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Representation of the Elliptic Algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ and the Coset Virasoro Modules

Hitoshi Konno
Department of Mathematics, Faculty of Integrated Arts & Sciences,
Hiroshima University, Higashi Hiroshima 739-8521, Japan

Abstract

Following [1], we review the level-k representation of the elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$. We stress that $U_{x,p}(\widehat{\mathfrak{sl}}_2)$-modules have a natural direct sum decomposition into the irreducible coset-Virasoro-modules associated with $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-k}$.

1 Introduction

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was first introduced in [2] as an elliptic deformation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ in the Drinfeld realization. From the beginning it was recognized that the level-1 elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ contains the algebra of the screening currents of the deformed Virasoro algebra[3] as a subalgebra, and it was conjectured that the level-k (≥ 1) elliptic algebra provides the screening currents for the deformation of the Virasoro algebra associated with the coset $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-k}$.

On the other hand, we conjectured that the level-k elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ provides the symmetry to formulate the fusion of the restricted Andrew-Baxter-Forrester model (RSOS model) in the sense of Jimbo-Miwa in [6]. This conjecture was based on the work by Lukyanov-Pugai[4] on the boson formulation of the RSOS model. There it was shown that the screening currents determining the vertex operators coincide with those of the deformed Virasoro algebra.

The latter conjecture was verified in [7] by establishing the connection between $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. Roughly speaking, the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ provides the Drinfeld realization of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. The coalgebra structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ is enough strong to determine the vertex operators and the full spectrum of the fusion RSOS models. There, we also extended $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ to those associated with an arbitrary affine Lie algebra $\mathfrak{g}$. We then conjectured that $U_{q,p}(\mathfrak{g})$ plays the same role in the face model associated with $\mathfrak{g}$. See [8, 9] for the higher rank cases.

The purpose of this paper is to review the level-k representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ stressing the fact that the level-k $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module has a natural direct sum decomposition into the coset Virasoro-modules associated with $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-k}$. Here we take a q-parafermion formulation, which makes the structure of modules more clear than the free field formulation given in [2].

1.1 Notations

Let $x^{2s}=e^{-\frac{2\pi i}{\tau(s)}}$ and $z=x^{2u}$. We define the symbols $[u]^{(s)}$ by

$$[u]^{(s)}=x^{\frac{u^2}{s}-u}\Theta_{x^{2s}}(x^{2u})=C\vartheta_1\left(\frac{u}{s}\middle|\tau(s)\right),\quad C=x^{-\frac{s}{4}}e^{-\frac{\pi i}{4}}\tau(s)^{1/2}.$$

with

$$\Theta_p(z)=(z;p)_\infty(p/z;p)_\infty(p;p)_\infty$$
$$(z;p_1,p_2,\cdots,p_m)_\infty=\prod_{n_1,n_2,\cdots,n_m=0}^{\infty}(1-zp_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}).$$

We also set $[[n]]_x=\frac{x^n-x^{-n}}{x-x^{-1}}$. For the other notations, we follows [1].

2 The Elliptic Algebra $U_{x,p}(\widehat{\mathfrak{sl}_2})$

Definition 2.1. *[2] The elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}_2})$ is an associative algebra of the currents $E(v),F(v)$, $K(v)$ and the central element c satisfying the following relations:*

$$\begin{aligned}
E(v_1)E(v_2)&=\frac{[v_1-v_2+1]^*}{[v_1-v_2-1]^*}E(v_2)E(v_1)\\
F(v_1)F(v_2)&=\frac{[v_1-v_2-1]}{[v_1-v_2+1]}F(v_2)F(v_1)\\
K(v_1)K(v_2)&=\rho(v_1-v_2)K(v_2)K(v_1)\\
K(v_1)E(v_2)&=\frac{[v_1-v_2+\frac{1-r^*}{2}]^*}{[v_1-v_2-\frac{1+r^*}{2}]^*}E(v_2)K(v_1)\\
K(v_1)F(v_2)&=\frac{[v_1-v_2-\frac{1+r}{2}]}{[v_1-v_2+\frac{1-r}{2}]}F(v_2)K(v_1),
\end{aligned}$$

$$\begin{aligned}
[E(v_1),F(v_2)]&=\frac{1}{x-x^{-1}}\left(\delta\left(x^{-c}\frac{z_1}{z_2}\right)H^+\left(v_2+\frac{c}{4}\right)-\delta\left(x^{c}\frac{z_1}{z_2}\right)H^-\left(v_2-\frac{c}{4}\right)\right),\\
H^\pm(v)&=\kappa K\left(v\pm\left(\frac{r}{2}-\frac{c}{4}\right)+\frac{1}{2}\right)K\left(v\pm\left(\frac{r}{2}-\frac{c}{4}\right)-\frac{1}{2}\right)
\end{aligned}$$

