_M = \frac{1}{h}\frac{\partial}{\partial r}h\frac{\partial}{\partial r} + \frac{1}{h}\frac{\partial}{\partial \theta}\frac{1}{h}\frac{\partial}{\partial \theta}.$$

Note that when $M = H^2 = \mathbf{R} \times (0,\infty)$,

$$ds^2 = y^{-2}(dx^2 + dy^2), \quad \Delta_{H^2} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

and $h(r,\theta) = \sinh r$.

In what follows we consider the case that h is independent of θ. So Gaussian curvature K is given by

$$K(r) = -\frac{h''}{h}.$$

It is known that

Theorem 1.1 *([?]) If $K(r) \geq 0$, $\forall r \geq 0$, then $\sigma(-\Delta_M) = [0,\infty)$ and $-\Delta_M$ has no eigenvalues. If $K(r) \leq 0$, $\forall r \geq 0$ and $\lim_{r\to\infty} K(r) = -\mu^2 < 0$, then $\sigma(-\Delta_M) = [\frac{\mu^2}{4},\infty)$ and $-\Delta_M$ has no eigenvalues.*

We give two remarks.

(i) If $K(r) < 0$ near ∞ and K takes a positive value, then there exists eigenvalues, in general. For example, it may occur that $h(r) \sim \exp(-r)$ as $r \to \infty$.

(ii) If $\lim_{r\to\infty} K(r) = -\infty$, then $\sigma(-\Delta_M) = \sigma_d(-\Delta_M)$.

Our purpose is to investigate the spectrum of the Dirac operator on M.

2 Main results

Let H_D be the Dirac operator on M. From Chernoff [?], it follows that H_D is essentially self-adjoint on $C_0^\infty(M)^2$. H_D has the following representation in the geodesic polar coordinates given in the previous section..

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H_D = \sigma^1 D_r + \sigma^2 \frac{1}{h(r)}(D_\theta - \sigma^3 \frac{h'}{2}) + m\sigma^3$$

It holds that

$$\sigma^i \sigma^j = \sqrt{-1}\sigma^k, \quad (i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2).$$

Theorem 2.1 *Suppose that the following condition holds.*
(A − 1) *$K(r) \geq 0$ on $[0,\infty)$ Then, $\sigma(H_D) = \mathbf{R}$ and there are no eigenvalues.*

Theorem 2.2 *Suppose that the following condition holds.*
(A − 2) $K(r) \leq 0$ *on* $[0, \infty)$ *and there exists* $R > 0$ *and* $\alpha > 0$ *such that*

$$K(r) < -\alpha, \quad \forall r > R.$$

Then, $\sigma(H_D) = \mathbf{R}$ *and there are no eigenvalues.*

Remark

$$H_D^2 = -\frac{1}{h}\partial_r h \partial_r + \frac{1}{h^2}(D_\theta - \frac{h'}{2}\sigma^3)^2 + \frac{K(r)}{2}. \tag{2.1}$$

3 Proof of Theorem ??

Lemma 3.1 *If* $K(r) \geq 0$ *on* $[0, \infty)$*, then* $h' \geq 0$ *on* $[0, \infty)$*. Hence,* $h(r) \leq r$ *for all* $r \geq 0$*.*

Proof: Suppose that there were $r_0 > 0$ such that $h'(r_0) < 0$. From that $h'' \leq 0$, it follows that

$$h'(r) \leq h'(r_0) = -\alpha < 0 \; \forall r \geq r_0.$$

It holds that if $r \geq r_0$ then

$$h(r) - h(r_0) = h'(\xi)(r - r_0) \leq -\alpha(r - r_0).$$

Therefore, there exists r_1 such that

$$h(r_1) = 0.$$

This is a contradiction.

The final part of our statement follows from that

$$1 = h'(0) \geq h'(r) \geq 0$$

and that $h(0) = 0$. Q.E.D.

