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Green functions and heat kernels of second order ordinary differential operators with discontinuous complex coefficients

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Abstract

We consider the operator $Bu \equiv (r(x))^{-1}Au$ where

$$(Au)(x) \equiv -\frac{d}{dx}\left(a(x)\frac{du}{dx}+b_1(x)u\right)+b_2(x)\frac{du}{dx}+c(x)u, (-\infty < x < \infty$$

with discontinuous bounded complex-valued coefficients. Under some additional condition, we estimate the kernel function (Green functions) of $(B-\lambda)^{-1}$ and the kernel for e^{-tB}.

1 Basic Assumptions and Notations

Consider an ordinary differential operator $A \in L^2(\mathbf{R})$:

$$(Au)(x) \equiv -(a(x)u'+b_1(x)u)'+b_2(x)u'+c(x) \quad (1)$$

$$\equiv -\frac{d}{dx}\left(a(x)\frac{du}{dx}+b_1(x)u\right)+b_2(x)\frac{du}{dx}+c(x)u \qquad (-\infty < x < \infty) \quad (2)$$

with

$$\mathrm{Dom}(A)=\{u\in H^1(\mathbf{R}); a(x)du/dx+b(x)u\in H^1(\mathbf{R})\}$$

Here

$$a(\bullet), b_1(\bullet), b_2(\bullet), c(\bullet) \in L^\infty(\mathbf{R})$$

are complex-valued and may be discontinuous and we assume there exist two positive constants $\theta_a \in (0,\pi/2)$ and $a_0>0$ such that

$$|\arg(a(x))| \le \theta_a, \quad \Re(a(x)) \ge a_0$$

We also consider another operator B with the same domain:

$$(Bu)(x) \equiv \frac{(Au)(x)}{r(x)}, \mathrm{Dom}(B)=\mathrm{Dom}(A)$$

where $r(x) \in L^\infty(\mathbf{R})$ is a scale function for which there exist also two positive constants $\theta_r \in (0,\pi/2)$ and $r_0>0$ such that

$$|\arg(r(x))| \leqq \theta_r, \quad \Re(r(x)) \geqq r_0$$

We will further assume later that$0 < \theta_a + \theta_r < \pi/2$.

Our problem is the solvability of $Bu - \lambda u = f \in L^2(\mathbf{R})$ and the representation of the solution by a Green function. Equivalently, we have only to consider the solvability of

$$Au - \lambda r(x)u = r(x)f(x) \in L^2(\mathbf{R}).$$

We also consider the kernel of the analytic semigroup e^{-tB}.

We sometimes omit $(\mathbf{R})$ of $L^(\mathbf{R}), L^\infty(\mathbf{R}), H^1(\mathbf{R}), \cdots$ for simplicity. And we generally denote constants by $k, k_0, k_1, \cdots$.

2 Functions with compact support in Dom(A)

Just as the domain H^2 of the opreator $-d^2/dx^2$ is itself a Hilbert space, the domain Dom(A) of A can be regarded as the Banach space (actually a Hilbert space).

Definition For $u \in \mathrm{Dom}(A)$, we define

$$\|u\|_{\mathrm{Dom}(A)} \equiv \left\{(\|u\|_{H^1})^2 + (\|a(x)u' + b_1(x)u\|_{H^1})^2\right\}^{1/2}$$

Theorem 1 *The domain* Dom(A) *of* A *is itself a Banach space with norm* $\| \bullet \|_{\mathrm{Dom}(A)}$.

Proof. We have only to consider the completeness. Let $\{u_n\}$ be a Cauchy sequence with $\| \bullet \|_{\mathrm{Dom}(A)}$. Then u_n and $a(x)u_n' + b_1(x)u_n$ are both Cauchy sequences in H^1. Hence there exist $u, v \in H^1$ such that

$$u_n \to u, \quad a(x)u_n' + b_1(x)u_n \to v \text{ in } H^1.$$

The first one means $a(x)u_n' + b_1(x)u_n \to a(x)u' + b_1(x)u$ in L^2. Therefore we have $a(x)u' + b_1(x)u = v \in H^1$ and $u \in \mathrm{Dom}(A)$. Q.E.D.

We will prove $C_0(\mathbf{R}) \cap \mathrm{Dom}(A)$ is dense in Dom(A) with norm $\| \bullet \|_{\mathrm{Dom}(A)}$. We first define cut-off functions in the next three lemmas.

Lemma 2 *Fix* $\rho(x) \in C_0^\infty$ *such that*

$$\rho(x) = \begin{cases} > 0 & (0 < x < 1) \\ = 0 & (x \le 0, x \ge 1). \end{cases}$$

Then

$$c_n = \int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)} dx \ne 0, \quad n = 0, \pm 1, \pm 2, \cdots$$