Here $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$, $z_i=x^{2v_i}$ $(i=1,2)$, $r^*=r-c$, $p=x^{2r}$ and $p^*=x^{2r^*}$, and we set $[u]=[u]^{(r)}$, $[u]^*=[u]^{(r^*)}$. The constant κ is given by

$$\kappa=\frac{\xi(x^{-2};p^*,x)}{\xi(x^{-2};p,x)},\qquad \xi(z;p,x)=\frac{(x^2z;p,x^4)_\infty(px^2z;p,x^4)_\infty}{(x^4z;p,x^4)_\infty(pz;p,x^4)_\infty}$$

and the scalar function $\rho(v)$ is given by

$$\rho(v)=\frac{\rho^{+*}(v)}{\rho^+(v)},\quad \rho^+(v)=z^{\frac{1}{2r}}x^{\frac{1}{2}}\frac{(px^2z;p,x^4)_\infty^2(z^{-1};p,x^4)_\infty(x^4z^{-1};p,x^4)_\infty}{(pz;p,x^4)_\infty(px^4z;p,x^4)_\infty(x^2z^{-1};p,x^4)_\infty^2},$$

where $\rho^{+*}(v)=\rho^+(v)|_{r\to r^*}$.

The elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}_2})$ is realized by tensoring the quantum affine algebra $U_x(\widehat{\mathfrak{sl}_2})$ and a Heisenberg algebra [7]. For this realization, it is convenient to introduce the Drinfeld realization of $U_x(\widehat{\mathfrak{sl}_2})$.

3 Quantum Affine Algebra $U_x(\widehat{\mathfrak{sl}}_2)$

3.1 Definition

Definition 3.1. **(The Drinfeld Realization of $U_x(\widehat{\mathfrak{sl}}_2)$)** *The quantum affine algebra $U_x(\widehat{\mathfrak{sl}}_2)$ is an associative algebra generated by $h, a_m, x_n^{\pm} (m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z}), d$ and the central element c satisfying the relations*

$$
\begin{aligned}
&[h,d]=0,\ [d,a_n]=na_n,\ [d,x_n^{\pm}]=nx_n^{\pm},\\
&[h,a_n]=0,\ [h,x^{\pm}(z)]=\pm 2x^{\pm}(z),\\
&[a_n,a_m]=\frac{[\![2n]\!]_x[\![cn]\!]_x}{n}x^{-c|n|}\delta_{n+m,0},\\
&[a_n,x^+(z)]=\frac{[\![2n]\!]_x}{n}x^{-c|n|}z^n x^+(z),\\
&[a_n,x^-(z)]=-\frac{[\![2n]\!]_x}{n}z^n x^-(z),\\
&(z-x^{\pm 2}w)x^{\pm}(z)x^{\pm}(w)=(x^{\pm 2}z-w)x^{\pm}(w)x^{\pm}(w),\\
&[x^+(z),x^-(w)]=\frac{1}{x-x^{-1}}\left(\delta(x^{-c}z/w)\psi(x^{c/2}w)-\delta(x^{c}z/w)\varphi(x^{-c/2}w)\right),
\end{aligned}
$$

where $x^{\pm}(z), \psi(z)$ and $\varphi(z)$ denote the Drinfeld currents defined by

$$
\begin{aligned}
x^{\pm}(z) &= \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n},\\
\psi(x^{c/2}z) &= x^h \exp\left((x-x^{-1})\sum_{n>0} a_n z^{-n}\right),\\
\varphi(x^{-c/2}z) &= x^{-h} \exp\left(-(x-x^{-1})\sum_{n>0} a_{-n} z^{n}\right).
\end{aligned}
$$

3.2 Parafermion Realization of the Level-k $U_x(\widehat{\mathfrak{sl}}_2)$

It is standard to realize the level-k ($c=k$) $U_x(\widehat{\mathfrak{sl}}_2)$ in terms of a q-deformed $\mathbb{Z}_k$-parafermion (through this paper we take x as q) and the Drinfeld boson a_n[10, 2] (see also [11] for the CFT case).