Now we are going to prove Theorem ??. First of all we shall show that 0 is not any eigenvalue. We shall use Fourier series expansion. Let

$$f(r, \theta) = \sum_{k=-\infty}^{\infty} f_k(r) e^{ik\theta}$$

and

$$H_{D,k} = \sigma^1(D_r + \frac{h'}{2hi}) + \sigma^2\frac{k}{h}.$$

Then it is seen that

$$H_D f = \sum_{k=-\infty}^{\infty} e^{ik\theta} H_{D,k} f_k(r).$$

Set $\tilde{f}_k = f_k\sqrt{h}$. Then it is easily seen that

$$H_{D,k} f_k = \lambda f_k$$

is equivalent to

$$\sigma^1 H_{D,k} f_k = \{D_r + i\sigma^3\frac{k}{h}\}\tilde{f}_k = \lambda\sigma^1\tilde{f}_k.$$

Let

$$\tilde{H}_{D,k} = D_r + i\sigma^3\frac{k}{h}.$$

Suppose that $H_D f = 0$ for $f \in [L^2(M)]^2$, which implies

$$\tilde{H}_{D,k}\tilde{f}_k = 0, \quad \tilde{f}_k \in [L^2([0,\infty) \times [0,2\pi))]^2.$$

Namely

$$\begin{pmatrix} \frac{d}{dr} - \frac{k}{h} & 0 \\ 0 & \frac{d}{dr} + \frac{k}{h} \end{pmatrix} \tilde{f}_k = 0,$$

Define $\tilde{f}_k =^t (\tilde{f}_k^+, \tilde{f}_k^-)$ and suppose that $\tilde{f}_k^+ \neq 0$. Then

$$\tilde{f}_k^+ = C_{k,+} \exp\left\{\int_1^r \frac{k}{h(s)} ds\right\} \in L^2([0,\infty)).$$

If $k \geq 0$ and $r \geq 1$, then

$$\exp\left[\int_1^r \frac{k}{h(s)} ds\right] \geq r^k.$$

Therefore, $C_{k.+} = 0$ for $k \geq 0$.

If $k < 0$ and $r \leq 1$, then

$$\exp\left[\int_1^r \frac{k}{h(s)}ds\right] \geq r^k.$$

Hence, we have $C_{k.+} = 0$ for $k < 0$. Similarly, we can verify that $C_{k,-} = 0$ for any k. Therefore we can conclude that $f_k = 0$ for all $k \in \mathbf{Z}$, so that 0 is not any eigenvalue.

We remark that $\sigma^1 H_{D,k}$ is in the limit point case at infinity for each $k \in \mathbf{Z}$ and that it is in the limit point case at the origin if and only if $k \in \mathbf{Z}\backslash\{0\}$. In particular $\sigma^1 H_{D,k}$ with $k \neq 0$ is essentially self-adjoint on $C_0^\infty((0,\infty))^2$.

For $\lambda \in \mathbf{R}$, let

$$H_D f = \lambda f, \quad f \in [L^2(M)]^2.$$

Since H_D is elliptic, it follows that f is smooth on M. Let $M_{s,t} = [s,t] \times [0,2\pi)$ with measure $hdrd\theta$ and $\Gamma_r = \{r\} \times [0,2\pi)$ with measure $hd\theta$. We note that if f and g belong to $C_0^\infty(M_{s,t})$, then

$$\langle(\frac{\partial}{\partial r} + \frac{h'}{2h})f, g\rangle_{M_{s,t}} = -\langle f, (\frac{\partial}{\partial r} + \frac{h'}{2h})g\rangle_{M_{s,t}}.$$

In view of

$$H_D = \sigma^1(D_r + \frac{h'}{2ih}) + \sigma^2\frac{1}{h}D_\theta,$$

we see that

$$\begin{aligned} 2\mathrm{Re}\langle h(\frac{\partial}{\partial r} + \frac{h'}{2h})f,\ H_D f\rangle_{M_{s,t}} &= 2\mathrm{Re}\langle h(\frac{\partial}{\partial r} + \frac{h'}{2h})f,\ \sigma^2\frac{1}{h}D_\theta f)\rangle_{M_{s,t}} \\ &= \mathrm{Re}\left[\int h(f, \sigma^2\frac{1}{h}D_\theta f)hd\theta\right]_{r=s}^t. \end{aligned}$$

On the other hand

$$2\mathrm{Re}\langle h(\frac{\partial f}{\partial r} + \frac{h'}{2h}f),\ \lambda f\rangle_{M_{s,t}} = -\langle h' f, \lambda f\rangle_{M_{s,t}} + \mathrm{Re}\left[\int h^2\lambda\|f\|^2 d\theta\right]_{r=s}^t.$$