Moreover there exists a constant $k > 1$ *such that*

$$k^{-1} \le |c_n| \le k \quad (n = 0, \pm 1, \pm 2, \cdots)$$

Proof. The assumption on $a(x)$ implies

$$(k_0)^{-1} \le \Re\frac{1}{a(x)} \le k_0 \quad (-\infty < x < \infty)$$

with some constant $k_0 > 1$. Taking account of $\rho(x) \ge 0$, we have

$$(k_0)^{-1}\int_{-\infty}^{\infty} \rho(x-n)dx \le \Re\int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)}dx \le k_0\int_{-\infty}^{\infty} \rho(x-n)dx,$$

that is,

$$(k_0)^{-1}\int_{-\infty}^{\infty} \rho(x)dx \le \Re c_n \le k_0\int_{-\infty}^{\infty} \rho(x)dx.$$

or

$$(k_0)^{-1}k_1 \le \Re c_n \le k_0 k_1 \tag{3}$$

if we put $k_1 = \int_{-\infty}^{\infty} \rho(x)dx$. On the other hand, $\rho(x) \ge 0$ and the convexity of the set $\{z \in \mathbf{C}; |\arg Z| \le \theta_a < \pi/2\}$ implies

$$\left|\arg\int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)}dx\right| \le \sup_x |\arg\frac{1}{a(x)}| \le \theta_a < \pi/2$$

that is,

$$|\arg c_n| \le \theta_a < \pi/2 \tag{4}$$

From these, we have the claim of the present lemma. Q.E.D.

Now the next two lemmas are clear.

Lemma 3 *Let $\rho(x)$ and c_n $(n = 0, \pm 1, \cdots$ be the same as in the previous lemma. Then*

$$\phi_n(x) \equiv c_n^{-1}\int_{-\infty}^{x} \frac{\rho(y-n)}{a(y)}dy,$$

$$\psi_n(x) \equiv c_n^{-1}\int_{x}^{\infty} \frac{\rho(y-n)}{a(y)}dy$$

satisfy

$$\phi_n(x) = \begin{cases} 0 & (x \le n) \\ 1 & (x \ge n+1), \end{cases}$$

$$\psi_n(x) = \begin{cases} 1 & (x \le n) \\ 0 & (x \ge n+1). \end{cases}$$

In addition, the functions

$$a(x)\phi_n' = (c_n)^{-1}\rho(x-n), a(x)\psi_n' = -(c_n)^{-1}\rho(x-n) \quad (n = 0, \pm 1, \cdots)$$

belong to $C_0^\infty(\mathbf{R})$ and form a bounded set in $B^1(\mathbf{R})$.

Lemma 4 *Let $\phi_m(x)$ and $\psi_n(x)$ be the same as in the previous lemma. The functions*

$$\phi_{mn}(x) \equiv \phi_m(x)\psi_n(x)$$

with the integer parameter $n \geq m+1$ satisfies

$$\phi_{mn}(x) = \begin{cases} 1 & (m+1 \leq x \leq n) \\ 0 & (x \leq m, \quad x \geq n+1) \end{cases}$$

In addition, two families of support compact functions

$$\{\phi_{mn}(x),\}, \{a(x))\phi'_{mn}\}$$

are bounded subsets in $W^{1,\infty}(\mathbf{R})$ and $B^1(\mathbf{R})$, respectively.

Using Lemma 4, we can prove the next theorem.

Theorem 5 *The set $C_0(\mathbf{R}) \cap \mathrm{Dom}(A)$ is dense in $\mathrm{Dom}(A)$ with norm $\| \bullet \|_{\mathrm{Dom}(A)}$.*

Proof. Fix $u \in \mathrm{Dom}(A)$ arbitrariliy. Set

$$u_{mn} \equiv \phi_{mn}(x)u(x) \in C_0(\mathbf{R}) \cap L^2(\mathbf{R})$$

where $\phi_{mn}(x)$ is the function in the previous lemma. Recalling $\phi_{mn}(x) \in W^{1,\infty}$ and $\{a(x)\phi'_{mn}\}(x) \in B^1$, we know

$$\begin{aligned} u'_{mn} &= \phi'_{mn}(x)u + \phi_{mn}(x)u' \\ a(x)u'_{mn} + b_1(x)u_{mn} &= a(x)\phi'_{mn}(x)u + \phi_{mn}(x)\{a(x)u' + b_1(x)u\} \\ \{a(x)u'_{mn} + b_1(x)u_{mn}\}' &= (a(x)\phi'_{mn}(x))'u + a(x)\phi'_{mn}(x)u'(x) \\ &\quad + \phi'_{mn}(x)\{a(x)u' + b_1(x)u\} \\ &\quad + \phi_{mn}(x)\{a(x)u' + b_1(x)u\}' \end{aligned}$$

are all in $L^2(\mathbf{R})$, i.e., $u_{mn} \in \mathrm{Dom}(A)$. The previous lemma states $\{\phi_{mn}\}$ and $\{a(x))\phi'_{mn}\}$ are bounded subsets in $W^{1,\infty}(\mathbf{R})$ and $B^1(\mathbf{R})$, respectively. Note also

$$\mathrm{supp}\phi'_{mn} \subset [m, m+1] \cup [n, n+1]$$

Therefore $\phi_{mn}(x) \to 1$ $(m \to -\infty, n \to \infty)$ implies

$$\begin{aligned} u_{mn} &\to u(x) \\ u'_{mn} &\to u'(x) \\ a(x)u'_{mn} + b_1(x)u_{mn} &\to \{a(x)u' + b_1(x)u\} \\ \{a(x)u'_{mn} + b_1(x)u_{mn}\}' &\to \{a(x)u' + b_1(x)u\}' \end{aligned}$$

all in $L^2(\mathbf{R})$. This means $u_{mn} \in C_0(\mathbf{R}) \cup \mathrm{Dom}(A)$ converges to u in the sense of the norm $\| \bullet \|_{\mathrm{Dom}(A)}$. Q.E.D.