The q-deformed $\mathbb{Z}_k$-parafermion algebra is conveniently introduced through the q-deformed $\mathcal{Z}$-algebra associated with the level-k Drinfeld currents of $U_x(\widehat{\mathfrak{sl}}_2)$. The algebraic structure of the q-deformed $\mathcal{Z}$-algebra [13] is quite parallel to the classical case[12]. The deformed $\mathcal{Z}$-algebra is generated by $\mathcal{Z}_{\pm,n}$ ($n \in \mathbb{Z}$) whose generating functions $\mathcal{Z}_{\pm}(z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\pm,n} z^{-n}$ are defined by

$$
\begin{aligned}
\mathcal{Z}_+(z) &= \exp\left\{-\sum_{n>0}\frac{1}{[\![kn]\!]_x}a_{-n}z^n\right\} x^+(z)\exp\left\{\sum_{n>0}\frac{1}{[\![kn]\!]_x}a_n z^{-n}\right\},\\
\mathcal{Z}_-(z) &= \exp\left\{\sum_{n>0}\frac{x^{kn}}{[\![kn]\!]_x}a_{-n}z^n\right\} x^-(z)\exp\left\{-\sum_{n>0}\frac{x^{kn}}{[\![kn]\!]_x}a_n z^{-n}\right\}.
\end{aligned}
$$

The $\mathcal{Z}$-algebra commutes with the Drinfeld bosons a_n, $n \neq 0$. Then the level-k highest-weight $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\lambda_\ell)$ with highest weight $\lambda_\ell = (k-\ell)\Lambda_0 + \ell\Lambda_1$ ($\ell = 0, 1, .., k$) has the structure

$$V(\lambda_\ell) = \mathcal{F}^a \otimes \Omega_\ell, \tag{3.1}$$

where $\mathcal{F}^a = \mathbb{C}[a_{-n}\ (n > 0)]$. The space Ω_ℓ is called the vacuum space defined by

$$\Omega_\ell = \left\{ v \in V(\lambda_\ell) \mid a_n v = 0\ (n > 0) \right\}.$$

The space Ω_ℓ is spanned by the vectors $v_\ell(\varepsilon_1, .., \varepsilon_s; n_1, .., n_s)$ ($s \geq 0, \varepsilon_j \in \{\pm\}, n_s \leq 0, n_{s-1} + n_s \leq 0, .., n_1 + \cdots + n_s \leq 0$) given by

$$\prod_{1 \leq i < j \leq s} \left[\frac{(x^{-2} x^{k + \frac{\varepsilon_i + \varepsilon_j}{2} k}; x^{2k})_\infty}{(x^2 x^{k + \frac{\varepsilon_i + \varepsilon_j}{2} k}; x^{2k})_\infty} \right]^{\varepsilon_i \varepsilon_j} \mathcal{Z}_{\epsilon_1}(z_1) \cdots \mathcal{Z}_{\epsilon_s}(z_s) \cdot 1 \otimes e^{\frac{\ell}{2}\alpha}$$
$$= \sum_{n_1, .., n_s \in \mathbb{Z}} v_\ell(\epsilon_1, .., \epsilon_s; n_1, .., n_s) z_1^{-n_1} \cdots z_s^{-n_s},$$

where α denotes the simple root of $\mathfrak{sl}_2$. The e^λ are the formal symbols satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$. We define the action of h on e^λ by $h \cdot e^\lambda = (h, \lambda) e^\lambda$ with (,) being an invarinat symmetric bilinear form. The action of $\mathcal{Z}_{\pm,n}$ is defined as follows.

$$\mathcal{Z}_{\pm,n} \cdot (f \otimes e^{\ell \frac{\alpha}{2}}) = \begin{cases} \mathcal{Z}_{\pm,n} f \otimes e^{\ell \frac{\alpha}{2}} & n \leq 0 \\ [\mathcal{Z}_{\pm,n}, f] \otimes e^{\ell \frac{\alpha}{2}} & n \geq 1 \end{cases}$$

for $f \in \mathbb{C}[\mathcal{Z}_{+,n}, \mathcal{Z}_{-,n}\ (n \leq 0)]$. The weight of $v_\ell(\varepsilon_1, .., \varepsilon_s; n_1, .., n_s)$ is $\lambda_\ell + \sum_{j=1}^s \varepsilon_j \alpha$ and its degree is $-\frac{\ell(\ell+2)}{4(k+2)} + n_1 + \cdots + n_s$.