Let Π be the projection onto the subspace spanned by $e^{ik\theta}$, $k \in \mathbf{Z}\backslash\{0\}$. Thus,

$$\begin{aligned}\lambda\langle h'f, f\rangle &= \mathrm{Re}\left[\int \lambda h|f|^2 d\Gamma_r\right]_{r=s}^{t} - \mathrm{Re}\left[\int (f, \sigma^2 D_\theta f) d\Gamma_r\right]_{r=s}^{t} \\ &= \mathrm{Re}\left[\int \lambda h|f|^2 d\Gamma_r\right]_{r=s}^{t} - \mathrm{Re}\left[\int (f, \sigma^2 (D_\theta - \frac{h'}{2}\sigma^3)\Pi f) d\Gamma_r\right]_{r=s}^{t},\end{aligned} \tag{3.1}$$

where $d\Gamma = hd\theta$. From (??) it follows that if $f \in C_0^\infty(M\backslash\{0\})$, then

$$\|\frac{1}{h}(D_\theta - \frac{h'}{2}\sigma^3)f\|_M^2 \leq (H_D^2 f, f)_M = (H_D f, H_D f)_M.$$

Then if $H_D f \in L^2(M)$, then it holds that

$$\frac{1}{h}(D_\theta - \frac{h'}{2}\sigma^3)\Pi f \in L^2(M).$$

Since $f \in [L^2(M)]^2$, there exists a sequence $\{t_j\}$ such that

$$\lim_{j\to\infty} t_j = \infty, \quad \lim_{j\to\infty} \int_{r=t_j} h|f|^2 d\Gamma_{t_j} \leq \lim_{j\to\infty} \int_{r=t_j} r|f|^2 d\Gamma_{t_j} = 0$$

and

$$\lim_{j\to\infty} \int_{r=t_j} \frac{1}{h}|(D_\theta - \frac{h'}{2}\sigma^3)\Pi f|^2 d\Gamma_{t_j} \leq \lim_{j\to\infty} \int_{r=t_j} \frac{r}{h^2}|(D_\theta - \frac{h'}{2}\sigma^3)\Pi f|^2 d\Gamma_{t_j} = 0.$$

Hence

$$\begin{aligned}&\lim_{j\to\infty} \left|\int_{r=t_j} (f, \sigma^2 (D_\theta - \frac{h'}{2}\sigma^3)\Pi f) d\Gamma_r\right| \\ &\qquad \leq \frac{1}{2}\lim_{j\to\infty} \int_{r=t_j} \{h|f|^2 + \frac{1}{h}|\sigma^2(D_\theta - \frac{h'}{2}\sigma^3)\Pi f|^2\} d\Gamma_r = 0.\end{aligned}$$

Similarly there exists a sequence $\{s_j\}$ such that

$$\lim_{j\to\infty} s_j = 0, \quad \lim_{j\to\infty} \int_{r=s_j} h|f|^2 d\Gamma_r = 0$$

and

$$\lim_{j\to\infty} \int_{r=s_j} (f, \sigma^2 (D_\theta - \frac{h'}{2}\sigma^3)\Pi f) d\Gamma_r = 0.$$

Taking the limit of the both sides of (??) with $t = t_j$ and s_j, we can conclude that if $\lambda \neq 0$,

$$\langle h' f, f \rangle = 0.$$

This implies that $f = 0$ on a non-empty open interval I. By virtue of the unique continuation property, we see that $u = 0$ on M. This means that every non-zero real number is not any eigenvalue.

Finally we shall show that the spectrum of H_D coincides $\mathbf{R}$. To prove this, we consider the case when $k = 0$.

$$\sigma^1(\tilde{H}_{D,0} - \lambda)\tilde{f}_0 = \begin{pmatrix} \frac{d}{dr} & -i\lambda \\ -i\lambda & \frac{d}{dr} \end{pmatrix} \tilde{f}_0 = 0, \quad \tilde{f}_0 =^t (f_+, f_-)$$

gives

$$\frac{d^2}{dr^2} f_\pm + \lambda^2 f_\pm = 0.$$

Therefore, it it not difficult to find a sequence $\{u_j\}$ such that

$$\|u_j\|_{L^2(M))^2} = 1, \quad \lim_{j \to \infty} \|(H_{D,0} - \lambda)u_j\|_{L^2(M))^2} = 0.$$

Hence we have $\sigma(H_D) = \mathbf{R}$.

