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INVERSE PROBLEMS FOR SCHRÖDINGER OPERATORS ON HYPERBOLIC SPACES AND $\overline{\partial}$-THEORY

筑波大学・数理物質科学研究科　磯崎　洋　(HIROSHI ISOZAKI)
GRADUATE SCHOOL OF PURE AND APPLIED SCIENCES,
UNIVERSITY OF TSUKUBA, TSUKUBA, 305-8571, JAPAN

This paper is a brief exposition of a method recently introduced by the author for solving the inverse problem for Schrödinger operators by using the hyperbolic space as a tool. In the first part, we explain the fundamental issues of inverse problems and the basic idea of this hyperbolic space approach. In the second part, representation formulas of the potential in terms of a $\overline{\partial}$-equation are shown. In the third part, we give an application to the numerical computation related to a practical problem in the medical science.

Part 1. Hyperbolic space approach to the inverse problem

1. Basic ideas

1.1. IBVP and ISP. There are two fundamental issues in inverse problems for Schrödinger operators : the inverse boundary value problem (IBVP) and the inverse scattering problem (ISP). In IBVP, we take a bounded domain Ω in $\mathbf{R}^n$ and consider the following Dirichlet problem

$$(-\Delta + q)u = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \partial\Omega. \tag{1}$$

The Dirichlet-Neumann map, called the *D-N map* hereafter, is defined by

$$\Lambda_q f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}, \tag{2}$$

ν being the outer unit normal to the boundary. In IBVP, we aim at reconstructing q from Λ_q. An important application of this IBVP is in the medical science, where one tries to reconstruct the electric conductivity of a body from the surface measurement.

The ISP is concerned with the movement of quantum mechanical particles and waves. For Schrödinger operators $H_0 = -\Delta$, $H = H_0 + V$ on $\mathbf{R}^n$, where V is a rapidly decaying potential, one observes the behavior at infinity of solutions to the Schrödinger equation $(H - \lambda)\varphi = 0$ in the following way :

$$\varphi(x, \lambda, \omega') \sim e^{i\sqrt{E}\omega'\cdot x} - C(E)\frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}}A(E; \omega, \omega'), \tag{3}$$

as $r = |x| \to \infty$, $\omega = x/r$, $\omega' \in S^{n-1}$. In ISP, we try to reconstruct V from the *scattering amplitude* $A(E; \theta, \omega)$. We are concerned here only with the fixed energy problem, namely, the reconstruction of V from the scattering amplitude of arbitrarily given fixed positive energy.

These two problems are known to be equivalent, and are solved affimatively when $n \geq 3$ by Sylvester-Uhlmann [SyUh], Nachman [Na1] and Khenkin-Novikov [KheNo].

Essentially only one method has been used so far for solving IBVP and ISP. In IBVP it is called the *method of complex geometrical optics*, or *exponentially growing solution*, and in ISP it is called *Faddeev's Green function.* This latter has the following form

$$(2\pi)^{-n}\int_{\mathbf{R}^n}\frac{e^{i(x-y)\cdot\xi}}{\xi^2+2z\gamma\cdot\xi-\lambda^2}d\xi, \tag{4}$$

whose important feature is that it contains an artificial direction $\gamma\in S^{n-1}$ and that it is analytic with respect to $z\in\mathbf{C}_+=\{z\in\mathbf{C};\operatorname{Im}z>0\}$.

Recently a new method for solving the inverse problem has been proposed in [Is2], which uses the hyperbolic manifold as a tool. Lets us explain the basic deas.

1.2. **The hyperbolic space approach.** Let Ω be a bounded domain in $\mathbf{R}^n, n\geq 2$, with smooth boundary. Suppose we are given the boundary value problem (1) for the Schrödinger equation. Without loss of generality, we can assume that

$$\overline{\Omega}\subset\mathbf{R}^n_+=\{(x,x_n);x_n>0\}. \tag{5}$$

1st step. As the first step, let us notice that :

IBVP in the Euclidean space and that in the hyperbolic space are equivalent.

This can be easily observed in the 2-dimensional case. In fact by multiplying the Schrödinger equation

$$-\Delta u+qu=0$$

in $\mathbf{R}^2$ by x_2^2, we have

$$-x_2^2\Delta u+x_2^2qu=0,$$

which is just the Schrödinger equation in $\mathbf{H}^2$. Therefore the D-N maps $\widetilde{\Lambda}_q$ in $\mathbf{R}^2$ and $\Lambda_{x_2^2q}$ in $\mathbf{H}^2$ are related as follows

$$\widetilde{\Lambda}_q=x_2\Lambda_{x_2^2q}.$$

If $n\geq 3$, putting $u=x_n^{(2-n)/2}v$, we are led to the equation

$$(-x_n^2\partial_n^2+(n-2)x_n\partial_n-x_n^2\Delta_x+V)v=0, \tag{6}$$

where $V=x_n^2q-n(n-2)/4$, and $\partial_n=\partial/\partial x_n$. Note that

$$\Delta_g=x_n^2\partial_n^2-(n-2)x_n\partial_n+x_n^2\Delta_x$$

is the Laplace-Beltrami operator on the hyperbolic space $\mathbf{H}^n$ realized in the upper half space $\mathbf{R}^n_+$. Therefore the Dirichlet problem (1) in a domain $\Omega\subset\mathbf{R}^n$ is equivalent to (6) in $\Omega\subset\mathbf{H}^n$.

2nd step. The next step is to use the gauge transformation $v=e^{i\theta\cdot x}u$ to introduce a parameter θ in the above equation. Then we get the following equation

$$(-x_n^2\partial_n^2+(n-2)x_n\partial_n-x_n^2(\partial_x+i\theta)^2+V)u=0 \tag{7}$$

in $\Omega\subset\mathbf{H}^n$, $\theta\in\mathbf{R}^{n-1}$.

3rd step. In the 3rd step, we consider the action of simple discrete groups. We take a sufficiently large lattice Γ of rank $n-1$ in $\mathbf{R}^{n-1}$ so that Ω is contained in one coordinate patch of the quotient space $\Gamma\backslash\mathbf{H}^n$. Then the above equation (7) can be regarded as that on a domain in $\Gamma\backslash\mathbf{H}^n$. Here one should note that the operator

$$H_0(\theta)=-x_n^2\partial_n^2+(n-2)x_n\partial_n-x_n^2(\partial_x+i\theta)^2 \tag{8}$$

is just the Floquet operator in the theory of periodic Schrödinger equation.

4th step. IBVP and ISP are also equivalent on the hyperbolic manifold $\Gamma\backslash\mathbf{H}^n$. Hence, we can construct the scattering amplitude for the Floquet operator from the D-N map. By passing to the Fourier series, the Green's function of the Floquet operator is written by modified Bessel functions, $K_{i\sigma}(\zeta x_n)$, $I_{i\sigma}(\zeta x_n)$, $\zeta = \sqrt{(\gamma^* + \theta)^2}$, where γ^* varies over the dual lattice of Γ. They are analytic with respect to θ for a suitable choice of the imaginary part of θ. (Let us remark that here we are taking the branch of $\sqrt{\cdot}$ in such a way that $\mathrm{Re}\sqrt{\cdot} \geq 0$ with cut along the negative real axis.) Therefore the scattering amplitude for the perturbed Floquet operator is also analytic with respect to θ.

5th step. We use the complex Born approximation. Putting $\theta = z\alpha$ for a suitable $\alpha \in \mathbf{R}^{n-1}$ and letting z tend to infinity along the imaginary axis, one can recover

$$\int e^{-ik\cdot x} e^{-itz} V(x, x_n) dx dx_n, \tag{9}$$

for $n \geq 3$, and

$$\int e^{-ik\cdot x_1} e^{-|k|x_2} V(x_1, x_2) dx_1 dx_2, \tag{10}$$

for $n = 2$ from the scattering amplitude. If $n \geq 3$, one can then recover q.

The above arguments in particular imply the following theorem.