Now let us consider the q-deformed $\mathbb{Z}_k$-parafermion. Define the basic $\mathbb{Z}_k$-parafermion currents $\Psi(z)$ and $\Psi^\dagger(z)$ through the following relations.

$$\mathcal{Z}_+(z) = \Psi(z) \otimes e^\alpha z^{\frac{h}{k}},$$
$$\mathcal{Z}_-(z) = \Psi^\dagger(z) \otimes e^{-\alpha} z^{-\frac{h}{k}},$$
$$[\Psi(z), \alpha] = [\Psi(z), h] = [\Psi^\dagger(z), \alpha] = [\Psi^\dagger(z), h] = 0.$$

To make this expression well-defined, $\Psi(z)$ and $\Psi^\dagger(z)$ should have their mode expansions depending on the weight of vectors on which they act. Namely, on the vector with weight λ such that $(h, \lambda) = m$, we have

$$\Psi(z) \equiv \Psi^+(z) = \sum_{n \in \mathbb{Z}} \Psi_{+, \frac{m}{k} - n} z^{-\frac{m}{k} + n - 1},$$
$$\Psi^\dagger(z) \equiv \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi_{-, \frac{m}{k} - n} z^{\frac{m}{k} + n - 1}.$$

The q-deformed $\mathbb{Z}_k$-parafermion algebra is generated by $\Psi_{+, \frac{m}{k} - n}$, $\Psi_{-, \frac{m}{k} - n}$ ($n \in \mathbb{Z}$). Its relations can be expressed as follows.

$$\left(\frac{z}{w}\right)^{\frac{2}{k}} \frac{(x^{-2} w/z; x^{2k})_\infty}{(x^{2+2k} w/z; x^{2k})_\infty} \Psi^\pm(z) \Psi^\pm(w) = \left(\frac{w}{z}\right)^{\frac{2}{k}} \frac{(x^{-2} z/w; x^{2k})_\infty}{(x^{2+2k} z/w; x^{2k})_\infty} \Psi^\pm(w) \Psi^\pm(z),$$
$$\left(\frac{z}{w}\right)^{-\frac{2}{k}} \frac{(x^{2+k} w/z; x^{2k})_\infty}{(x^{-2+k} w/z; x^{2k})_\infty} \Psi^\pm(z) \Psi^\mp(w) - \left(\frac{w}{z}\right)^{-\frac{2}{k}} \frac{(x^{2+k} z/w; x^{2k})_\infty}{(x^{-2+k} z/w; x^{2k})_\infty} \Psi^\mp(w) \Psi^\pm(z)$$
$$= \frac{1}{x - x^{-1}} \left(\delta\left(x^k \frac{w}{z}\right) - \delta\left(x^{-k} \frac{w}{z}\right) \right).$$

By construction, we have

Theorem 3.1. *The following currents $x^{\pm}(z)$ and operator d with h give a level-k representation of $U_q(\widehat{\mathfrak{sl}}_2)$.*

$$x^+(z) = \Psi(z) \ :\exp\left\{-\sum_{n\neq 0}\frac{1}{[\![kn]\!]_x}a_n z^{-n}\right\}: e^{\alpha} z^{\frac{1}{k}h}, \tag{3.2}$$

$$x^-(z) = \Psi^{\dagger}(z) \ :\exp\left\{\sum_{n\neq 0}\frac{x^{k|n|}}{[\![kn]\!]_x}a_n z^{-n}\right\}: e^{-\alpha} z^{-\frac{1}{k}h}, \tag{3.3}$$

$$d = d^{PF} + d^{a}, \tag{3.4}$$

where

$$d^{a} = -\sum_{m>0}\frac{m^2 x^{km}}{[\![2m]\!]_x[\![km]\!]_x}a_{-m}a_m - \frac{h^2}{4k} \tag{3.5}$$

and d^{PF} is an operator such that

$$d^{PF}\cdot 1\otimes e^{\frac{\ell}{2}\alpha} = -\frac{\ell(k-\ell)}{2k(k+2)}1\otimes e^{\frac{\ell}{2}\alpha},$$
$$[d^{PF},\Psi(z)] = -z\frac{\partial}{\partial z}\Psi(z), \quad [d^{PF},\Psi^{\dagger}(z)] = -z\frac{\partial}{\partial z}\Psi^{\dagger}(z).$$

