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Infrared Catastrophe and Carleman Operator

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1 Introduction

In this paper, we make a broad report on a mathematical mechanism of *infrared catastrophe* (*IR catastrophe*) considered in [9], where by IR catastrophe we mean the divergence of the average of the total number of *soft bosons*. The soft bosons are read as bosons in a ground state. In [2] we gave the definition of IR catastrophe by the statement that the ground state is not in the domain of the half of the boson number operator. We visualize the definition by introducing a tool to estimate the average of the total number of soft bosons. IR catastrophe is usually derived by using the pull-through formula [5, 13]. This formula was studied with the L^2-theoretical way in [4, 6], and with the operator-theoretical way in [3, 8]. We employ the latter way and derive the Carleman operator from the formula. Then, we can characterize IR catastrophe in terms of the domain of the Carleman operator. Recently, the Carleman operator is used in [10] to describe the necessary and sufficient condition for convergence of average of the total number of the soft bosons. We show it with our tool, and conversely, we also use the Carleman operator in order to argue IR catastrophe. In some literatures [4, 5, 8, 11], for a few models it was proved that IR catastrophe is caused by the infrared singularity condition [1, 2] and IR catastrophe results in absence of ground state. We grasp them inclusively in the light of properties of the domain of the Carleman operator.

2 Main Results

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}_{\mathrm{b}}(L^2(S))$ the boson Fock space over $L^2(S)$, where S is arbitrary open subset of $\mathbb{R}^d$, $\mathcal{F}_{\mathrm{b}}(L^2(S)) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n L^2(S)$. Here, for $n \in \mathbb{N}$, we denote the n-fold symmetric tensor product of $L^2(S)$ by $\otimes_s^n L^2(S)$ with convention $\otimes_s^0 L^2(S) := \mathbb{C}$. Set $\mathcal{F}_b := \mathcal{F}_b(L^2(\mathbb{R}^d))$ for simplicity. We consider the Hilbert space $\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b$, which is the state space of several models in quantum field theory with a standard inner product $(\ ,\ )_{\mathcal{F}}$. Let $h : S \longrightarrow [0, \infty)$ be Borel measurable such that $0 < h(k) < \infty$ for almost every (a.e.) $k \in S$ with respect to the d-dimensional Lebesgue measure. We use the same symbol for the multiplication operator by h, acting in $L^2(S)$. We denote by $d\Gamma(h)$ the second quantization of h [12, Sect. X.7].

Let A be a self-adjoint operator acting in $\mathcal{H}$ bounded from below, and $\omega : \mathbb{R}^d \longrightarrow [0, \infty)$ be continuous such that $0 < \omega(k) < \infty$ for every $k \in \mathbb{R}^d \setminus \{0\}$. The unperturbed Hamiltonian of our quantum filed model is defined by

$$H_0 := A \otimes I + I \otimes d\Gamma(\omega)$$

with domain $D(H_0) := D(A \otimes I) \cap D(I \otimes d\Gamma(\omega)) \subset \mathcal{F}$, where I denotes identity operator and $D(T)$ the domain of an operator T. The operator H_0 is self-adjoint and bounded from below. For $\psi \in \mathcal{F}_b(S)$, we introduce the following notation: $\psi = \oplus \sum_{n=0}^{\infty} \psi^{(n)}$, $\psi^{(n)} \in \otimes_s^n L^2(S)$; $n \in \{0\} \cup \mathbb{N}$.

We denote by $\mathcal{F}_{b,0}(L^2(S))$ the finite particle space, i.e., the set of all elements $\psi = \oplus \sum_{n=0}^{\infty} \psi^{(n)}$ in $\mathcal{F}_b(L^2(S))$ satisfying $\exists n_0 \in \mathbb{N}$ such that $\psi^{(n)} = 0$, $\forall n \geq n_0$. We set $\mathcal{F}_{b,0} := \mathcal{F}_{b,0}(L^2(\mathbb{R}^d))$. We denote the smeared annihilation operators acting in $\mathcal{F}_b(L^2(S))$ by $a(f)$ for every $f \in L^2(S)$, where we assume antilinearity in f [12, Sect. X.7]. On $\mathcal{F}_{b,0}(L^2(S))$, we get

$$\begin{aligned}(a(f)\psi)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1} \int_S \overline{f(k)} \psi^{(n+1)}(k, k_1, \cdots, k_n) dk \\ \in \otimes_s^n L^2(S), \qquad n \in \{0\} \cup \mathbb{N}.\end{aligned}$$

Since $a(f)$ is closable on $\mathcal{F}_{b,0}(L^2(S))$, we denote its closure by the same symbol. So, we always regard $a(f)$ as a closed operator.