Theorem 1.1. *Let $n \geq 3$, and Ω a contractible relatively compact open set in $\mathbf{H}^n$ with smooth boundary. Suppose 0 is not a Dirichlet eigenvalue of $-\Delta_g + V$. Then V is uniquely reconstructed from the D-N map.*

We are also interested in the inverse spectral problem on general hyperolic manifolds. Recall that any hyperbolic manifold is realized as $\Gamma\backslash\mathbf{H}^n$ for a discrete subgroup Γ of isometries on $\mathbf{H}^n$. By passing to the universal covering, to pick a bounded open contractible set Ω in $\Gamma\backslash\mathbf{H}^n$ means to take a bounded open set Ω in $\mathbf{R}^n_+$. Therefore Theorem 1.1 also holds with $\mathbf{H}^n$ replaced by any n-dimensional hyperbolic manifold.

Our next concern is the inverse scattering problem. Let us try to solve it by showing the equivalence of the knowledge of the scattering amplitude and that of the D-N map. However it depends on the structure of infinity. Consider the simplest case that Γ is the lattice of rank $n-1$ in $\mathbf{R}^{n-1}$. Then there are two infinities of $\Gamma\backslash\mathbf{H}^n$, at $x_n = 0$ and at $x_n = \infty$. The former is called the *regular infinity* and the latter the *cusp.*

Now let $\mathcal{M}$ be an n-dimensional connected Riemannian manifold having the following structure : $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_\infty$, where $\overline{\mathcal{M}_0}$ is compact, and $\mathcal{M}_\infty$ is diffeomorphic to $\mathbf{E} \times (0,1)$, $\mathbf{E} = \mathbf{R}^{n-1}/\Gamma$, Γ being a lattice of rank $n-1$ in $\mathbf{R}^{n-1}$. We assume that the Riemannian metric g of $\mathcal{M}$, when restricted to $\mathcal{M}_\infty$ is equal to that on $\Gamma\backslash\mathbf{H}^n$. We consider the Schrödinger operator

$$H = -\Delta_g + A, \tag{11}$$

where A is a formally self-adjoint 2nd order differential operator. We assume that for $j = 1, 2$ the coefficients of j-th covariant derivatives are in C^j, and that the multiplication operator term is bounded. Moreover we assume the following.

The supports of the coefficients of A are contained in a bounded contractible set Ω in $\mathcal{M}$.

By observing the asymptotic behavior at regular infinity of solutions to the Schrödinger equation $(H - \lambda)\psi = 0$ representing the scattering phenomena (more precisely by observing the asymptotic behavior of the resolvent at regular infinity), one can introduce the scattering amplitude. One can then show that

Theorem 1.2. *Let $n \geq 2$. Then from the scattering amplitude at the regular infinity we can construct the D-N map on Ω and vice versa.*

Of course this theorem holds when $\mathcal{M} = \mathbf{H}^n$. Using this theorem and the results already established for the inverse problem for the metric (see e.g. [LaTaUh] and the references therein), one can argue the reconstruction of the metric or the first or the zeroth order perturbations of $-\Delta_g$ from the scattering amplitude. The cusp requires a dfferent formulation. We shall elucidate the results for the cusp case in the next section.

1.3. **Floquet operators.** Let us compare the above approach with the method based on the Green function of Faddeev. Let $R_0(z)$ be the resolvent of $-\Delta$ in $\mathbf{R}^n$. Then for $t \in \mathbf{R}$ and $\gamma \in S^{n-1}$, the gauge transformed resolvent $e^{-it\gamma\cdot x}R_0(E+i\epsilon)e^{it\gamma\cdot x}$ is written as

$$e^{-it\gamma\cdot x}R_0(E+i\epsilon)e^{it\gamma\cdot x}f = (2\pi)^{-n}\iint_{\mathbf{R}^n}\frac{e^{i(x-y)\cdot\xi}}{(\xi+t\gamma)^2 - E - i\epsilon}f(y)d\xi dy. \tag{12}$$

If we let formally $\epsilon \to 0$ in (12), we get the expression (4) with $z = t$ and $\lambda^2 = E - t^2$. However the Green function (4) can not be obtained in this manner. In fact, if it were true, letting $G_{\gamma,0}(\lambda, t)$ be the operator having (4) as the integral kernel, the gauge transformed operator $\widetilde{R_{\gamma,0}}(\lambda,t) = e^{it\gamma\cdot x}G_{\gamma,0}(\lambda,t)e^{-it\gamma\cdot x}$ would be the outgoing resolvent. But as is shown in (4.2) of [Is1], it is outgoing in a half space of momentum and incoming in the opposite half space. Namely we have

$$\widetilde{R}_{\gamma,0}(\lambda,t) = R_0(E-i0)M_\gamma^{(+)}(t) + R_0(E+i0)M_\gamma^{(-)}(t),$$

where $E = \lambda^2 + t^2$ and

$$M_\gamma^{(\pm)}(t) = (\mathcal{F}_{x\to\xi})^{-1}F(\pm\gamma\cdot(\xi - t\gamma) \geq 0)\mathcal{F}_{x\to\xi},$$

$\mathcal{F}_{x\to\xi}$ being the Fourier transformation and $F(\cdots)$ the characteristic funtion of the set $\{\cdots\}$. In Faddeev's approach of inverse scattering, one constructs the scattering amplitude different from the physical one by using this direction dependent Green operator, which turns out to satisfy an integral equation having the usual scattering amplitude as input ([Fa], [Is1]).

Next let us consider the same problem in the flat torus $S^1 \times \mathbf{R}^1$. We expand the resolvent of the Floquet operator into the Fourier series. Then the part projected to e^{inx} is written as

$$\begin{aligned}(2\pi)^{-1}&\iint\frac{e^{i(y-y')\xi}}{|\xi|^2 - (E+i\epsilon-(n+\theta)^2)}\hat{f}_n(y')d\xi dy' \\ &= \frac{i}{2\sqrt{E+i\epsilon-(n+\theta)^2}}\int_{-\infty}^{\infty}e^{i\sqrt{E+i\epsilon-(n+\theta)^2}|y-y'|}\hat{f}_n(y')dy.\end{aligned} \tag{13}$$

Here $\hat{f}_n(y)$ is the Fourier coefficient of $f(x,y)$ with respect to x and the branch of $\sqrt{\cdot}$ is taken in such a way that $\mathrm{Im}\sqrt{\cdot} \geq 0$ with cut along the positive real axis. One then observes the same phenomenon as in the case of $\mathbf{R}^n$. In fact let $\epsilon \to 0$ in the above expression and define the operator $G_0^{(n)}(E,\theta)$ by the right-hand side of (13) for complex θ. When θ approaches to 0 along the positive imaginary axis, $G_0^{(n)}(E,0)$ is outgoing for $n > 0$ and incoming for $n < 0$. Therefore this has a property similar to that of the Faddeev Green operator on $\mathbf{R}^n$.

This is no longer the case when we pass to the hyperbolic quotient space $\Gamma\backslash\mathbf{H}^n$, where Γ is the lattice of rank $n-1$ in $\mathbf{R}^{n-1}$. In fact, letting $y = \log x_n$ and passing to the Fourier series in x, we are led to consider the equation

$$(-\partial_y^2 + e^{2y}(\gamma^* + \theta)^2 - \sigma^2)u = f. \tag{14}$$

The outgoing resolvent can be written by modified Bessel functions, and it has always a nice analyticity property with respect to θ.

The main barrier for the multi-dimensional inverse scattering is the existence of *exceptional points*. They are the points z for which $\widetilde{R_{\gamma,0}}(\lambda, z)V$ has -1 as an eigenvalue, namely the points where the perturbed Green operator

$$(1 + \widetilde{R_{\gamma,0}}(\lambda, z)V)^{-1}\widetilde{R_{\gamma,0}}(\lambda, z)$$

does not exist (see e.g. 3.3 of [Is1]). Eskin-Ralston ([EsRa]) overcame this difficulty by introducing a new Green's function slightly different from that of Faddeev and employing a family of scattering amplitudes as the spectral data. Our approach is similar to Eskin-Ralston's one in that we adopt the family of scattering amplitudes of Floquet operators as the spectral data. In short, in our hyperbolic space approach, the role of the artificial direction γ of the Faddeev Green operator is played by the Floquet parameter θ varying over the fundamental domain of the dual lattice of Γ.