We define the $\mathbb{Z}_{2k}$ charge of $\Psi_{\pm,\frac{m}{k}-n}$ and $1\otimes e^{\frac{\ell}{2}\alpha}$ to be ± 2 and ℓ mod $2k$ respectively. For example, the $\mathbb{Z}_{2k}$ charge of the vector

$$\Psi_{\varepsilon_1,\frac{\ell+2(\varepsilon_2+\cdots+\varepsilon_s)}{k}-n_1}\Psi_{\varepsilon_2,\frac{\ell+2(\varepsilon_3+\cdots+\varepsilon_s)}{k}-n_2}\cdots\Psi_{\varepsilon_s,\frac{\ell}{k}-n_s}\otimes e^{\frac{\ell}{2}\alpha} \tag{3.6}$$

is $\ell+2\sum_{j=1}^{s}\varepsilon_j$. Let us denote by $\mathcal{H}^{PF}_{\ell,M}$ the irreducible parafermion module of the $\mathbb{Z}_{2k}$ charge M defined by the relation

$$\Omega_\ell \ = \bigoplus_{\tilde{M}\in\ell+2\mathbb{Z}}\mathcal{H}^{PF}_{\ell,\tilde{M}}\otimes e^{\frac{\tilde{M}}{2}\alpha} = \bigoplus_{n\in\mathbb{Z}}\bigoplus_{\substack{M=0\\ \mathrm{mod}2k}}^{2k-1}\mathcal{H}^{PF}_{\ell,M}\otimes e^{\frac{M+2kn}{2}\alpha}. \tag{3.7}$$

Here $\mathcal{H}^{PF}_{\ell,M}=\{0\}$ for $M\not\equiv\ell \mod 2$ and $\mathcal{H}^{PF}_{\ell,M}=\mathcal{H}^{PF}_{\ell,M+2k}$. We also assume the symmetry[11]

$$\mathcal{H}^{PF}_{\ell,M} = \mathcal{H}^{PF}_{k-\ell,M+k} = \mathcal{H}^{PF}_{\ell,-M}.$$

The basic parafermion currents act on the space $\mathcal{H}^{PF}_{\ell,M}$ as the following linear operators.

$$\Psi(z) \ : \ \mathcal{H}^{PF}_{\ell,M}\to\mathcal{H}^{PF}_{\ell,M+2},$$
$$\Psi^{\dagger}(z) \ : \ \mathcal{H}^{PF}_{\ell,M}\to\mathcal{H}^{PF}_{\ell,M-2}.$$

The character of the q-$\mathbb{Z}_k$-parafermion space $\mathcal{H}^{PF}_{\ell,M}$ is known to be[11]

$$(x^4)^{-\frac{c^{PF}}{24}}\mathrm{tr}_{\mathcal{H}^{PF}_{\ell,M}}x^{-4d^{PF}} = \eta(\bar{\tau})\ c^{\lambda_\ell}_{\lambda_M}(\bar{\tau}) \tag{3.8}$$

where $c^{PF} = \frac{2(k-1)}{k+2}$. $c^{\lambda_\ell}_{\lambda_M}(\bar\tau)$ denotes the level-k string function and $\eta(\bar\tau)$ denotes Dedekind's η-function given by

$$\eta(\bar\tau) = (x^4)^{\frac{1}{24}}(x^4;x^4)_\infty,$$

where we set $e^{2\pi i\bar\tau} = x^4$.

From (3.1) and (3.7), the level-k irreducible highest-weight module $V(\lambda_\ell)$ of $U_x(\widehat{\mathfrak{sl}}_2)$ with highest weight λ_ℓ is realized as follows:

$$V(\lambda_\ell) = \mathcal{F}^a \otimes \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\substack{M=0 \\ \mathrm{mod} 2k}}^{2k-1} \mathcal{H}^{PF}_{\ell,M} \otimes e^{(M+2kn)\frac{\alpha}{2}}. \tag{3.9}$$

In particular, the highest-weight vector is given by

$$1 \otimes 1 \otimes e^{\ell\frac{\alpha}{2}}. \tag{3.10}$$

From (3.9), the normalized character of $V(\lambda_\ell)$ is evaluated as follows:

$$\begin{aligned} \chi^{(k)}_\ell(x^4,y) &= (x^4)^{-\frac{c}{24}}\mathrm{tr}_{V(\lambda_\ell)} x^{-4d} y^{\frac{h}{2}} \\ &= \sum_{n\in\mathbb{Z}} \sum_{\substack{M=0 \\ \mathrm{mod} 2k}}^{2k-1} c^{\lambda_\ell}_{\lambda_M}(\bar\tau) x^{4k(n+\frac{M}{2k})^2} y^{k(n+\frac{M}{2k})}. \end{aligned} \tag{3.11}$$

By setting $y = x^{-2}$, we reproduce the level-k principally specialized character of $V(\lambda_\ell)$.