In the later sections, we consider the perturbation A^μ of the operator A which is formally defined

$$(A^\mu)(x) \equiv e^{-\mu\Phi(x)} A(e^{\mu\Phi(x)} u(x))$$

where

$$\Phi(x) \equiv \int_0^x \frac{dy}{a(y)}.$$

The next theorem partially guarantees the appropriateness of the definition of A^μ.

Theorem 6 *Let $\mu \in \mathbf{C}$ be an arbitrarily fixed constant and*

$$\Phi(x) \equiv \int_0^x \frac{dy}{a(y)}.$$

Suppose $u \in \mathrm{Dom}(A) \cap C_0(\mathbf{R})$. Then

$$v(x) \equiv e^{-\mu\Phi(x)} u(x) \in \mathrm{Dom}(A).$$

Proof . Since $\Phi(x)$ is absolutely continuous and locally bounded,

$$\begin{aligned} v(x) &= e^{\mu\Phi(x)} u(x) \in L^2 \\ v'(x) &= \frac{\mu}{a(x)} e^{\mu\Phi(x)} u(x) + e^{\mu\Phi(x)} u'(x) \in L^2 \end{aligned}$$

as $u \in \mathrm{Dom}(A) \cap C_0(\mathbf{R}) \subset C_0(\mathbf{R}) \cap H^1(\mathbf{R})$. Moreover,

$$a(x)v' + b_1(x)v = \mu e^{\mu\Phi(x)} u(x) + e^{\mu\Phi(x)} u(x)(a(x)u' + b_1(x)u) \in H^1(\mathbf{R})$$

since$u \in \mathrm{Dom}(A) \cap C_0(\mathbf{R})$ implies

$$a(x)u' + b_1(x)u \in H^1(\mathbf{R}) \cap C_0(\mathbf{R})$$

by definition.

3 Sesquilinear form associated with A

Theorem 7 *The sesquilinear form $\alpha[u, v]$ defined as*

$$\alpha[u, v] = \int_{-\infty}^{\infty} \{(a(x)u' + b_1(x)u)\bar{v}' + b_2(x)u'\bar{v} + c(x)u\bar{v}\}\, dx,$$

$$\mathrm{Dom}(\alpha) = H^1(\mathbf{R})$$

is a closed sectorial form in $L^2(\mathbf{R})$. Moreover, A is the sectorial operator representing the sectorial form α, i.e.,

$$\alpha[u, v] = (Au, v)$$

for any $u \in \mathrm{Dom}(A)$ and any $v \in H^1$.

Proof. First, we prove the sectoriality. We begin with the first part of $\alpha[u,u]$:

$$\int_{-\infty}^{\infty} a(x)|u'|^2 dx = \gamma(u)\|u'\|_{L^2}^2$$

Here $\gamma(u)$ is in the closed convex hull of

$$\{a(x); x \in \mathbf{R}\} \subset \{|\arg(z)| \le \theta_a\} \cap \{\Re z \ge a_0\} \cap \{|z| \le |a(\bullet)|_{L^\infty}\}.$$

On the other hand,

$$\left| \int_{-\infty}^{\infty} \{b_1(x)u\bar{u}' + b_2(x)u'\bar{u} + c(x)u\bar{u}\}\, dx \right| \le \epsilon\|u'\|^2 + (k/\epsilon)\|u\|^2$$

with two constant $k > 0$ and $0 < \epsilon < 1$ where $0 < \epsilon < 1$ can be arbitrarily chosen. So, with appropriately chosen constant $K > 0$,

$$\alpha[u,u] + K(u,u)$$

takes values in the sector $\{|\arg z| \le \theta_a < \pi/2\}$. In other words, $\alpha[u,v]$ is a sectorial form. It is also shown that

$$|\alpha[u,u] + K(u,u)| \ge k_0(\|u\|^2 + \|u'\|^2)$$

for some constant $k_0 > 0$. Therefore Cauchy sequences in the sense of $\alpha[u,v]$ are the one in H^1 and it is a closed form.

Theorem 8 *The dual A^* of the operator A is*

$$(A^*v)(x) \equiv -\left(\overline{a(x)}v' + \overline{b_2(x)}v\right)' + \overline{b_1(x)}v' + \overline{c(x)}v$$

with

$$\mathrm{Dom}(A^*) = \{v \in H^1(\mathbf{R}); \overline{a(x)}dv/dx + \overline{b_2(x)}u \in H^1(\mathbf{R})\}$$

We omit tje proof.

In order to obtain later the exponential decay the Green functions, we will need the next perturbation of the operator A.