There are so many articles on the forward and inverse spectral problems on Riemannian manifolds that we quote here only those related to the continuous spectrum of hyperbolic manifolds. Lax-Phillips studied the scattering problem for the wave equation on hyperbolic manifolds, and Agmon [Ag] applied modern techniques of scattering theory to study the Laplacian related to number theory. In particular he derived the analytic continuation of the Eisenstein series from that of the resolvent. More general analytic continuation result was obtained by Mazzeo-Melrose [MaMe]. The problem of embedded eigenvalues was studied in a general setting by Mazzeo. The distibution of resonances and the asymptotics of scattering phase were computed by Guillopé and Zworski [GuZwo]. Melrose-Zworski, Perry and Hislop have shown that the scattering matrices are written down by pseudo-diffrential operators. Joshi and Sá Barreto [JoSaBa] investigated the symbol of this pseudo-differential operator and derived the asymptotics at infinity of perturbations from the scattering matrix at a fixed energy. Our approach is different from this work in that we are trying to recover the total perturbation from the knowledge of the scattering matrix.

2. Inverse scattering at the cusp

2.1. Arithmetic surface. The inverse spectral problem on the hyperbolic manifold depends largely on the structure of infinity. For example, the Laplace-Beltrami operator $-\Delta_g$ of the arithmetic surface $SL(2, \mathbf{Z})\backslash\mathbf{H}^2$ has the continuous spectrum $[1/4, \infty)$ with imbeded eigenvalues. The generalized eigenfunction associated with the spectrum $\lambda > 1/4$, the Maass wave form, has the following asymptotic expansion

$$\psi_\lambda(z) \sim x_2^s + \frac{B(1-s)}{B(s)}x_2^{1-s}, \quad \text{as} \quad x_2 \to \infty \tag{15}$$

where $z = x_1 + ix_2$, $s = 1/2 + i\sqrt{\lambda}$ and $B(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ (see e.g. [Te] p. 253). This means that when one fixes the energy, the S-matrix is a constant and that one can not expect to reconstruct the perturbation from the S-matrix of one fixed energy. This is because the continuous spectrum of $-\Delta_g$ is one-dimensional. In fact, the infinity of the arithmetic surface is at $x_2 = \infty$, and $-\Delta_g$ can be regarded as a compact perturbation of $-x_2^2(\partial_{x_1}^2 + \partial_{x_2}^2)$ on $(-1/2, 1/2) \times (2, \infty)$ with suitable boundary condition. If $-\partial_{x_1}^2$ is expanded into a Fourier series, the continuous spectrum arises only from the mode $n = 0$.

2.2. Inverse scattering at the cusp. Let $\mathcal{M}$ be an n-dimensional connected Riemannian manifold. Suppose $\mathcal{M}$ consists of two parts : $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_\infty$, where $\overline{\mathcal{M}_0}$ is compact, and $\mathcal{M}_\infty$ is diffeomorphic to $\mathbf{E} \times (1, \infty)$, $\mathbf{E} = \Gamma\backslash\mathbf{R}^{n-1}$, Γ being a lattice of rank $n-1$ in $\mathbf{R}^{n-1}$. We assume that the Riemannian metric g of $\mathcal{M}$, when restricted to $\mathcal{M}_\infty$, takes the following form :

$$g\big|_{\mathcal{M}_\infty} = (dy)^2 + e^{-2y}(dx)^2, \tag{16}$$

where $y \in (1, \infty)$ and $(dx)^2$ is the flat metric on $\mathbf{E}$. We consider the Schrödinger operator

$$H = -\Delta_g + A, \tag{17}$$

where A is a formally self-adjoint 2nd order differential operator. We assume that for $j = 1, 2$ the coefficients of j-th covariant derivatives are in C^j, and that the multiplication operator term is bounded. Moreover we assume that the supports of the coefficients of A are contained in a bounded contractible set Ω in $\mathcal{M}$. One can then construct a solution of the Schrödinger equation $(H - \lambda)\psi = 0$ which grows up exponentially at the cusp. By looking at the behavior of this solution at the cusp, one can define an analogue of the scattering amplitude $\mathbf{A}_c(\lambda)$.

Take a bounded contractible domain $\Omega \subset \mathcal{M}$ such that $A = 0$ outside Ω, and define the D-N map $\Lambda(A)$ for $H_D = -\Delta_g + A$ in Ω with Dirichlet boundary condition. Then we can show

Theorem 2.1. *Suppose $\lambda \notin \sigma_p(H) \cup \sigma_p(-\Delta_g) \cup \sigma_p(H_D)$. Then the scattering amplitude at the cusp $\mathbf{A}_c(\lambda)$ and the D-N map $\Lambda(A)$ determine each other.*

2.3. **Reconstruction of the metric.** Now let us look at briefly the inverse problem for the local perturbation of the metric. The basic examples in mind are $\mathbf{H}^n$ as the upper half space model, $\Gamma\backslash\mathbf{H}^n$ where Γ is the lattice of rank $n-1$ in $\mathbf{R}^{n-1}$, and $SL(2,\mathbf{Z})\backslash\mathbf{H}^2$.

First we consider the conformal deformation of the hyperbolic metric. Let $\mathcal{M}$ be one of the above hyperbolic manifolds. Suppose that the metric is deformed into $ds^2 = \rho(x)\sum_{i=1}^n (dx_i)^2$, where $\rho(x) = x_n^{-2}$ outside a compact set $K \subset \mathcal{M}$. We assume that there is a bounded contractible open set $\Omega \subset \mathcal{M}$ such that $K \subset \Omega$. For $\lambda \in \mathbf{R}$, consider the boundary value problem

$$\begin{cases} (-\Delta_g - \lambda)u = 0 & \text{in} \quad \Omega, \\ \quad u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{18}$$

This is rewritten as

$$-\nabla(\rho^{(n-2)/2}\nabla u) - \lambda\rho^{n/2}u = 0 \quad \text{in} \quad \Omega. \tag{19}$$

Letting $u = \rho^{(2-n)/4}v$, we have

$$-\Delta v + \left(\frac{\Delta\rho^\alpha}{\rho^\alpha} - \lambda\rho\right)v = 0 \quad \text{in} \quad \Omega, \tag{20}$$

where $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$ and $\alpha = (n-2)/4$. Let us recall that Ω is now identified with an open set in $\mathbf{R}^n_+$.

If $n \geq 3$, one can uniquely reconstruct $q = (\Delta\rho^\alpha)/\rho^\alpha - \lambda\rho$ from the knowledge of the D-N map on Ω. To recover ρ from q, letting $\varphi = \rho^{(n-2)/4}$, one must solve the non-linear equation

$$\begin{cases} (-\Delta + q)\varphi = -\lambda\varphi^{(n+2)/(n-2)} & \text{in} \quad \Omega, \\ \quad \varphi = x_n^{-(n-2)/2} \quad |\alpha| \leq 1 & \text{on} \quad \partial\Omega. \end{cases} \tag{21}$$

This equation has a unique positive solution. In fact, we have the following theorem.

Theorem 2.2. *Let $n \geq 2, p > 1$. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary. Let $\lambda > 0$ and $q(x) \in L^\infty(\Omega)$ be real-valued. Take $\varphi(x) > 0$ from $C^{2,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$. Then there exists a unique positive solution of the boundary value problem*

$$\begin{cases} -\Delta u + qu = -\lambda u^p & \text{in} \quad \Omega, \\ \quad u = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

For $\mathbf{H}^n$ or $\Gamma\backslash\mathbf{H}^n$, the D-N map and the scattering amplitude determine each other. (For $\Gamma\backslash\mathbf{H}^n$, in addition to the scattering amplitude at the regular infinity, we must take into account of the contribution from the one-dimensional continuous spectrum arising from the cusp). Therefore on these manifolds, the local conformal deformation of the metric is reconstructed from the scattering amplitudes. By Theorem 2.1, one can derive the same conclusion for the manifold whose infinity is the cusp. Thurston [Thu] gave such an example of 3-dimensional hyperbolic manifold.