4 The Level-k Representation of $U_{x,p}(\widehat{\mathfrak{sl}}_2)$

The elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ is realized by tensoring $U_x(\widehat{\mathfrak{sl}}_2)$ and a Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$ generated by P, e^Q with $[e^Q, P] = e^Q$.

Theorem 4.1. *[7] The elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ is realized by tensoring $U_x(\widehat{\mathfrak{sl}}_2)$ and the Heisenberg algebra $\mathbb{C}\{\hat{\mathcal{H}}\}$. The elliptic currents $E(u), F(u)$, $K(u)$ and $\hat d$ are given by*

$$\begin{aligned} K(u) &= k(z)e^Q z^{(\frac{1}{r}-\frac{1}{r^*})\frac{P}{2}+\frac{h}{2r}+\frac{1}{4}(\frac{1}{r^*}-\frac{1}{r})}, \\ E(u) &= u^+(z,p)x^+(z)e^{2Q}z^{\frac{1}{r^*}(-P+1)}, \\ F(u) &= x^-(z)u^-(z,p)z^{\frac{1}{r}(P+h-1)}, \\ \hat d &= d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1), \end{aligned}$$

where $k(z)$ and $u^\pm(z,p)$ are given by

$$k(z) = \exp\left(\sum_{n>0}\frac{[[n]]_x}{[[2n]]_x[[r^*n]]_x}a_{-n}(x^c z)^n\right)\exp\left(-\sum_{n>0}\frac{[[n]]_x}{[[2n]]_x[[rn]]_x}a_n z^{-n}\right),$$

$$u^+(z,p) = \exp\left(\sum_{n>0}\frac{1}{[[r^*n]]_x}a_{-n}(x^r z)^n\right), \quad u^-(z,p) = \exp\left(-\sum_{n>0}\frac{1}{[[rn]]_x}a_n(x^{-r}z)^{-n}\right).$$

The following commutation relations are important.

Proposition 4.2.

$$[K(u),P]=K(u),\ [E(u),P]=2E(u),\ [F(u),P]=0, \tag{4.1}$$
$$[K(u),P+h]=K(u),\ [E(u),P+h]=0,\ [F(u),P+h]=2F(u). \tag{4.2}$$

The level-k ($c=k$) representation of the elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ is realized by using the level-k $U_x(\widehat{\mathfrak{sl}}_2)$ and the Heisenberg algebra $\mathbb{C}\{\mathcal{H}\}$. It can be expressed in a compact form by using the level-k dressed Drinfeld boson α_n ($n \in \mathbb{Z}_{\neq 0}$) defined by

$$\alpha_n = \begin{cases} a_n & \text{for } n>0 \\ \frac{[[rn]]_x}{[[r^*n]]_x} x^{k|n|} a_n & \text{for } n<0 \end{cases} \tag{4.3}$$

with $r^* = r-k$. This satisfies the following commutation relation.

$$[\alpha_m, \alpha_n] = \frac{[[2m]]_x [[km]]_x}{m} \frac{[[rm]]_x}{[[r^*m]]_x} \delta_{m+n,0}. \tag{4.4}$$

From Theorems 3.1 and 4.1, we get the following free field realization of the elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$

Theorem 4.3. *The level-k elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}_2})$ is realized by the following currents.*

$$\begin{aligned}
K(u) &= :\exp\left(-\sum_{m\neq 0} \frac{[[m]]_x}{[[2m]]_x[[rm]]_x} \alpha_m z^{-m}\right): e^Q z^{(\frac{1}{r}-\frac{1}{r^*})\frac{P}{2}+\frac{h}{2r}+\frac{1}{4}(\frac{1}{r^*}-\frac{1}{r})}, \\
E(u) &= \Psi(z) :\exp\left(-\sum_{m\neq 0} \frac{1}{[[2m]]_x} \alpha_m z^{-m}\right): e^{2Q+\alpha} z^{\frac{h}{2}-\frac{P}{r^*}+\frac{1}{r^*}+\frac{1}{2}}, \\
F(u) &= \Psi^\dagger(z) :\exp\left(\sum_{m\neq 0} \frac{[[r^*m]]_x}{[[2m]]_x[[rm]]_x} \alpha_m z^{-m}\right): e^{-\alpha} z^{(-\frac{1}{2}+\frac{1}{r})h+\frac{P}{r}-\frac{1}{r}+\frac{1}{2}}, \\
\widehat{d} &= d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1).
\end{aligned}$$