We define our total Hamiltonian by

$$H_{\text{QFT}} = H_0 + H_{\text{I}}$$

be a self-adjoint operator acting in $\mathcal{F}$ and describing a quantum field model. If H_{QFT} has a (normalized) ground state, we denote it by Ψ_{QFT}.

When the operator-theoretical pull-through formula on ground states holds in the same way as in [8] and it has the form of (2.1) below, we define an operator $B_{\mathrm{PT}}(k)$ by

$$a(f)\Psi_{\mathrm{QFT}} = -\int_{\mathbb{R}^d} \overline{f(k)}\left(H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}) + \omega(k)\right)^{-1} B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}dk, \quad (2.1)$$
$$\forall f \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),$$

where $E_0(H_{\mathrm{QFT}})$ is the ground state energy of H_{QFT}, i.e., $E_0(H_{\mathrm{QFT}}) := \inf \sigma(H_{\mathrm{QFT}})$ and $\sigma(T)$ denotes the spectrum of a closed operator T. We are preparing the work on the operator-theoretical pull-through formula for several models in quantum field theory [3].

We set

$$\widehat{H}_{\mathrm{QFT}} := H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}).$$

We assume the following conditions for $B_{\mathrm{PT}}(k)$:

(PT1) $B_{\mathrm{PT}}(k)$ is determined for every $k \in \mathbb{R}^d \setminus \{0\}$ as an operator acting in $\mathcal{F}$ and $B_{\mathrm{PT}}(\cdot)\Psi$ is measurable for every $\Psi \in D(H_0)$ (i.e., for every $\Phi \in \mathcal{F}$ and $\Psi \in D(H_0)$ $(\Phi\,,\, B_{\mathrm{PT}}(\cdot)\Psi)_{\mathcal{F}} : \mathbb{R}^d \longrightarrow \mathbb{C}$ is a measurable function).

(PT2) $B_{\mathrm{PT}}(k)\left(\widehat{H}_{\mathrm{QFT}} + \omega(k)\right)^{-1}$ is bounded for every $k \in \mathbb{R}^d \setminus \{0\}$.

For every $\varepsilon \geq 0$, we set $\mathbb{R}^d_{<\varepsilon} := \left\{k \in \mathbb{R}^d \,|\, |k| < \varepsilon\right\}$ and $\mathbb{R}^d_{>\varepsilon} := \left\{k \in \mathbb{R}^d \,|\, |k| > \varepsilon\right\}$. Following this decomposition, for every $f \in L^2(\mathbb{R}^d)$ we define $f_{<\varepsilon} \in L^2(\mathbb{R}^d_{<\varepsilon})$ and $f_{>\varepsilon} \in L^2(\mathbb{R}^d_{>\varepsilon})$ by $f_{<\varepsilon} := \chi_{|k|<\varepsilon} f$ and $f_{>\varepsilon} := \chi_{|k|>\varepsilon} f$, where $\chi_{|k|<\varepsilon}$ and $\chi_{|k|>\varepsilon}$ are characteristic functions defined by $\chi_{|k|<\varepsilon}(k) := 1$ (if $|k| < \varepsilon$); $:= 0$ (otherwise), and $\chi_{|k|>\varepsilon}(k) := 1$ (if $|k| > \varepsilon$) ; $:= 0$ (otherwise. As is well known, there exists a unitary operator U_ε such that

$$U_\varepsilon \mathcal{F}_{\mathrm{b}} = \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_{<\varepsilon})) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_{>\varepsilon})),$$
$$U_\varepsilon d\Gamma(\omega)U_\varepsilon^* = d\Gamma(\omega_{<\varepsilon}) \otimes I + I \otimes d\Gamma(\omega_{>\varepsilon}),$$
$$U_\varepsilon d\Gamma(1)U_\varepsilon^* = d\Gamma(1_{<\varepsilon}) \otimes I + I \otimes d\Gamma(1_{>\varepsilon}).$$

We define the number operator N acting in $\mathcal{F}$ by $N := I \otimes d\Gamma(1)$, where 1 in $d\Gamma(1)$ denotes the constant function $1(k) = 1$. Moreover, for every $\varepsilon > 0$, we define $N(\varepsilon)$

acting in $\mathcal{F}_\mathrm{b}$ by $N(\varepsilon) := U_\varepsilon^* \left(I \otimes d\Gamma(1_{>\varepsilon})\right) U_\varepsilon$, where $I \otimes d\Gamma(1_{>\varepsilon})$ acts in $\mathcal{F}_\mathrm{b}(L^2(\mathbb{R}^d_{<\varepsilon})) \otimes \mathcal{F}_\mathrm{b}(L^2(\mathbb{R}^d_{>\varepsilon}))$, and an operator $N_{>\varepsilon}$ acting in $\mathcal{F}$ by $N_{>\varepsilon} := I \otimes N(\varepsilon)$. We sometimes denote N by $N_{>0}$.