When $n = 2$, the inverse boundary value problem for (20) has not been solved yet except for the cases of generic or small perturbations. One remedy is to consider the negative energy $\lambda < 0$. In this case one can construct a positive function $c(x)$ such that

$$\left\{\begin{array}{ll} \dfrac{\Delta\sqrt{c}}{\sqrt{c}} = -\lambda\rho & \text{in} \quad \Omega, \\ c = 1 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{22}$$

Using this $c(x)$, one can convert the boundary value problem (19) to the conductivity problem

$$\nabla(c\nabla u) = 0 \quad \text{on} \quad \Omega. \tag{23}$$

The inverse boundary value problem for (23) was solved by Nachman [Na2].

Now let us remark that one can construct the scattering amplitude at the cusp for the negative energy in the same way as above and Theorem 2.1 also holds for this case. Therefore one can determine the local conformal perturbation of the metric from the scattering amplitude at the cusp for negative energy.

Let us finally consider the general deformation of the metric. Let us assume that we know a-priori the perturbation is done only on a compact set K, and also suppose that K is contained in a bounded contractible open set Ω. Fix $\lambda > 0$ and consider the Schrödinger operator

$$H = -\Delta_g + \lambda\chi_\Omega, \tag{24}$$

where χ_Ω is the characteristic function of Ω, and Δ_g is the Laplace-Beltrami operator for the perturbed metric. As above, the knowledge of the scattering amplitude at a regular infinity or cusp determines the D-N map for $H - \lambda$ on Ω, which turns out to be the D-N map of $-\Delta_g$. If $n \geq 3$, one can reconstruct the perturbed metric by using the results of Lee-Uhlmann [LeUh] or Lassas, Taylor and Uhlmann [LaTaUh]. If $n = 2$, by using the result of Nachman [Na2] one can reconstruct $\sqrt{\det(g_{ij})}g^{ij}$. For two metrics g and $\overline{g}$, $\sqrt{\det(g_{ij})}g^{ij} = \sqrt{\det(\overline{g}_{ij})}\overline{g}^{ij}$ is equivalent to that g and $\overline{g}$ are conformal. Therefore the scattering amplitudes associated with two metrics g and $\overline{g}$ coincide if and only if g and $\overline{g}$ are conformal. Let us remark that in 2-dimensions there is a difference between the conductivity problem and the Laplace-Beltrami operator, since the latter is conformally invariant. Therefore the best we can expect is to reconstruct the conformal class of the metric. One can also deal with the case of many cusps.

Part 2. The $\overline{\partial}$-theory

3. The $\overline{\partial}$-equation in the inverse scattering problem

For the Schrödinger operator in $\mathbf{R}^n$, the scattering amplitude $\widetilde{A}(E;\theta,\omega)$ is observed from the asymptotic behavior of the solution to the Schrödinger equation

$$(-\Delta + V(x))\varphi = E\varphi \tag{25}$$

in the following manner :

$$\varphi(x; E, \omega) \sim e^{i\sqrt{E}\omega\cdot x} + \widetilde{C}_E\frac{e^{i\sqrt{E}r}}{r^{(n-1)/2}}\widetilde{A}(E;\theta,\omega) \tag{26}$$

as $r = |x| \to \infty, \theta = x/r$. This φ is obtained by solving the Lippman-Schwinger equation :

$$\varphi(x) = e^{i\sqrt{E}\omega\cdot x} - \int_{\mathbf{R}^n} G_0(x-y, E)V(y)\varphi(y)dy, \tag{27}$$

where $G_0(x,E)$ is the Green function for $-\Delta - E$ defined by

$$G_0(x,E) = (2\pi)^{-n}\int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{\xi^2 - E - i0} d\xi. \tag{28}$$

Here and in the sequel for $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n$, we denote $\zeta^2 = \sum_{i=1}^n \zeta_i^2$.

The inverse problem for the Schrödinger operator aims at constructing $V(x)$ from the scattering amplitude. When $n = 1$, the well-known theory of Gel'fand-Levitan-Marchenko provides us with the necessary and sufficient condition for a function $A(E)$ to be the scattering amplitude of a Schrödinger operator and an algorithm for the reconstruction of $V(x)$.

The multi-dimensional inverse problem has not been solved yet completely as in the 1-dimensional case. The main difficulty arises from the overdeterminacy ; the scattering amplitude $\tilde{A}(E;\theta,\omega)$ is a function of $2n-1$ parameters while the potential $V(x)$ depends on n variables. Therefore for a function $f(E,\theta,\omega)$ on $(0,\infty)\times S^{n-1}\times S^{n-1}$ to be the scattering amplitude associated with a Schrödinger operator, f must satisfy a sort of compatibililty conditon, which is still unknown. However, there is a series of deep results related to inverse problems in multi-dimensions, the main idea of which consists in using exponentially growing solutions for the Schrödinger equation (25). In the inverse scattering problem, it is commonly called the $\overline{\partial}$-theory ([Na1], [Na2], [KheNo]), although the pioneering work of Faddeev [Fa] does not bear this term.

In the $\overline{\partial}$-approach of inverse scattering, instead of $\widetilde{A}(E;\theta,\omega)$, one uses Faddeev's scattering amplitude :

$$A(\xi,\zeta) = \int_{\mathbf{R}^n} e^{-ix\cdot(\xi+\zeta)} V(x)\psi(x,\zeta)dx, \quad \xi \in \mathbf{R}^n, \quad \zeta \in \mathbf{C}^n \tag{29}$$

where $\zeta^2 = E$, and $\psi(x,\zeta)$ is a solution to the equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbf{R}^n} G(x-y,\zeta)V(y)\psi(y,\zeta)dy, \tag{30}$$

$G(x,\zeta)$ being Faddeev's Green function defined by

$$G(x,\zeta) = (2\pi)^{-n}\int_{\mathbf{R}^n} \frac{e^{ix\cdot(\xi+\zeta)}}{\xi^2 + 2\zeta\cdot\xi} d\xi. \tag{31}$$

This function $A(\xi,\zeta)$ has the following features :

(*i*) It is natural to regard $A(\xi,\zeta)$ as a function on the fiber bundle $\mathcal{M} = \cup_\xi \{\xi\} \times \mathcal{V}_\xi$, where ξ varies over the base space $\mathbf{R}^n$ and the fiber $\mathcal{V}_\xi$ is defined by

$$\mathcal{V}_\xi = \{\zeta \in \mathbf{C}^n; \zeta^2 = E, \xi^2 + 2\zeta\cdot\xi = 0\}. \tag{32}$$

As a 1-form on $\mathcal{M}$, it satisfies a $\overline{\partial}$-equation

$$\overline{\partial}_\zeta A(\xi,\zeta) = -(2\pi)^{1-n}\int_{\mathbf{R}^n} A(\xi-\eta,\zeta+\eta)A(\eta,\zeta)\eta\delta(\eta^2 + 2\zeta\cdot\eta)d\eta. \tag{33}$$

(*ii*) When $n \geq 3$, the Fourier transform of the potential V is recoverd from $A(\xi,\zeta)$ in the following way :

$$\hat{V}(\xi) = (2\pi)^{-n/2} \lim_{|\zeta|\to\infty, \zeta\in\mathcal{V}_\xi} A(\xi,\zeta). \tag{34}$$

Consequently, by virtue of a generalization of Bochner-Martinelli's formula on $\mathcal{V}_\xi$, we have an integral representation of $V(x)$ in terms of $A(\xi,\zeta)$.

(*iii*) The $\overline{\partial}$-equation characterizes the Faddeev scattering amplitude. Namely, the equation (33) is a necessary and sufficient condition for a function $A(\xi,\zeta)$ on the fiber bundle $\mathcal{M}$ to be the scattering amplitude associated with a Schrödinger operator on $\mathbf{R}^n$.