Now let us consider the $U_{x,p}(\widehat{\mathfrak{sl}}_2)$-modules $\widehat{V}(\lambda_\ell) = \bigoplus_{m\in\mathbb{Z}} V(\lambda_\ell)\otimes e^{-mQ}$. From (3.9), we have

$$\widehat{V}(\lambda_\ell) = \bigoplus_{m\in\mathbb{Z}} \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\substack{M=0 \\ \mathrm{mod}\ 2k}}^{2k-1} \mathcal{F}_{M;m,\ell,n}, \tag{4.5}$$

where we set

$$\mathcal{F}_{M;m,\ell,n} = \mathcal{F}^a \otimes \mathcal{H}^{PF}_{\ell,M} \otimes e^{(M+2kn)\frac{\alpha}{2}} \otimes e^{-mQ}. \tag{4.6}$$

Let r be generic and note that

$$P|_{\mathcal{F}_{M;m,\ell,n}} = m, \qquad P+h|_{\mathcal{F}_{M;m,\ell,n}} = M+m+2kn. \tag{4.7}$$

From (4.6) and (3.8), the character of the space $\mathcal{F}_{M;m,\ell,n}$ is evaluated as follows.

$$(x^4)^{-\frac{c_{Vir}}{24}} \mathrm{tr}_{\mathcal{F}_{M;m,\ell,n}} x^{-4\widehat{d}} = c^{\lambda_\ell}_{\lambda_M}(\bar{\tau})\, x^{\frac{(mr-(M+m+2kn))r^*)^2}{krr^*}},$$

where $c_{Vir} = \frac{3k}{k+2}\left(1 - \frac{2(k+2)}{rr^*}\right)$. One should note that this coincides with the one point function of the $k \times k$ fusion SOS model with the central height $a = M + m + 2kn$. Hence one may make the following identification:

$$\text{the fusion SOS space of states}: \quad \mathcal{H}^{(\ell)}_{m,a} \longleftrightarrow \mathcal{F}_{M;m,\ell,n} \qquad a = M + m + 2kn,\ M \equiv \ell \mod 2,$$

$$\text{the corner Hamiltonian}: \quad H^{(\ell)}_{m,a} \longleftrightarrow -\widehat{d} - \frac{1}{24}c_{Vir}.$$

Furthermore let us set

$$\mathcal{F}_{m,\ell}(n) \equiv \bigoplus_{\substack{M=0 \\ \mathrm{mod}\ 2k}}^{2k-1} \mathcal{F}_{M;m,\ell,n}.$$

When r is generic, $\mathcal{F}_{m,\ell}(n)$ is irreducible. The character of the space $\mathcal{F}_{m,\ell}(n)$ is evaluated as follows.

$$(x^4)^{-\frac{c_{Vir}}{24}} \mathrm{tr}_{\mathcal{F}_{m,\ell}(n)} x^{-4\widehat{d}} = \sum_{\substack{M=0 \\ \mathrm{mod}\ 2k}}^{2k-1} c^{\lambda_\ell}_{\lambda_M}(\bar{\tau})\, x^{\frac{(mr-(m+M+2kn)r^*)^2-k^2}{krr^*}}. \tag{4.8}$$

This coincides with the character of the irreducible Virasoro module $Vir_{m,a}$ ($a \equiv \ell + m \mod 2$) associated with the coset $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-2}$. See §5 for the cases $k = 1$ and 2.

In addition, when r ($> k+2$) is an integer, $\mathcal{F}_{m,\ell}(n)$ is reducible. We observed that the BRST resolution of the complex formed by $\mathcal{F}_{m,\ell}(n)$ yields the irreducible coset Virasoro *minimal* module $Vir_{m,a}$, ($a \equiv \ell + m \mod 2$)[1]. These considerations leads us to the following conjecture:

Conjecture 4.4. *The space $\mathcal{F}_{m,\ell}(n)$ is isomorphic to the irreducible coset (deformed) Virasoro module $Vir_{m,a}$, ($a \equiv m + \ell \mod 2$) associated with $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-2}$ whose central charge and highest weight are $c_{Vir} = \frac{3k}{k+2}\left(1 - \frac{2(k+2)}{4rr^*}\right)$ and $h_{m,a} = \frac{\ell(k-\ell)}{2k(k+2)} + \frac{(mr-ar^*)^2-k^2}{4krr^*}$, respectively.*

5 The Level-1 and 2 Cases

Here we consider the level-1 and 2 cases explicitely. In this section we assume r is generic.