Definition. A^μ is defined to be a peturbation of A :

$$(A^\mu u)(x) \equiv (Au)(x) + -2\mu u' + \mu c_1(x)u + \mu^2 c_2(x)u$$

with perturbation parameter $\mu \in \mathbf{C}$. where

$$c_1(x) = \frac{-b_1(x) + b_2(x)}{a(x)}, c_2(x) = -\frac{1}{a(x)} \in L^\infty.$$

The corresponding sesqilinear form is denoted by

$$\alpha^\mu[u,v] \equiv \alpha[u,v] - 2\mu(u',v) + \mu(c_1(x)u,v) + \mu^2(c_2(x)u,v).$$

Remark. A^μ is formally obtained as

$$(A^\mu u)(x) = e^{-\mu\Phi(x)} A(e^{\mu\Phi(x)} u)$$

Next is one of the Sobolev inequalities.

Lemma 9 *For arbitrary* $u \in W^{1,2}(\mathbf{R})$,

$$\|u\|_{L^\infty} \leqq \sqrt{2}\|u\|_{L^2}^{1/2}\|u'\|_{L^2}^{1/2}.$$

Proof. For any $x \in \mathbf{R}$,

$$\{u(x)\}^2 = \int_{-\infty}^{x} 2u(t)u'(t)dt$$

Hence

$$|u(x)|^2 \leqq 2\left(\int_{-\infty}^{\infty} |u(t)|^2\right)^{1/2}\left(\int_{-\infty}^{\infty} |u'(t)|^2\right)^{1/2}$$

Q.E.D.

Lemma 10 *Arbitrary* $z, w \in \mathbf{C} \setminus \{0\}$ *satisfy*

$$|z - w| \geq (\sin(|\theta|/2))(|z| + |w|).$$

Here

$$\theta = \arg(z) - \arg(w) = \arg(z/w) \in [-\pi, \pi)$$

Proof. Applying the cosine theorem to the triangle with vertices $0, z, w$, we have

$$\begin{aligned} |z-w|^2 &= |z|^2 + |w|^2 - 2|z||w|\cos\theta \\ &= \frac{1-\cos\theta}{2}(|z|+|w|)^2 + \frac{1+\cos\theta}{2}(|z|-|w|)^2 \\ &\geq \sin^2(\frac{\theta}{2})(|z|+|w|)^2. \end{aligned}$$

Q.E.D.

Theorem 11 *The sesquilinear form*

$$\alpha_\lambda[u, v] \equiv \alpha[u, v] - \lambda(r(x)u, v)$$

is a closed form with $\mathrm{Dom}(\alpha_\lambda) = \mathrm{Dom}(\alpha) = W^{1,2}$. *Let also* $\theta_a + \theta_r < \omega < \pi/2$ *for some* $\omega \in (0, \pi/2)$. *Then*

$$|\alpha_\lambda[u, u]| \geq k_0\|u'\|_{L^2}^2 + k_1|\lambda|\|u\|_{L^2}^2, \quad u \in \mathrm{Dom}(\alpha_\lambda) = W^{1,2}$$

for λ *which satisfies*

$$|\arg(\lambda)| \geq \omega, |\lambda| \geq k_2.$$

Here k_0, k_1 *and* k_2 *are positive constants which depend only on* $\omega, \theta_a, \theta_r, a_0, r_0, \|b_1(\bullet)\|_{L^\infty}, \|b_2(\bullet)\|_{L^\infty}, \|c(\bullet)\|_{L^\infty}$.

Proof. First notice that

$$\left|\arg\left(\int a(x)|u'|^2dx\right)\right| \leqq \theta_a$$

and

$$\left|\arg\left(\lambda\int r(x)|u'|^2dx\right)\right| \geqq \omega-\theta_r$$

Therefore the previous lemma is applicable and

$$\begin{aligned}\left|\int a(x)|u'|^2dx-\lambda\int r(x)|u'|^2dx\right| &\geqq \sin\frac{\omega-\theta_a-\theta_r}{2}\left(\left|\int a(x)|u'|^2dx\right|+|\lambda|\left|\int r(x)|u'|^2dx\right|\right)\\ &\geqq k_0(\|u'\|_{L^2}^2+|\lambda|\|u\|_{L^2}^2)\end{aligned}$$

for some constant $k_0>0$ dependent only on $\theta_a,\theta_r,\omega,a_0,r_0$. On the other hand,

$$\begin{aligned}\int\left(b_1(x)u\overline{u'}+b_2(x)u'\overline{u}+c(x)|u|^2\right)dx &\leqq k\left(\|u\|_{L^2}\|u'\|_{L^2}+(\|u\|_{L^2})^2\right)\\ &\leqq (k_0/2)\|u'\|_{L^2}^2+k_2\|u\|_{L^2}^2\end{aligned}$$

for some other constants k_2 dependent only on k_0 and $\|b_1(\bullet)\|_{L^\infty}, \|b_2(\bullet)\|_{L^\infty}, \|c(\bullet)\|_{L^\infty}$. Combining these two inequalities, we have

$$|\alpha_\lambda[u,u]| = |\alpha[u,u]-\lambda(r(\bullet)u,u)| \geqq (k_0/2)\|u'\|_{L^2}^2+(k_0|\lambda|-k_2)\|u\|_{L^2}^2$$

We have only to redefine the positive constantsk_0,k_1,k_2.