These ideas have been found and confirmed in various levels. For the details see [NaAb], [BeCo], [Na1], and especially the introduction of [KheNo]. We show a generalization of these results to the case of $\mathbf{H}^3$

4. Green operators

Let us construct a Green operator of

$$H_0(\theta) = y^2(-\partial_y^2 + (-i\partial_x + \theta)^2) + (n-2)y\partial_y. \tag{35}$$

For $\theta, \theta' \in \mathbf{C}^{n-1}$, we put

$$\theta \cdot \theta' = \sum_{i=1}^{n-1} \theta_i \theta_i', \quad \theta^2 = \theta \cdot \theta,$$

and define for $\xi \in \mathbf{R}^{n-1}$

$$\zeta(\xi, \theta) = \sqrt{(\xi + \theta)^2}, \tag{36}$$

where we take the branch of $\sqrt{\cdot}$ such that $\mathrm{Re}\sqrt{\cdot} \geq 0$, i.e. $\sqrt{z} = \sqrt{r}e^{i\varphi/2}$ for $-\pi < \varphi < \pi$. Let I_ν and K_ν be the modified Bessel functions of order ν. We put

$$G_0(y, y'; \zeta) = \begin{cases} (yy')K_\nu(\zeta y)I_\nu(\zeta y'), & y > y' > 0, \\ (yy')I_\nu(\zeta y)K_\nu(\zeta y'), & y' > y > 0, \end{cases} \tag{37}$$

and define the 1-dimensional Green operator by

$$G_0(\zeta)f(y) = \int_0^\infty G_0(y, y'; \zeta) f(y') \frac{dy'}{(y')^n}. \tag{38}$$

The n-dimensional Green operator is then defined by

$$\mathbf{G}_0(\theta)f(x, y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi} \left(G_0(\zeta(\xi, \theta))\hat{f}(\xi, \cdot)\right)(y) d\xi, \tag{39}$$

$$\hat{f}(\xi, y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} f(x, y) dx. \tag{40}$$

Let us remark that when $\theta \in \mathbf{R}^{n-1}$ and $\nu = i\sigma$ with $\sigma > 0$ (or $\sigma < 0$), $\mathbf{G}_0(\theta)$ is the incoming (or outgoing) Green operator of $H_0(\theta) - E$:

$$\mathbf{G}_0(\theta) = (H_0(\theta) - (E \mp i0))^{-1}, \tag{41}$$

where the right-hand side exists on a certain Banach space.

4.1. $\overline{\partial}$-equation. For $\theta = \theta_R + i\theta_I \in \mathbf{C}^{n-1}$, let $\overline{\partial_\theta}$ be defined as follows :

$$\overline{\partial_\theta} = \left(\frac{\partial}{\partial\overline{\theta}_1}, \cdots, \frac{\partial}{\partial\overline{\theta}_{n-1}}\right), \quad \frac{\partial}{\partial\overline{\theta}_j} = \frac{1}{2}\left(\frac{\partial}{\partial\theta_{Rj}} + i\frac{\partial}{\partial\theta_{Ij}}\right). \tag{42}$$

We are going to compute $\overline{\partial_\theta}\mathbf{G}_0(\theta)$. Note that if $f(z)$ is analytic, $f(\zeta(\xi, \theta))$ has singularities on the set $\{\theta \in \mathbf{C}^{n-1}\,;\, (\xi + \theta)^2 \leq 0\}$. The crucial lemma is the following.

Lemma 4.1. *Let $f(z)$ be an analytic function on $\{z \in \mathbf{C}\,;\, \mathrm{Re}\, z > 0\}$ satisfying*

$$\sup_{|z|<r} |f(z)| < \infty, \quad \forall r > 0.$$

For $\theta = \theta_R + i\theta_I \in \mathbf{C}^{n-1}$ such that $\theta_I \neq 0$ we put

$$r_\theta(\xi) = \sqrt{|\theta_I|^2 - |\xi + \theta_R|^2}, \tag{43}$$

$$M_\theta = \left\{\xi \in \mathbf{R}^{n-1}\,;\, \theta_I \cdot (\xi + \theta_R) = 0,\, |\xi + \theta_R| < |\theta_I|\right\}, \tag{44}$$

and define a compactly supported distribution $T_\theta(\xi)$ by

$$\langle T_\theta(\xi), \varphi(\xi)\rangle = \int_{M_\theta} \varphi(\xi)\frac{i(\xi+\overline{\theta})}{2|\theta_I|}dM_\theta(\xi), \quad \forall\varphi \in C^\infty(\mathbf{R}^{n-1}), \tag{45}$$

$dM_\theta(\xi)$ being the measure on M_θ induced from the Lebesgue measure $d\xi$ on $\mathbf{R}^{n-1}$. Then regarding $f(\zeta(\xi,\theta))$ as a distribution with respect to $\xi \in \mathbf{R}^{n-1}$ depending on a parameter $\theta \in \mathbf{C}^{n-1}$, we have for $\theta_I \neq 0$

$$\overline{\partial_\theta}\, f(\zeta(\xi,\theta)) = [f(ir_\theta(\xi)) - f(-ir_\theta(\xi))]\, T_\theta(\xi). \tag{46}$$

With the aid of this lemma and the well-known relation

$$I_\nu(ir) = e^{\nu\pi i} I_\nu(-ir) = e^{\nu\pi i/2} J_\nu(r), \tag{47}$$

$$K_\nu(ir) = e^{-\nu\pi i} K_\nu(-ir) - \pi i I_\nu(-ir), \tag{48}$$

J_ν being the Bessel function of order ν, one can show that the Green operator $\mathbf{G}_0(\theta)$ satisfies the following equation.

Theorem 4.2. *For $f \in C_0^\infty(\mathbf{H}^n)$, we have*

$$\begin{aligned} \overline{\partial}_\theta\, \mathbf{G}_0(\theta) f &= -\frac{\pi i}{(2\pi)^{(n-1)/2}} \iint_{M_\theta\times(0,\infty)} e^{ix\cdot k}\,(yy')^{(n-1)/2} \\ &\quad \cdot J_\nu(r_\theta(k)y) J_\nu(r_\theta(k)y')\hat{f}(k,y')\frac{i\,(k+\overline{\theta})}{2|\theta_I|}\frac{dM_\theta(k)dy'}{(y')^n}. \end{aligned}$$

4.2. **Perturbed Green operator.** From now on we restrict the space dimension to 3. For $s > 0$, we introduced the function space $\mathcal{W}_s^{(\pm)}$ by

$$\mathcal{W}_s^{(-)} \ni u \Longleftrightarrow \int_{\mathbf{R}^3_+} \frac{y}{(1+|\log y|)^{2s}}|u(x,y)|^2\frac{dxdy}{y^3} < \infty, \tag{49}$$

$$\mathcal{W}_s^{(+)} \ni f \Longleftrightarrow \int_{\mathbf{R}^3_+} \frac{(1+|\log y|)^{2s}}{y}(1+|x|)^{2s}|f(x,y)|^2\frac{dxdy}{y^3} < \infty. \tag{50}$$

Suppose that V satisfies

$$|V(x,y)| \leq C(1+|x|)^{-2s}(1+|\log y|)^{-2s}(1+y)^{-2}y \tag{51}$$

for some $s > 1$. Then we have the following theorem.

Theorem 4.3. *Let $\mathbf{G}_V(\theta)$ be defined by*

$$\mathbf{G}_V(\theta) = (1+\mathbf{G}_0(\theta)V)^{-1}\mathbf{G}_0(\theta)$$

for sufficiently large $|\theta_I|$. Then there exists a constant $C_s > 0$ such that

$$\|\mathbf{G}_V(\theta)\|_{\mathbf{B}(\mathcal{W}_s^{(+)};\mathcal{W}_s^{(-)})} \leq C_s\left(\frac{\log\tau}{\tau}\right)^{1/2}, \quad |\theta_I| > C_s.$$

Lemma 4.4. *The following equalities hold :*

$$\begin{aligned} \overline{\partial_\theta}\, \mathbf{G}_V(\theta) &= (1+\mathbf{G}_0(\theta)V)^{-1}\left(\overline{\partial_\theta}\,\mathbf{G}_0(\theta)\right)(1-V\mathbf{G}_V(\theta)) \\ &= (1-\mathbf{G}_V(\theta)V)\left(\overline{\partial_\theta}\,\mathbf{G}_0(\theta)\right)(1-V\mathbf{G}_V(\theta)). \end{aligned}$$