Lemma 2.1 *[7] If* $\Psi \in D(H_0)$, *then*

$$\Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2}). \tag{2.2}$$

Symbolically, the average of the number of bosons in the state $\Psi \in \mathcal{F}$ is give by $\int_{\mathbb{R}^d} \|a(k)\Psi\|_{\mathcal{F}}^2 dk$, of which justification is in [9]. Then, we set

$$D_{\mathrm{CNB}} := \left\{ \Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2}) \,\middle|\, \sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 < \infty \right\}.$$

As we also justify the following in [9],

$$\text{if } \Psi \in D_{\mathrm{CNB}}, \quad \text{then } \int_{\mathbb{R}^d} \|a(k)\Psi\|_{\mathcal{F}}^2 dk = \sup_{\varepsilon>0} \int_{|k|>\varepsilon} \|a(k)\Psi\|_{\mathcal{F}}^2 dk < \infty$$

by Lebesgue's monotone convergence theorem. Thus, D_{CNB} is the state space such that the average of the number of bosons in the state in D_{CNB} converges.

We call $\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2$ *the average of the total number of soft bosons*, provided that $\Psi_{\mathrm{QFT}} \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$. By Lemma 2.1, we can *always* estimate the average of the total number of soft bosons and give a decision whether it is convergent or divergent so long as $D(H_{\mathrm{QFT}}) = D(H_0)$. We call the divergence of the average of the total number of soft bosons *infrared catastrophe* (*IR catastrophe*).

As is well known, $\mathcal{F}$ is unitarily equivalent to $\bigoplus_{n=0}^\infty L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})$, where $L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})$ is the Hilbert space of square integrable $\mathcal{H}$-valued, symmetric functions on $\mathbb{R}^{dn} = (\mathbb{R}^d)^n$ with convention $L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})\lceil_{n=0} := \mathcal{H}$. Moreover, $L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})$ is unitarily equivalent to $\mathcal{H}\otimes\left(\otimes_\mathrm{s}^n L^2(\mathbb{R}^d)\right)$. We often identify $\mathcal{F}$ as $\mathcal{F} = \bigoplus_{n=0}^\infty \mathcal{H}\otimes\left(\otimes_\mathrm{s}^n L^2(\mathbb{R}^d)\right) = \bigoplus_{n=0}^\infty L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})$ in this paper. Then, through this identification we can denote all $\Psi \in \mathcal{F}$ by $\Psi = \oplus\sum_{n=0}^\infty \Psi^{(n)} = \Psi^{(0)} \oplus \Psi^{(1)} \oplus \cdots \oplus \Psi^{(n)} \oplus \cdots$, where $\Psi^{(n)} \in \mathcal{H} \otimes \left(\otimes_\mathrm{s}^n L^2(\mathbb{R}^d)\right) = L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})$. For each $n_1, n_2 \in \mathbb{N}$, we introduce the following notation: $\oplus\sum_{n=n_1}^{n_2} \Psi^{(n)} = 0 \oplus \cdots 0 \oplus \Psi^{(n_1)} \oplus \cdots \oplus \Psi^{(n_2)} \oplus 0 \oplus \cdots \in \mathcal{H} \otimes \mathcal{F}_{\mathrm{b},0}$.

Similarly, we use the following identification.

$$\mathcal{F} = \mathcal{F}_\mathrm{b}(L^2(\mathbb{R}^d_{<\varepsilon};\mathcal{H})) \otimes \mathcal{F}_\mathrm{b}(L^2(\mathbb{R}^d_{>\varepsilon};\mathcal{H})), \tag{2.3}$$

where $L^2(S;\mathcal{H})$ denotes the Hilbert space of square integrable $\mathcal{H}$-valued functions on an open set $S \subset \mathbb{R}^d$.

Lemma 2.2 *$D_{\text{CNB}} = D(N^{1/2})$ and*

$$\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2$$

for $\Psi \in D_{\text{CNB}}$.