Corollary *The sesquilinear form*

$$\alpha_\lambda^\mu[u,v]\equiv\alpha^\mu[u,v]-\lambda(r(x)u,v)$$

is a closed form with $\mathrm{Dom}(\alpha_\lambda\mu)=\mathrm{Dom}(\alpha)=H^1$. *Let also* $\theta_a+\theta_r<\omega<\pi/2$ *for some* $\omega\in(0,\pi/2)$. *Then*

$$|\alpha_\lambda^\mu[u,u]|\geq k_0\|u'\|_{L^2}^2+(k_1|\lambda|-k_2|\mu|^2)\|u\|_{L^2}^2,\quad u\in\mathrm{Dom}(\alpha_\lambda)=H^1$$

for λ *and* μ *which satisfy*

$$|\arg(\lambda)|\geq\omega, |\lambda|\geq k_3, |\mu|\leq k_4|\lambda|^{1/2}.$$

Here k_0,k_1,k_2,k_3 *and* k_4 *are positive constants which depend only on* $\omega,\theta_a,\theta_r,a_0,r_0,\|b_1(\bullet)\|_{L^\infty}$, $\|b_2(\bullet)\|_{L^\infty}, \|c(\bullet)\|_{L^\infty}$.

Proposition 12 *Let* $\theta_a+\theta_r<\omega<\pi/2$. *Suppose that* $|\arg\lambda|\geq\omega$ *and that*$|\lambda|$ *is sufficiently large. Then, for any* $f(\bullet)\in L^2$,

$$(A-\lambda r(\bullet))u(x)\equiv Au(x)-\lambda r(x)u=f(x),$$

has a unique solution $u\in\mathrm{Dom}(A_\lambda)=\mathrm{Dom}(A)$ *and it satisfies*

$$\|u\|_{L^2}\leq k_1|\lambda|^{-1}\|f\|_{L^2}, \|u'\|_{L^2}\leq k_1|\lambda|^{-1/2}\|f\|_{L^2}, \|u\|_{L^\infty}\leq k_1|\lambda|^{-3/4}\|f\|_{L^2}$$

Proof. By the preceding theorem 11,

$$k_0\|u'\|_{L^2}^2 + k_1|\lambda|\|u\|_{L^2}^2 \leqq |\alpha_\lambda[u,u]| = |(f,u)| \leqq \|f\|\|u\|.$$

Therefore there exists a unique solution$u \in \mathrm{Dom}(A)$. We also have

$$k_1|\lambda|\|u\|_{L^2}^2| \leq \|f\|\|u\|,$$

hence

$$\|u\| \leq (k_1)^{-1}|\lambda|^{-1}\|f\|.$$

Back to the original inequality, we obtain

$$k_0\|u'\|_{L^2}^2 \leq \|f\|\|u\| \leq (k_1)^{-1}|\lambda|^{-1}\|f\|^2,$$

hence

$$\|u'\|_{L^2} \leq (k_0k_1)^{-1/2}|\lambda|^{-1/2}\|f\|.$$

Finally, we have

$$\|u\|_{L^\infty} \leq \sqrt{2}\|u\|_{L^2}\|u'\|_{L^2} \leq \sqrt{2}k_0^{-1/4}k_1^{-3/4}|\lambda|^{-3/4}\|f\|_{L^2}$$

Q.E.D.

Corollary *Let $\theta_a + \theta_r < \omega_0 < \omega < \pi/2$. Suppose that $|\arg\lambda| \geq \omega$ and $|\lambda|$is sufficiently large. Suppose also that $|\mu| \leq k_0|\lambda|^{1/2}$ with some constant $k_0 > 0$. Then, for any $f(\bullet) \in L^2$,*

$$(A^\mu - \lambda r(\bullet))u(x) \equiv A^\mu u(x) - \lambda r(x)u = f(x),$$

has a unique solution $u \in \mathrm{Dom}(A_\lambda^\mu) = \mathrm{Dom}(A)$ and it satisfies

$$\|u\|_{L^2} \leq k_1|\lambda|^{-1}\|f\|_{L^2}, \|u'\|_{L^2} \leq k_1|\lambda|^{-1/2}\|f\|_{L^2}, \|u\|_{L^\infty} \leq k_1|\lambda|^{-3/4}\|f\|_{L^2}$$

Proposition 13 *Let $\theta_a + \theta_r < \omega < \pi/2$ for some $\omega \in (0,\pi/2)$ Suppose that $|\arg\lambda| > \omega$ and $|\lambda|$is sufficiently large. Then, for any $f(\bullet) \in L^2$,*

$$(A - \lambda r(\bullet))u(x) \equiv Au(x) - \lambda r(x)u = (f(x))',$$

has a unique solution $u \in$ $\mathrm{Dom}(\alpha_\lambda) = \mathrm{Dom}(\alpha) = H^1$ and it satisfies

$$\|u\|_{L^2} \leq k_1|\lambda|^{-1/2}\|f\|_{L^2}, \|u'\|_{L^2} \leq k_2\|f\|_{L^2}, \|u\|_{L^\infty} \leq k_4|\lambda|^{-1/4}\|f\|_{L^2}$$