5. $\overline{\partial}$-THEORY FOR SCATTERING AMPLITUDES

5.1. **Scattering matrix in quantum mechanics.** The wave function associated with the Schrödinger operator in quantum mechanics on $\mathbf{R}^3$ is a *bounded* solution to the equation $(-\Delta + V(x))\phi = E\phi$. It is also the case for the hyperbolic space $\mathbf{H}^3$. Suppose $\nu = i\sigma$, $\sigma \in \mathbf{R} \setminus \{0\}$. Then the wave function for the equation

$$H\phi := \left[-y^2(\partial_y^2 + \Delta) + y\partial_y + V(x,y)\right]\phi = E\phi \tag{52}$$

is defined as follows. Let for $\eta \in \mathbf{R}^2$

$$\begin{aligned} \phi_0(x,y,\eta) &= e^{ix\cdot\eta} y K_\nu(|\eta| y), \\ \phi(x,y,\eta) &= \phi_0(x,y,\eta) - v, \\ v(x,y,\eta) &= \mathbf{G}_V(0)[V(x,y)\phi_0(x,y,\eta)], \\ E &= 1 - \nu^2. \end{aligned}$$

Then ϕ solves (52), behaves like $e^{ix\cdot\eta}(c_1 y^{(1+i\sigma} + c_2 y^{1-i\sigma})$ as $y \to 0$, and gives an eigenfunction expansion associated with H. By observing the behavior of the Fourier transform of v with respect to x, we get

$$\hat{v}(\xi,y,\eta) \sim (2\pi)^{-1}\left(\frac{|\xi|}{2}\right)^{i\sigma} \frac{y^{i\sigma+1}}{\Gamma(i\sigma+1)} \tilde{A}(\xi,\eta), \quad y \to 0. \tag{53}$$

This $\tilde{A}(\xi,\eta)$ is (after a suitable unitary transformation) the scattering amplitude in the quantum mechanical scattering problem.

5.2. **Exponentially growing solutions.** In the $\overline{\partial}$-approach, contrary to the above quantum mechanical problem, we seek exponentially growing solutions to the equation (52). We put for $\eta \in \mathbf{R}^2$ and $\theta \in \mathbf{C}^2$,

$$\psi_0(x,y;\eta,\theta) = e^{ix\cdot\theta}\Psi_0(x,y;\eta,\theta), \tag{54}$$

$$\Psi_0(x,y;\eta,\theta) = e^{ix\cdot\eta} y I_\nu(\zeta(\eta,\theta)y). \tag{55}$$

It satisfies the Schrödinger equation

$$H_0\psi_0 := \left[-y^2(\partial_y^2 + \Delta_x) + y\partial_y\right]\psi_0 = E\psi_0, \tag{56}$$

and behaves like $e^{ix\cdot(\theta+\eta)}y^{1+\nu}$ as $y \to 0$. Hence if $\theta = 0$ and $y \to 0$, ψ_0 is bounded. However it grows up exponentially as $y \to \infty$.

We seek a solution of the perturbed Schrödinger equation

$$(H_0 + V(x,y))\psi = E\psi, \tag{57}$$

which behaves like ψ_0 at infinity. It is defined as

$$\psi(x,y;\eta,\theta) = \psi_0(x,y;\eta,\theta) - e^{ix\cdot\theta}u, \tag{58}$$

$$u = \mathbf{G}_V(\theta)\left[V(x,y)\Psi_0(x,y;\eta,\theta)\right]. \tag{59}$$

Since $\mathbf{G}_V(\theta) = \mathbf{G}_0(\theta) - \mathbf{G}_0(\theta)V\mathbf{G}_V(\theta)$, by passing to the Fourier transformation with respect to x, we have (at least formally)

$$\hat{u}(\xi,y;\theta) \sim (2\pi)^{-1} y K_\nu(\zeta(\xi,\theta)y) A(\xi,\eta;\theta), \quad y \to \infty, \tag{60}$$

$$\begin{aligned} A(\xi,\eta;\theta) = \; & \int_{\mathbf{R}^3_+} e^{-ix\cdot\xi} y I_\nu(\zeta(\xi,\theta)y) V(x,y)\Psi_0(x,y;\eta,\theta)\tfrac{dxdy}{y^3}, \\ & - \int_{\mathbf{R}^3_+} e^{-ix\cdot\xi} y I_\nu(\zeta(\xi,\theta)y) V(x,y) u(x,y;\eta,\theta)\tfrac{dxdy}{y^3}. \end{aligned} \tag{61}$$

Our scattering amplitude will be defined to be this $A(\xi,\eta;\theta)$.

5.3. **Scattering amplitudes and the $\overline{\partial}$-equation.** The potential $V(x,y)$ is assumed to satisfy the following condition.

There exist $\alpha > 2$ and $\beta > 3/2$ such that for any $N > 0$

$$|V(x,y)| \leq C_N(1+|x|)^{-\alpha}y^{\beta}e^{-Ny} \tag{62}$$

holds on $\mathbf{R}^3_+$ for a constant $C_N > 0$.

We put

$$\Psi_I^{(0)}(x,y;\xi,\theta) = \zeta(\xi,\theta)^{-\nu}e^{ix\cdot\xi}yI_\nu(\zeta(\xi,\theta)y), \tag{63}$$

$$\Psi_I(x,y;\xi,\theta) = \Psi_I^{(0)}(x,y;\xi;\theta) - \left(\mathbf{G}_V(\theta)(V\Psi_I^{(0)}(\xi,\theta))\right)(x,y), \tag{64}$$

$$\Psi_J^{(0)}(x,y;\xi,\theta) = r_\theta(\xi)^{-\nu}e^{ix\cdot\xi}yJ_\nu(r_\theta(\xi)y), \tag{65}$$

$$\Psi_J(x,y;\xi,\theta) = \Psi_J^{(0)}(x,y;\xi;\theta) - \left(\mathbf{G}_V(\theta)(V\Psi_J^{(0)}(\xi,\theta))\right)(x,y), \tag{66}$$

where $\Psi_I^{(0)}(\xi,\theta) = \Psi_I^{(0)}(x,y;\xi,\theta)$, $\Psi_J^{(0)}(\xi,\theta) = \Psi_J^{(0)}(x,y;\xi,\theta)$.

Definition 5.1. *We define the scattering amplitude by*

$$A(\xi,\eta;\theta) = \int_{\mathbf{R}^3_+} \Psi_I^{(0)}(x,y;-\xi,-\theta)V(x,y)\Psi_I(x,y;\eta,\theta)\frac{dxdy}{y^3}. \tag{67}$$

The potential V is reconstructed from this scattering amplitude in the following way.

Theorem 5.2. *Let $\alpha = \theta_I/|\theta_I|$. Suppose $\alpha\cdot(\xi+\theta_R) > 0$, $\alpha\cdot(\eta+\theta_R) > 0$. Then*

$$\lim_{|\theta_I|\to\infty} |\theta_I|^{1+2\nu}A(\xi,\eta;\theta) = \frac{e^{i\nu\pi}}{\pi}\int_{\mathbf{R}^3_+} e^{-ix\cdot(\xi-\eta)}\cosh(ay)V(x,y)\frac{dxdy}{y^2},$$

where $a = \alpha\cdot(\xi-\eta)$.

We next compute $\overline{\partial_\theta}A(\xi,\eta;\theta)$.

Theorem 5.3. *For all $\xi,\eta \in \mathbf{R}^2$, we have*

$$\overline{\partial_\theta}\Psi_I(x,y;\xi,\theta) = -\frac{1}{8\pi}\int_{M_\theta} \Psi_I(x,y;k,\theta)A(k,\xi;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k). \tag{68}$$

$$\overline{\partial_\theta}A(\xi,\eta;\theta) = -\frac{1}{8\pi}\int_{M_\theta} A(\xi,k;\theta)A(k,\eta;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k). \tag{69}$$

5.4. **Integral representation of the potential.** The above $\overline{\partial}$-equation enables us to derive integral representations of the potential $V(x,y)$ in terms of $A(\xi,\eta;\theta)$.