When a ground state Ψ_{QFT} of H_{QFT} exists, we can define an $\mathcal{F}$-valued function $K_{\text{PT}} : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathcal{F}$ by

$$K_{\text{PT}}(k) := \left(\widehat{H}_{\text{QFT}} + \omega(k)\right)^{-1} B_{\text{PT}}(k)\Psi_{\text{QFT}}, \quad k \in \mathbb{R}^d \setminus \{0\}, \tag{2.4}$$

since $\left(\widehat{H}_{\text{QFT}} + \omega(k)\right)^{-1} B_{\text{PT}}(k)$ is bounded for every $k \in \mathbb{R}^d \setminus \{0\}$ by (PT2). K_{PT} defined by (2.4) is measurable by (PT1). For the ground state Ψ_{QFT}, we define the *maximal Carleman operator* $T_{\text{PT}} : \mathcal{F} \to L^2(\mathbb{R}^d)$ *for the ground state* by

$$D(T_{\text{PT}}) := \left\{\Phi \in \mathcal{F} \,\middle|\, (K_{\text{PT}}(\cdot)\,,\,\Phi)_{\mathcal{F}} \in L^2(\mathbb{R}^d)\right\}, \tag{2.5}$$

$$(T_{\text{PT}}\Phi)(k) := (K_{\text{PT}}(k)\,,\,\Phi)_{\mathcal{F}}, \quad \forall k \in \mathbb{R}^d \setminus \{0\}; \quad \forall \Phi \in D(T_{\text{PT}}). \tag{2.6}$$

Remark 2.1 (1) If $K_{\text{PT}} \in L^2(\mathbb{R}^d; \mathcal{F})$, then the maximal Carleman operator T_{PT} for the ground state is Hilbert-Schmidt [14, Theorems 6.12 and 6.13].

(2) If H_{QFT} is the Hamiltonian of the Pauli-Fierz model, T_{PT} is Hilbert-Schmidt.

Theorem 2.3 *Assume $D(H_{\text{QFT}}) = D(H_0)$ and there exists a ground state Ψ_{QFT} of H_{QFT}. Then, $\Psi_{\text{QFT}} \in D_{\text{CNB}} = D(N^{1/2})$ if and only if T_{PT} is Hilbert-Schmidt.*

Theorem 2.4 *Suppose that $D(H_{\text{QFT}}) = D(H_0)$. If a ground state Ψ_{QFT} of H_{QFT} exists, then*

$$D(T_{\text{PT}}) \supset D_{\text{CNB}} = D(N^{1/2}).$$

Theorem 2.4 immediately implies the following important statement:

Corollary 2.5 *Suppose that $D(H_{\text{QFT}}) = D(H_0)$. If $\Psi_{\text{QFT}} \notin D(T_{\text{PT}})$, then IR catastrophe occurs.*

As corollaries of Theorem 2.4, we can prove Dereziński and Gérard's lemma [4, Lemma 2.6] in the weak topology and we obtain a generalization of [2, Theorem 3.4]. We note that we can make another device when we do not have $g/\omega \notin L^2(\mathbb{R}^d)$ in decomposition (2.7) below. See Corollary 2.10.

Corollary 2.6

(i) *Assume that* $D(H_{\mathrm{QFT}}) = D(H_0)$. *In addition, assume there exist a measurable function* $g : \mathbb{R}^d \longrightarrow \mathbb{C}$ *and an operator* $J_{\mathrm{err}}(k)$ *acting in* $\mathcal{F}$ *for every* $k \in \mathbb{R}^d \setminus \{0\}$ *such that*

$$B_{\mathrm{PT}}(k) = g(k) I \otimes I + J_{\mathrm{err}}(k) \tag{2.7}$$

for $k \in \mathbb{R}^d \backslash \{0\}$ *with* $g/\omega \notin L^2(\mathbb{R}^d)$. *If a ground state* Ψ_{QFT} *exists, then* $\omega(\cdot)^{-1} J_{\mathrm{err}}(\cdot)$ $\Psi_{\mathrm{QFT}} \notin L^2(\mathbb{R}^d; \mathcal{F})$.

(ii) *Assume* $B_{\mathrm{PT}}(k)$ *is independent of all* $k \in \mathbb{R}^d \setminus \{0\}$, *i.e.,* $B_{\mathrm{PT}}(k) = B_{\mathrm{PT}}$. *Moreover, assume* H_{QFT} *and* B_{PT} *are strongly commutable. If a ground state* Ψ_{QFT} *exists, then* $B_{\mathrm{PT}} \Psi_{\mathrm{QFT}} = 0$.

Corollary 2.6 can be read as the following. When we do not have $g/\omega \notin L^2(\mathbb{R}^d)$ decomposition (2.7), we have Corollary 2.10.