Proof. Note that

$$k_0\|u'\|_{L^2}^2 + k_1|\lambda|\|u\|_{L^2}^2 \leq |\alpha_\lambda[u,u]| = |(f',u)| = |(f,u')| \leq \|f\|\|u'\|$$

in the present case. Similarly to the preceding theorem, we have first

$$k_0\|u'\|_{L^2}^2 \leq \|f\|\|u'\|$$

hence,

$$\|u'\|_{L^2} \le k_0{}^{-1}\|f\|.$$

Back to the original inequality, we obtain

$$k_1|\lambda|\|u\|_{L^2}^2 \le \|f\|\|u'\| \le (k_0)^{-1}|\lambda|^{-1}\|f\|^2,$$

hence

$$\|u\|_{L^2} \leqq k_0{}^{-1}|\lambda|^{-1/2}\|f\|.$$

Finally, we have

$$\|u\|_{L^\infty}\| \leqq \sqrt{2}\|u\|_{L^2}\|u'\|_{L^2} \leqq \sqrt{2}k_0^{-1/4}k_1^{-3/4}|\lambda|^{-1/4}\|f\|_{L^2}$$

$$\|u'\|_{L^2} \leqq k_0{}^{-1}\|f\|_{L^2}.$$

Corollary *Let $\theta_a+\theta_r < \omega_0 < \omega < \pi/2$. Suppose that $|\arg\lambda| > \pi-\omega$ and $|\lambda|$ is sufficiently large. Suppose also that $|\mu| \le k_0|\lambda|^{1/2}$ with some constant $k_0 > 0$. Then, for any $f(\bullet) \in L^2$,*

$$(A^\mu - \lambda r(\bullet))u(x) \equiv A^\mu(x) - \lambda r(x)u = (f(x))',$$

has a unique solution $u \in$ $\mathrm{Dom}(\alpha_\lambda) = \mathrm{Dom}(\alpha) = H^1$ satisfying

$$\|u\|_{L^2} \le k_1|\lambda|^{-1/2}\|f\|_{L^2}, \|u'\|_{L^2} \le k_2\|f\|_{L^2}, \|u\|_{L^\infty} \le k_1|\lambda|^{-1/4}\|f\|_{L^2}$$

Proposition 14 *Let the assumption be the same as in the previous two Propositions. Then there exists a kernel function $R_\lambda(x,\xi)$ which represents the solution $u = (A-\lambda r)^{-1}f$:*

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)f(\xi)d\xi$$

with the estimate

$$|R_\lambda(x,\xi)| \le k_0|\lambda|^{-1/2}$$

for some constant $k_0 > 0$.

Proof. Since $u \in H^1 \subset B^0$ is a continuous function and

$$|u(x)| \le \|u\|_{B^0} \le \|u\|_{H^1} \le k_1|\lambda|^{-3/4}\|f\|_{L^2}$$

for an arbitrarily fixed x. Thus $L^2 \to \mathbf{C} : f(\bullet) \to u(x)$ is turned out to be a bounded functional. Hence the Riesz theorem asserts that there exists $R_\lambda(x,\bullet) \in L^2$ dependent on $x \in \mathbf{R}$ such that

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)f(\xi)d\xi$$

and $\|R_\lambda(x,\bullet)\|_{L^2} \le k_1|\lambda|^{-3/4}$

Now we consider the solution $v \in H^1 \subset B^0$ of $(A-\lambda)v = g', \quad g \in L^2$. By the previous theorem, $L^2 \to \mathbf{C} : f(\bullet) \to v(x)$ with an arbitrarily fixed x is also a bounded functional and

$$|v(x)| \le \|v\|_{B^0} \le \|v\|_{H^1} \le k_2|\lambda|^{-1/4}\|f\|_{L^2}$$

with another constant $k_2 > 0$. So there exists again another kernel $S_\lambda(x,\xi)$ such that

$$v(x) = \int_{-\infty}^{\infty} S_\lambda(x,\xi)g(\xi)d\xi$$

and $\|S_\lambda(x,\bullet)\|_{L^2} \le k_2|\lambda|^{-1/4}$ We look into the relation of $R_\lambda(x,\xi)$ and $S_\lambda(x,\xi)$. For an arbitrary $g \in C_0^\infty$, the solution $v \in H^1$ of $(a-\lambda r)v = g'$ can be written in two ways.

$$v(x) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)g'(\xi)d\xi,$$

$$v(x) = \int_{-\infty}^{\infty} S_\lambda(x,\xi)g(\xi)d\xi.$$

Since $g \in C_0^\infty$ is arbitrary , $S_\lambda(x,\xi) \in L^2$ is a distribution derivative of $R_\lambda(x,\xi)$ with respect to ξ. Thus $R_\lambda(x,\bullet) \in H^1 \subset B^0$. By Lemma?,

$$\|R_\lambda(x,\bullet)\|_{L^\infty} \le \|R_\lambda(x,\bullet)\|_{L^\infty}^{1/2}\|S_\lambda(x,\bullet)\|_{L^\infty}^{1/2} \le k_2|\lambda|^{-1/2}$$

Corollary *Let the assumption be the same as in the corollaries of the two previous two propositions. Then there exists a kernel function $R_\lambda^\mu(x,\xi)$ which represents the solution $u = (A^\mu - \lambda r)^{-1}f$:*

$$u(x) = \int_{-\infty}^{\infty} R_\lambda^\mu(x,\xi)f(\xi)d\xi$$

with the estimate

$$|R_\lambda^\mu(x,\xi)| \le k_0|\lambda|^{-1/2}$$

for some constant $k_0 > 0$.