Let $\alpha,\alpha^\perp \in S^1$ be such that $\alpha\cdot\alpha^\perp = 0$. For a sufficiently large constant $T_0 > 0$, let Ω be the set of $\theta = \theta_R + i\theta_I \in \mathbf{C}^2$ satisfying the following condition :

$$|\theta_R| < 1, \quad \alpha\cdot\theta_I > T_0, \quad |\alpha^\perp\cdot\theta_I| < 1. \tag{70}$$

Let us note that for $\theta \in \Omega$

$$\frac{\theta_I}{|\theta_I|} \to \alpha \quad \text{as} \quad |\theta_I| \to \infty. \tag{71}$$

By virtue of the Bochner-Martinell formula and (69), we then have

Theorem 5.4. *Let ξ,η be such that $\theta_I\cdot(\xi+\theta_R)>0$, $\theta_I\cdot(\eta+\theta_R)>0$, $\forall\theta\in\Omega$. Then letting $\theta^{4-2\nu}=(\theta^2)^{2-\nu}$, $K(\theta)=\theta_1 d\theta_2-\theta_2 d\theta_1$, $L(\theta)=d\theta_1\wedge d\theta_2$, and $a=\alpha\cdot(\xi-\eta)$, we have for $\theta_0\in\Omega$,*

$$
\begin{aligned}
&\int_{\mathbf{R}^3_+} e^{-ix\cdot(\xi-\eta)}\cosh(ay)V(x,y)\frac{dxdy}{y^2}\\
&= \frac{e^{-i\nu\pi}}{2}(\theta_0)^{4-2\nu}A(\xi,\eta;\theta_0)\\
&\quad-\frac{e^{-i\nu\pi}}{4}\int_{\partial\Omega}A(\xi,\eta;\theta)\frac{\theta^{4-2\nu}K(\overline{\theta}-\overline{\theta_0})}{|\theta-\theta_0|^4}\wedge L(\theta)\\
&\quad-\frac{e^{-i\nu\pi}}{32\pi}\int_\Omega\left(\int_{M_\theta}A(\xi,k;\theta)A(k,\eta;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k)\right)N(\theta),\\
&N(\theta)=d\overline{\theta}\wedge\frac{\theta^{4-2\nu}K(\overline{\theta}-\overline{\theta_0})}{|\theta-\theta_0|^4}\wedge L(\theta),
\end{aligned}
$$

where the integral is performed in the sense of improper integral.

5.5. **Restriction to lower dimensional submanifolds.** Let us recall that in the Euclidean case, the Faddeev scattering amlitude $A(\xi,\zeta)$ is first defined on a 7-dim. manifold $\mathbf{R}^3\times\{\zeta\in\mathbf{C}^3\,;\,\zeta^2=E\}$, and then restricted to the 5-dim. manifold $\cup_\xi\{\xi\}\times\mathcal{V}_\xi$. In the hyperbolic space case, $A(\xi,\eta;\theta)$ is a function on a 8-dim. manifold $\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{C}^2$. However, noting the formula

$$e^{-ix\cdot k}\mathbf{G}_0(\theta)e^{ix\cdot k}=\mathbf{G}_0(\theta+k),\quad \forall k\in\mathbf{R}^2, \tag{72}$$

and the resulting equation

$$A(\xi-k,\eta-k;\theta+k)=A(\xi,\eta;\theta),\quad \forall k\in\mathbf{R}^2, \tag{73}$$

one can see that $A(\xi,\eta;\theta)$ actually depends on 6 parameters. Let us restrict $A(\xi,\eta;\theta)$ to a 5-dim. manifold.

In the Euclidean case, the fibre $\mathcal{V}_\xi$ defined by (32) has a natural complex structure. The condition $\xi^2+2\zeta\cdot\xi=0$ stems from the singularities of the integrand of the Green function (31). In the hyperbolic space case, the corresponding singularities appear from $\sqrt{(\xi+\theta)^2}$, which gives rise to the condition $\mathrm{Im}\,(\xi+\theta)^2=2\theta_I\cdot(\xi+\theta_R)=0$. Since the set of all θ satisfying this condition is of 3-dimension, we should look for a 2-dim. submanifold for θ. We try a simple choice of $\mathbf{C}\hat{\xi}_\perp$ to be defined below. Note that this set is not included in the above set of singularities.

For $\xi=(\xi_1,\xi_2)\in\mathbf{R}^2\setminus\{0\}$, we put

$$\hat{\xi}_\perp=\left(-\frac{\xi_2}{|\xi|},\frac{\xi_1}{|\xi|}\right) \tag{74}$$

and for $z\in\mathbf{C}$, we define

$$\theta(\xi,z)=z\hat{\xi}_\perp. \tag{75}$$

For $\xi\in\mathbf{R}^2\setminus\{0\}$, $z\in\mathbf{C}$ such that $\mathrm{Re}\,z\neq0$ and $|\mathrm{Im}\,z|$ is sufficiently large, and $k\in M_{\theta(\xi,z)}$, we put

$$B_{II}(\xi,z)=z^{2+2\nu}A\Big(\frac{\xi}{2},-\frac{\xi}{2};\theta(\xi,z)\Big), \tag{76}$$

$$B_{IJ}(\xi,k,z)=z^{2+2\nu}A\Big(\frac{\xi}{2},k;\theta(\xi,z)\Big), \tag{77}$$

$$B_{JI}(k,\xi,z) = z^{2+2\nu} A\Big(k, -\frac{\xi}{2}; \theta(\xi,z)\Big). \tag{78}$$

Since $\mathrm{Re}\, z \neq 0$, $\pm\xi/2 \notin M_{\theta(\xi,z)}$. Note that $B_{II}(\xi,z)$ is a function on (an open set of) the product space $\mathbf{R}^2 \times \mathbf{C}$ and $B_{IJ}(\xi,k,z)$, $B_{JI}(k,\xi,z)$ are functions on (an open set of) the line bundle with base space $\mathbf{R}^2 \times \mathbf{C}$ and fibre $M_{\theta(\xi,z)}$. Or it may be better to regard $\mathbf{R}^2$ as base space and $\mathbf{C}\hat{\xi}_\perp \times M_{\theta(\xi,z)}$ as fibre.

Lemma 5.5. *The following equation holds :*

$$\overline{\partial_z} B_{II}(\xi,z) = \frac{i\,\epsilon(z)}{8\pi z^{2+2\nu}} \int_{M_\theta} B_{IJ}(\xi,k,z) B_{JI}(k,\xi,z) r_\theta(k)^{2\nu} dM_\theta(k),$$

where $\theta = \theta(\xi,z)$ *and* $\epsilon(z) = 1$ *if* $\mathrm{Im}\, z > 0$, $\epsilon(z) = -1$ *if* $\mathrm{Im}\, z < 0$.

Take $T_0 > 0$ large enough and put

$$D = \{z = t + i\tau\,; 1 < t < 2,\ T_0 < \tau < \infty\}. \tag{79}$$

Theorem 5.6. *For* $w \notin \overline{D}$, *we have in the sense of improper integral*

$$e^{i\nu\pi} \int_{\mathbf{R}^3_+} e^{-ix\cdot\xi} V(x,y) \frac{dxdy}{y^2} = \pi i \int_{\partial D} \frac{B_{II}(\xi,z)}{z-w} dz - \frac{1}{8} \int_D F(\xi,z) \frac{dz \wedge d\overline{z}}{z^{2+2\nu}(z-w)},$$

$$F(\xi,z) = \int_{M_\theta} B_{IJ}(\xi,k,z) B_{JI}(k,\xi,z) r_\theta(k)^{2\nu} dM_\theta(k),$$

where $\theta = \theta(\xi,z)$.

5.6. **Radon transform.** Let Π be a 2-dimensional plane orthogonal to $\{y=0\}$, and $d\Pi_E$ be the measure induced on Π from the Euclidean metric $(dx)^2 + (dy)^2$. By Theorem 5.6 one can reconstruct

$$\int_\Pi V(x,y) \frac{d\Pi_E}{y^2} \tag{80}$$

from $B_{II}(\xi; z), B_{IJ}(\xi,k;z), B_{JI}(k,\xi;z)$. Let S be any hemisphere in $\mathbf{R}^3_+$ with center at $\{y=0\}$ and take an isometry on $\mathbf{H}^3$ mapping S to Π. Then from the Faddeev scattering amplitude of $H_\phi = \phi \circ H \circ \phi^{-1}$, one can recover (80). Therefore one can recover $\int_S V(x,y)dS$, dS being the measure on S induced from the hyperbolic metric. If one knows the scattering amplitude $A^{(\phi)}(\xi,\eta;\theta)$ of H_ϕ for all ϕ, one can then reconstruct $V(x,y)$ by virtue of the inverse Radon transform on $\mathbf{H}^3$. For this to be possible, one must be able to compute $A^{(\phi)}(\xi,\eta;\theta)$ for all ϕ from a given Faddeev scattering amplitude. This does not seem to be an obvious problem in general. If V is compactly supported, however, this is possible via the Dirichlet-Neumann map.