4 Proof of Theorem ??

Lemma 4.1 *Suppose that $K \leq 0$ on $[0, \infty)$. Then $h'(r) \geq 1$ and $h(r) \geq r$ for all $r \in [0, \infty)$.*

Lemma 4.2 *Suppose that $K \leq 0$ on $[0, \infty)$ and there exists $\mu > 0$ such that $\lim_{r \to \infty} K(r) = -\mu^2$. Then,*

$$\lim_{r \to \infty} \frac{h'}{h} = \mu, \quad \lim_{r \to \infty} \left(\frac{h'}{h} \right)' = 0.$$

Lemma 4.3 *Let $\mu > 0$. If $f \in C^1$ satisfies*

$$\frac{df}{dr} = \mu^2 - f^2(r), \; f(s) \geq 0$$

for all $r \geq s$, *then*

$$f(r) = \mu \tanh(\mu(r-s)+r_0), \quad \text{if } f(s) < \mu,$$

$$f(r) = \mu \coth(\mu(r-s)+r_0), \quad \text{if } f(s) > \mu,$$

$$f(r) = \mu, \quad \text{if } f(s) = \mu.$$

Lemma 4.4 *Suppose that* $K \leq 0$ *on* $[0,\infty)$ *and there exists* $\mu > 0$ *such that* $K(r) < -\mu^2$ *for* $r > R$. *Then,*

$$\frac{h'}{h} \geq \tanh(1)\mu, \quad \text{for } r > \frac{1}{\mu} + R.$$

In particular there exist $M > 0$ *and* δ *such that*

$$h(r) \geq M \exp(\delta \mu r), \; r > \frac{1}{\mu} + R.$$

Proof: Let $v(r) = h'(r)/h(r)$ and consider $w(r) = v(r) - f(r)$ with $v(R) = f(R)$. Then,

$$w'(r) = K(r) - \mu^2 - v^2(r) + f^2(r).$$

Thus $w(R) = 0$ and $w'(R) = -K(R) - \mu^2 > 0$. We are going to prove that $w(r) > 0$ for all $r > R$. Suppose that there were $r' > R$ such that $w(r') = 0$. Let $r_0 (> R)$ be the smallest one of such r'. Then, we see that $w'(r_0) = -K(r_0) - \mu^2 > 0$. Then $w(r)$ must take both a negative value and a positive value in (R, r_0). This is a contradiction to the choice of r_0. Therefore, we conclude that $w(r) > 0$ for all $r \geq R$. $v(R) = f(R)$ means that there exists $r_0 > 0$ such that

$$\frac{h'}{h} > f(r) \geq \mu \tanh(\mu(r-R)+r_0)$$

for all $r \geq R$. Hence,

$$\frac{h'}{h} > \mu \tanh(1+r_0) > \mu \tanh(1).$$

for all $r \geq R + \frac{1}{\mu}$. Q.E.D.

Now we are going to prove Theorem ??. Recall

$$\tilde{H}_{D,k} = \sigma^1 D_r + \frac{k}{h}\sigma^2$$

and suppose that for $\lambda \in \mathbf{R}\backslash\{0\}$ $\tilde{u}_k \in L^2((0,\infty))$ satisfies $\tilde{H}_{D,k}\tilde{u}_k = \lambda\tilde{u}_k$.

Let $\chi \in C_0^\infty(\mathbf{R})$ be a nonnegative cut-off function supported in $[s-1, t+1]$ such that

$$\chi(r) = 1, \quad r \in [s,t]$$

and

$$\sup|\chi'(r)| \leq 1.$$

In addition, $\varphi \in C^3(\mathbf{R}_+;\mathbf{R})$ satisfies $\varphi' > 0$. The vector function $w_k = \chi(r)e^{\varphi}\tilde{u}_k$ satisfies

$$\{\frac{1}{i}\sigma^1(\partial_r - \frac{k}{h}\sigma^3 - \varphi') - \lambda\}w_k = f_\chi, \tag{4.1}$$

where

$$f_\chi = \frac{1}{i}\sigma^1\chi' e^{\varphi}\tilde{u}_k.$$

In view of $1/h(r) \leq M\exp(-\delta\mu r)$, we can use the standard virial theorem to estimate

$$0 = 2\mathrm{Re}\int_s^t \langle r\partial_r w_k,\ (\tilde{H}_{D,k} - \lambda)w_k\rangle dr.$$

Integration by parts implies that

Lemma 4.5

$$\int_{s-1}^{t+1}\left[\langle\partial_r(r\lambda w_k, w_k)\rangle + 2\mathrm{Re}\langle r\{i\sigma^1(\varphi' + \frac{k}{h}\sigma^3)\}w_k, \partial_r w_k\rangle\right]dr = \int_{s-1}^{t+1}\langle rf_\chi, \partial_r w_k\rangle dr.$$