Theorem 15 *Let the assumption be the same as in the two theorems. The kernel function $R_\lambda(x,\xi)$ which represents the solution $u = (A-\lambda r)^{-1}f$:*

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)f(\xi)d\xi$$

has the estimate

$$|R_\lambda(x,\xi)| \le k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|x-\xi|}$$

for some constant $k_1, k_2 > 0$.

Proof. Let μ be as in the corollaries of the Theorems. Let $u \in \mathrm{Dom}(A) \cap C_0$ be arbitrarily fixed. Then

$$e^{-\mu\Phi(x)}u(x) \in \mathrm{Dom}(A)$$

where

$$\Phi(x) = \int_0^x \frac{dy}{a(y)}$$

as in Theorem 6. Now putting

$$f = (A - \lambda r)u,$$

we have

$$\begin{aligned}
(A - \lambda R)e^{\mu\Phi(x)}(e^{-\mu\Phi(x)}u(x)) &= f(x) \\
e^{\mu\Phi(x)}(A^\mu - \lambda r)(e^{-\mu\Phi(x)}u(x)) &= f(x) \\
(A^\mu - \lambda r)(e^{-\mu\Phi(x)}u(x)) &= e^{-\mu\Phi(x)}f(x).
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-\mu\Phi(x)}u(x) &= \int_{-\infty}^{\infty} R_\lambda^\mu(x,\xi)e^{-\mu\Phi(\xi)}f(\xi)d\xi \\
u(x) &= e^{\mu\Phi(x)}\int_{-\infty}^{\infty} R_\lambda^\mu(x,\xi)e^{-\mu\Phi(\xi)}f(\xi)d\xi \\
u(x) &= \int_{-\infty}^{\infty} e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x,\xi)f(\xi)d\xi.
\end{aligned}$$

On the other hand, $u = (A - \lambda r)^{-1}f$ can be written as

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)f(\xi)d\xi.$$

Hence

$$\int_{-\infty}^{\infty} e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x,\xi)f(\xi)d\xi = \int_{-\infty}^{\infty} R_\lambda(x,\xi)f(\xi)d\xi$$

for all $f = (A - \lambda r)u$ with $u \in \mathrm{Dom}(A) \cap C_0$. Such f form a dense subset in L^2. Therefore

$$R_\lambda(x,\xi) \equiv e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x,\xi).$$

Recalling that μ with $|\mu| \leq k_0|\lambda|^{1/2}$ is arbitrary and using the Corollary of the previous Proposition 14,

$$|R_\lambda(x,\xi)| \leq k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|\Phi(x)-\Phi(\xi)|}.$$

Noticing

$$\Re(1/a(y)) \geq k_0$$

with a certain constant $k_0 > 0$,

$$|\Phi(x) - \Phi(\xi)| \geq |\Re\Phi(x) - \Phi(\xi)| = |\Re \int_x^\xi \frac{dy}{a(y)}| \geq k_0|x - \xi|.$$

Combining these, we finally obtain

$$|R_\lambda(x,\xi)| \leq k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|x-\xi|}.$$

Q.E.D.

Corollary *There exists a kernel function $\tilde{R}_\lambda(x,\xi)$ of $(B-\lambda)^{-1}$ where $Bu(x) = (r(x))^{-1}Au(x)$:*

$$(B-\lambda)^{-1}f(x) = \int_{-\infty}^{\infty} \tilde{R}_\lambda(x,\xi)f(\xi)d\xi$$

Moreover

$$|\tilde{R}_\lambda(x,\xi)| \leq k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|x-\xi|}$$

with constants k_1, k_2.

Proof. Since $Bu - \lambda u = f \in L^2$ is equivalent to

$$Au - \lambda r(x)u = r(x)f \in L^2,$$

we have

$$u(x) = (B-\lambda)^{-1}f(x) = (A-\lambda r)^{-1}(rf) = \int_{-\infty}^{\infty} R_\lambda(x,\xi)r(\xi)f(\xi)d\xi.$$

Therefore, we have only to put $\tilde{R}_\lambda(x,\xi) = R_\lambda(x,\xi)r(\xi)$. Q.E.D.

Theorem 16 *Let the assumption be the same as the preceding theorem and its corollary. Then*

$$\begin{aligned} \left|\frac{\partial R_\lambda}{\partial x}(x,\xi)\right| &\leq k_1e^{-k_2|\lambda|^{1/2}|x-\xi|} \\ \left|\frac{\partial \tilde{R}_\lambda}{\partial x}(x,\xi)\right| &\leq \tilde{k}_1e^{-\tilde{k}_2|\lambda|^{1/2}|x-\xi|}. \end{aligned}$$

for some constants $k_1, k_2, \tilde{k}_1, \tilde{k}_2 > 0$.

We omit the proof.