Part 3. Applications to numerical computation

6. Detection of inclusions

6.1. **Dirichlet-Neumann map.** Let Ω be a bounded open set with smooth boundary in $\mathbf{R}^\nu$ with $\nu = 2, 3$, and consider the following boundary value problem

$$\begin{cases} \nabla\cdot(\gamma(x)\nabla v) = 0 & \text{in} \quad \Omega, \\ v = f & \text{on} \quad \partial\Omega. \end{cases} \tag{81}$$

We assume that $\inf_{x\in\Omega} \gamma(x) > 0$. It is well-known that one can reconstruct $\gamma(x)$ from the Dirichlet-Neumman map $\Lambda_\gamma : f \to \gamma\partial v/\partial n\Big|_{\partial\Omega}$, where v is the solution to (81) and n is the outer unit normal to $\partial\Omega$. In practical applications (e.g. in medical sciences), $\gamma(x)$ represents the electric conductivity of the body Ω. In this case, these theorems guarantee the uniqueness and the reconstruction of $\gamma(x)$ by the experimental data from all part of the surface of the body. However, it is often important to extract informations of $\gamma(x)$ from the local knowledge of the D-N map Λ_γ. In this section, we consider the problem of the detection of location of inclusions inside the 2 or 3-dimensional body Ω. Let us assume that $\gamma(x)$ is a bounded perturbation of $\gamma_0(x) \in C^\infty(\overline{\Omega})$. Namely there exists an open subset $\Omega_1 \subset \Omega$ such that $\overline{\Omega_1} \subset \Omega$ (we denote this property $\Omega_1 \subset\subset \Omega$) and

$$\gamma(x) = \begin{cases} \gamma_1(x), & x \in \Omega_1 \\ \gamma_0(x), & x \in \Omega_0 := \Omega \setminus \Omega_1, \end{cases} \tag{82}$$

with $\gamma_1(x) \in L^\infty(\Omega_1)$. Let

$$\Lambda_0 : f \to \gamma_0\Big(\frac{\partial u}{\partial n}\Big)\Big|_{\partial\Omega}, \quad \Lambda : f \to \gamma\Big(\frac{\partial v}{\partial n}\Big)\Big|_{\partial\Omega} \tag{83}$$

be the associated DN maps, where v is the solution to (81) and u solves the equation (81) with γ replaced by γ_0. We assume that the background conductivity $\gamma_0(x)$ is known on whole Ω and try to recover the location of Ω_1 from the local knowledge of Λ. No regularity is assumed on $\gamma_1(x)$, however we assume that for any $p \in \Omega_1$, there exist constants $C,\ \epsilon > 0$ such that

$$C^{-1} < \gamma_1(x) - \gamma_0(x) < C \quad \text{if} \quad |x-p| < \epsilon. \tag{84}$$

Although our principal purpose is to study discontinous perturbations, we allow $\gamma(x)$ to be a smooth function. Our main results are the following two theorems.

Theorem 6.1. *Take x_0 from the outside of the convex hull of Ω. We choose $\epsilon > 0$ small enough so that $x_0 \notin U_\epsilon :=$ the ϵ-neighborhood of the convex hull of Ω. Take an arbitrary constant $R > 0$. Then there exists $u_\tau(x) \in C^\infty(U_\epsilon)$ depending on a large parameter $\tau > 0$ (and also on R) having the following properties.*
(1) $\nabla \cdot (\gamma_0(x) \nabla u_\tau(x)) = 0$ *on* Ω.
(2) Let $K_\pm$ be any compact sets such that

$$K_+ \subset \{x \in U_\epsilon; |x - x_0| < R\}, \quad K_- \subset \{x \in U_\epsilon; |x-x_0| > R\}.$$

Then there exists a constant $\delta > 0$ such that for large $\tau > 0$

$$\int_{K_+} |u_\tau(x)|^2 dx \geq e^{\delta\tau}, \quad |u_\tau(x)| \leq e^{-\delta\tau} \quad \text{on} \quad K_-.$$

(3) Let $f_\tau(x) = u_\tau(x)\big|_{\partial\Omega}$. Then if $R <$ dis$\,(x_0, \Omega_1)$, there exists a $\delta > 0$ such that for large $\tau > 0$

$$0 \leq ((\Lambda - \Lambda_0) f_\tau, f_\tau) < e^{-\delta\tau}. \tag{85}$$

(4) If $R >$ dis$\,(x_0, \partial\Omega_1)$, there exists a $\delta > 0$ such that for large $\tau > 0$

$$((\Lambda - \Lambda_0) f_\tau, f_\tau) > e^{\delta\tau}. \tag{86}$$

In order to deal with the case $R = \text{dis}\,(x_0, \partial\Omega_1)$, we assume Ω_1 to satisfy the following cone condition.

(87) For any $p \in \partial\Omega_1$, there exists an open cone $C_p \subset \Omega_1$ with vertex p.

The following jump condition is also necessary.

For any $p \in \partial\Omega_1$, there exists $\epsilon > 0$ such that

$$\gamma(x) > \gamma_0(x) + \epsilon \text{ if } x \in \Omega_1, |x-p| < \epsilon. \tag{88}$$

Theorem 6.2. *Suppose $R = \text{dis}\,(x_0, \partial\Omega_1)$. Then*

$$\liminf_{\tau\to\infty} \tau^3 \left((\Lambda - \Lambda_0) f_\tau, f_\tau\right) > 0.$$

It will be useful to give an approximate form of the above $u_\tau(x)$. Suppose that $\Omega \subset\subset \mathbf{R}^3_+ = \{x = (x_1, x_2, x_3)\,; x_3 > 0\}$ and $x_0 = 0$. Then if $\gamma_0(x) = 1$, $u_\tau(x)$ is approximately equal to

$$\sqrt{\frac{\tau}{x_3}} y_3 e^{-\tau y_1} H^{(1)}_{1/2}(\tau y_3) \tag{89}$$

$$\begin{aligned} y_1 &= \frac{x_1^2 + x_2^2 + x_3^2 - R^2}{(x_1 + R)^2 + x_2^2 + x_3^2}, \\ y_3 &= \frac{2x_3 R}{(x_1 + R)^2 + x_2^2 + x_3^2}. \end{aligned}$$

In the 2-dimensional case, $u_\tau(x)$ is approximately equal to

$$\sqrt{\tau y_2} e^{-\tau y_1} H^{(1)}_{1/2}(\tau y_2), \tag{90}$$

$$\begin{aligned} y_1 &= \frac{x_1^2 + x_2^2 - R^2}{(x_1 + R)^2 + x_2^2}, \\ y_2 &= \frac{2x_2 R}{(x_1 + R)^2 + x_2^2}. \end{aligned}$$

Here $H^{(1)}_{1/2}(z)$ is the Hankel function of the first kind :

$$H^{(1)}_{1/2}(z) = -i\sqrt{\frac{2}{\pi z}} e^{iz}. \tag{91}$$

One can also use $z^{-1/2} \sin z$ or $z^{-1/2} \cos z$ instead of $H^{(1)}_{1/2}(z)$.

For the proof of the above results, we first imbed the boundary value problem in the upper half space. We then use a hyperbolic isometry to transform a hemisphere centered at the plane $\{x_3 = 0\}$ to the vertical plane $\{x_1 = 0\}$. The construction is thus reduced to the case where the sphere is replaced by the plane.

For the 2-diemnsional problem, this sort of idea was used by Ikehata-Siltanen [IkSi] via the function theory of one complex variable and the fractional linear transformation. In the 3-dimensional case, their roles are played by the hyperbolic space and isometries in terms of quaternions.

The above boundary data has the interesting property that its support is essentially contained in a part of the surface. This enables us to know the location of inclusions by a localized data of the boundary. We hope it to be usefull in practical problems.

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