Lemma 4.6 *Let* $\lambda < 0$.

$$\begin{aligned}
-I_1 &= -2\mathrm{Re}\int_{s-1}^{t+1}\langle ir\sigma^1\varphi' w_k, \partial_r w_k\rangle dr \\
&\geq \int_{s-1}^{t+1}\{\frac{r\varphi'}{-\lambda} - o(1)\}\|\partial_r w_k\|^2 dr + \frac{-1}{\lambda}\int_{s-1}^{t+1}(r(\varphi')^2)'\|w_k\|^2 dr \\
&\quad - \int_{s-1}^{t+1}\frac{r\varphi'}{(-\lambda)}\|i\sigma^1 f_\chi\|^2 dr - \int_{s-1}^{t+1} o(1)\varphi'\|w_k\|^2 dr
\end{aligned} \tag{4.2}$$

Proof: Multiplying (??) by $i\sigma^1$ and squaring it, we have

$$
\begin{aligned}
&-2\mathrm{Re}((-\lambda)i\sigma^1 w_k, \partial_r w_k) \\
&= \|\partial_r w_k\|^2 + \|\partial_r w_k - i\sigma^1 f_\chi\|^2 - 2\mathrm{Re}\langle \frac{k}{h}\sigma^3 w_k, \partial_r w_k\rangle - 2\mathrm{Re}\langle \varphi' w_k, \partial_r w_k\rangle.
\end{aligned}
$$

$$
\begin{aligned}
-I_1 = \int_{s-1}^{t+1} \frac{r\varphi'}{(-\lambda)}\Big[&\{\|\partial_r w_k\|^2 + \|\partial_r w_k - i\sigma^1 f_\chi\|^2 - \|i\sigma^1 f_\chi\|^2\} \\
&- 2\mathrm{Re}\langle \varphi' w_k, \partial_r w_k\rangle\Big] dr - 2\mathrm{Re}\int_{s-1}^{t+1} \langle r\varphi' \frac{k}{h}\sigma^3 w_k, \partial_r w_k\rangle
\end{aligned}
\tag{4.3}
$$

$$
-2\mathrm{Re}\int_{s-1}^{t+1} \frac{r\varphi'}{(-\lambda)}\langle \varphi' w_k, \partial_r w_k\rangle dr = \frac{-1}{\lambda}\int_{s-1}^{t+1} \langle (r(\varphi')^2)' w_k, w_k\rangle dr.
$$

Q.E.D.

Proposition 4.7 *Let $\lambda < 0$. Let*

$$
k_\varphi = \varphi'(\varphi' + 2r\varphi'\varphi'') - o(1)\varphi'.
$$

Then

$$
\begin{aligned}
-\lambda\int_{s-1}^{t+1} \|w_k\|^2 dr + \int_{s-1}^{t+1} \frac{k_\varphi}{-\lambda}\|w_k\|^2 dr + \int_{s-1}^{t+1} \{\frac{r\varphi'}{-\lambda} - o(1)\}\|\partial_r w_k\|^2 dr \\
\leq \int_{s-1}^{t+1} \frac{r\varphi'}{(-\lambda)}\|i\sigma^1 f_\chi\|^2 dr.
\end{aligned}
\tag{4.4}
$$

Once these weighted L^2 estimates are verified, we can repeat the same argument as in [?] to prove that there exists $R > 0$ such that $\tilde{u}_k = 0$ in (R, ∞). Applying unique continuation property, we conclude that $u = 0$ in M, so that every non-zero real number is not any eigenvalue.

Now we shall show that 0 is not any eigenvalue. $\tilde{H}_{D,k}\tilde{u}_k = 0$ means that

$$
\begin{pmatrix} \frac{d}{dr} - \frac{k}{h} & 0 \\ 0 & \frac{d}{dr} + \frac{k}{h} \end{pmatrix}\begin{pmatrix} \tilde{u}_k^+ \\ \tilde{u}_k^- \end{pmatrix} = 0.
$$

It holds that

$$
\tilde{u}_k^\pm = C_\pm \exp\{\pm\int_R^r \frac{k}{h(r)} dr\}.
$$

Under the our assumption we see that $1/h(r)$ is integrable over $[R,\infty)$. This implies that $C_{\pm}$ must be zero in order that $\tilde{u}_k^{\pm}$ belong to $L^2([R,\infty))$. Therefore, 0 is not any eigenvalue of H_D. Q.E.D.

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