Corollary 2.7

(i) *Assume* $D(H_{\mathrm{QFT}}) = D(H_0)$ *and that decomposition (2.7) with* $g/\omega \notin L^2(\mathbb{R}^d)$ *in Corollary 2.6 holds. Then, there is no ground state* Ψ_{QFT} *satisfying* $\omega(\cdot)^{-1} J_{\mathrm{err}}(\cdot)$ $\Psi_{\mathrm{QFT}} \in L^2(\mathbb{R}^d; \mathcal{F})$.

(ii) *Assume* $B_{\mathrm{PT}}(k)$ *is independent of all* $k \in \mathbb{R}^d \setminus \{0\}$, *i.e.,* $B_{\mathrm{PT}}(k) = B_{\mathrm{PT}}$. *If* H_{QFT} *and* B_{PT} *are strongly commutable, then there is no ground state* Ψ_{QFT} *satisfying* $B_{\mathrm{PT}} \Psi_{\mathrm{QFT}} \neq 0$.

Theorem 2.8 *Assume* $D(H_{\mathrm{QFT}}) = D(H_0)$. *In addition, assume the following conditions:*

(Ass1) *A ground state* Ψ_{QFT} *exists.*

(Ass2) *There exist a function* $\lambda_{\mathrm{IR}} \in L^2(\mathbb{R}^d)$ *and an operator* $B_{\mathrm{IR}}(k)$ *acting in* $\mathcal{F}$ *for every* $k \in \mathbb{R}^d$ *such that*

(Ass2-1) $B_{\mathrm{PT}}(k) = \lambda_{\mathrm{IR}}(k) B_{\mathrm{IR}}(k)$ on $D(B_{\mathrm{PT}}(k)) = D(B_{\mathrm{IR}}(k))$ for $k \neq 0$,

(Ass2-2) $\lambda_{\mathrm{IR}}/\omega \notin L^2(K)$ for all neighborhoods K of $k = 0$,

(Ass2-3) $B_{\mathrm{IR}}(k) \Psi_{\mathrm{QFT}} \longrightarrow B_{\mathrm{IR}}(0) \Psi_{\mathrm{QFT}}$ as $k \to 0$.

If $\Phi \in D(T_{\mathrm{PT}})$ *satisfies*

$$\frac{1}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\mathrm{QFT}}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^d),$$

then $(\Phi\,,\,B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}=0$.

From this theorem, we obtain the following generalization of the method in [8].

Corollary 2.9 *Assume* $D(H_{\mathrm{QFT}})=D(H_0)$ *and (Ass2). Then, there is no ground state* Ψ_{QFT} *in* $\mathcal{F}$ *such that*

$$\frac{1}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\mathrm{QFT}}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^d), \qquad \forall\Phi \in D(N^{1/2}), \tag{2.8}$$

and $B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}} \neq 0$.

When we do not have $g/\omega \notin L^2(\mathbb{R}^d)$ in decomposition (2.7), we have the following corollary:

Corollary 2.10 *Assume* $D(H_{\mathrm{QFT}}) = D(H_0)$ *and (Ass2). In addition, assume there exist a measurable function* $g:\mathbb{R}^d \longrightarrow \mathbb{C}$ *and an operator* $J_{\mathrm{err}}(k)$ *acting in* $\mathcal{F}$ *for every* $k\in\mathbb{R}^d\setminus\{0\}$ *such that*

$$B_{\mathrm{PT}}(k)=g(k)I\otimes I+J_{\mathrm{err}}(k) \tag{2.9}$$

for $k\in\mathbb{R}^d\backslash\{0\}$*. Then, there is no ground state satisfying* $\omega(\cdot)^{-1}J_{\mathrm{err}}(\cdot)\,\Psi_{\mathrm{QFT}}\in L^2(\mathbb{R}^d;\mathcal{F})$ *and* $B_{\mathrm{IR}}(0)\Psi_{\mathrm{QFT}}\neq 0$.

Corollary 2.9 implies the following.

Corollary 2.11 *Assume* $D(H_{\mathrm{QFT}}) = D(H_0)$ *and (Ass2). Then, there is no ground state* Ψ_{QFT} *in* $D(T_{\mathrm{PT}})$ *such that* $(\Psi_{\mathrm{QFT}}\,,\,B_{IR}(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}}\neq 0$.

For the Nelson model, (2.8) holds, provided its ground state exists. Thus, since Corollary 2.9 works, the Nelson model has no ground state in $\mathcal{F}$ [8]. We can consider the genelarized spin-boson model and the model describing several kinds of polaron as some examples for the above general theory [9].

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