Theorem 17 *The analytic semigroup* e^{-tB} *generated by*

$$Bu(x) = (r(x))^{-1}(Au)(x)$$

has a kernel function $G(x,y;t)$ *with estimate*

$$|G(x,\xi;t)| \le k_0 e^{k_1 t} e^{-k_2|t|^{-1}|x-\xi|^2}, (x,\xi) \in \mathbf{R}^2, |\arg t| \le \pi/2 - \omega$$

with constants $k_0, k_1, k_2 > 0$.

Proof. The kernel function $\tilde{R}_\lambda(x,\xi)$ of $(B-\lambda)^{-1}$ is expressed by the kernel $\tilde{R}_\lambda(x,\xi)$ with estimate

$$|\tilde{R}_\lambda(x,\xi)| \le k_1|\lambda|^{-1/2} e^{-k_2|\lambda|^{1/2}|x-\xi|}$$

for

$$\{\lambda; |\arg\lambda| \ge \omega', |\lambda| \ge k_3\}$$

with constants $\omega' \in (\theta_a + \theta_r\omega)$, , $k_1, k_2, k_3 > 0$.

By a standard argument, $B + k_0$ with some $k_0 > 0$ has a kernel which has a similar estimate in

$$\{\lambda; |\arg\lambda| \ge \omega'\}$$

We have only to discuss this $B + k_0$ and $e^{-t(B+k_0)}$, rewriting $B + k_0$ as B from now on. We recall the formula:

$$e^{-tB} = \frac{-1}{2\pi}\int_\Gamma e^{-\lambda t}(B-\lambda)^{-1}d\lambda.$$

with the integral path

$$\Gamma = \{\lambda = \rho e^{i\omega'}; \infty > \rho \ge 0\} \cup \{\lambda = \rho e^{i\omega'}; 0 \le \rho < \infty\}$$

The corresponding kernel function is

$$G(x,\xi;t) = \frac{-1}{2\pi i}\int_\Gamma e^{-\lambda t}(\tilde{R}_\lambda(x,\xi)d\lambda.$$

We modify the integral path to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$\begin{aligned} \Gamma_1 &= = \{\lambda = \rho e^{i\omega'}; \infty > \rho \ge k|t|^{-2}|x-\xi|^2\} \\ \Gamma 2 &= \{\lambda = k|t|^{-2}|x-\xi|^2 e^{i\theta}; \omega' \le \theta \le 2\pi - \theta\} \\ \Gamma_3 &= = \{\lambda = \rho e^{-i\omega'}; k|t|^{-2}|x-\xi|^2 \le \rho < \infty\} \end{aligned}$$

Here the constant $k > 0$ is chosen so small that

$$|\lambda||t| = k|t|^{-1}|x-\xi|^2 \le 2^{-1}|k|^{1/2}k_2|t|^{-1}|x-\xi|^2 = 2^{-1}k_2|\lambda|^{1/2}|x-\xi|$$

holds on the path Γ_2. We estimate the integral on each path.

$$\begin{aligned}\left|\frac{-1}{2\pi i}\int_{\Gamma_1} e^{-\lambda t}\tilde{R}_\lambda(x,\xi)d\lambda\right| &\le k_0\int_{\frac{k|x-\xi|^2}{|t|^2}}^{\infty} e^{-k_1\rho|t|}|\rho|^{-1/2}e^{-k_1|\rho|^{1/2}|x-\xi|}d\rho \\ &\le k_0 e^{-k_1k^{1/2}|t|^{-1}|x-\xi|^2}\int_{\frac{k|x-\xi|^2}{|t|^2}}^{\infty} e^{-k_1\rho|t|}|\rho|^{-1/2}d\rho \\ &\le k_0 e^{-k_1k^{1/2}|t|^{-1}|x-\xi|^2}\int_0^{\infty} e^{-k_1\rho|t|}|\rho|^{-1/2}d\rho \\ &\le k_0 e^{-k_1k^{1/2}|t|^{-1}|x-\xi|^2}O(|t|^{-1/2})\end{aligned}$$

Similarly,

$$\left|\frac{-1}{2\pi i}\int_{\Gamma_3} e^{-\lambda t}\tilde{R}_\lambda(x,\xi)d\lambda\right| \le k_0|t|^{-1/2}e^{-k_1|t|^{-1}|x-\xi|^2}$$

with some constants $k_0, k_1 > 0$. Finally, holds on the path Γ_2. We estimate the integral on each path.

$$\begin{aligned}\left|\frac{-1}{2\pi i}\int_{\Gamma_2} e^{-\lambda t}\tilde{R}_\lambda(x,\xi)d\lambda\right| &\le k_0\int_{\Gamma_2} e^{2^{-1}k_2|\lambda|^{1/2}|x-\xi|}|\rho|^{-1/2}e^{-k_2|\lambda|^{1/2}|x-\xi|}d|\lambda| \\ &\le k_0\int_{\omega'}^{2\pi-\omega'} e^{-k_3|t|^{-1}|x-\xi|^2}(|t|^{-1}|x-\xi|^2)^{1/2}|T|^{-1/2}d\theta \\ &\le k_0|t|^{-1/2}e^{-k_4|t|^{-1}|x-\xi|^2}\end{aligned}$$