5.1 The Level-1 Case

In this case, the parafermion theory is trivial, i.e. $\mathcal{H}^{PF}_{\ell,M} = \mathbb{C}1$ for $M \equiv \ell \mod 2$, otherwise $= \{0\}$. Hence the level-1 $U_{x,p}(\widehat{\mathfrak{sl}}_2)$-modules $\widehat{V}(\lambda_\ell)$ with $\lambda_\ell = (1-\ell)\Lambda_0 + \Lambda_1$ ($\ell = 0, 1$) are given by

$$\widehat{V}(\lambda_\ell) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_{\ell;m,\ell,n},$$
$$\mathcal{F}_{\ell;m,\ell,n} = \mathcal{F}^a \otimes e^{(\ell+2n)\frac{\alpha}{2}}$$

In this case $\mathcal{F}_{m,\ell}(n) = \mathcal{F}_{\ell;m,\ell,n}$. Noting $c_0^0(\bar{\tau}) = c_1^1(\bar{\tau}) = 1/\eta(\bar{\tau})$, we have the normalized character

$$(x^4)^{-\frac{c_{Vir}}{24}} \mathrm{tr}_{\mathcal{F}_{m,\ell}(n)} x^{-4\hat{d}} = \frac{1}{\eta(\bar{\tau})} x^{\frac{(mr-(m+\ell+2n)r^*)^2-1}{rr^*}}$$

for $\ell = 0, 1$, where $r^* = r - 1$. This coincides with the character of the irreducible Virasoro module $Vir_{m,a}$ ($a = \ell + m \mod 2$) with the central charge $c_{Vir} = 1 - \frac{6}{rr^*}$ and the highest weight $h_{m,a} = \frac{(mr-ar^*)^2-1}{4rr^*}$.

5.2 The Level-2 Case

The level-2 $U_{x,p}(\widehat{\mathfrak{sl}}_2)$-modules $\widehat{V}(\lambda_\ell)$ with $\lambda_\ell = (2-\ell)\Lambda_0 + \ell\Lambda_1$ ($\ell = 0, 1, 2$) are given by

$$\widehat{V}(\lambda_\ell) = \bigoplus_{m\in\mathbb{Z}} \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\substack{M=0,1,2,3\\ \mathrm{mod} 4}} \mathcal{F}_{M;m,\ell,n}. \tag{5.1}$$

Note that as other realization, $\mathcal{F}_{M;m,\ell,n}$ can be realized by using the dressed Drinfeld boson α_m at the level-2 and the fermion operators, Neveu-Schwartz type $\Psi^{NS}(z)$ and Ramond type $\Psi^R(z)$. We follows the notations in [14, 15]. We have

$$\begin{aligned}
\mathcal{F}_{0;m,0,n} &= \mathcal{F}^a \otimes \mathcal{F}^{NS}_{even} \otimes e^{2n\alpha} \otimes e^{-mQ}, \\
\mathcal{F}_{2;m,0,n} &= \mathcal{F}^a \otimes \mathcal{F}^{NS}_{odd} \otimes e^{(2n+1)\alpha} \otimes e^{-mQ}, \\
\mathcal{F}_{0;m,2,n} &= \mathcal{F}^a \otimes \mathcal{F}^{NS}_{odd} \otimes e^{2n\alpha} \otimes e^{-mQ}, \\
\mathcal{F}_{2;m,2,n} &= \mathcal{F}^a \otimes \mathcal{F}^{NS}_{even} \otimes e^{(2n+1)\alpha} \otimes e^{-mQ}, \\
\mathcal{F}_{M;m,1,n} &= \mathcal{F}^a \otimes \left(\mathcal{F}^R_{even} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}^R_{odd} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \otimes e^{(\frac{M}{2}+2n)\alpha} \otimes e^{-mQ}.
\end{aligned} \tag{5.2}$$

and $\mathcal{F}_{M;m,\ell,n} = \{0\}$ for $M \not\equiv \ell \mod 2$. The both type of fermion field $\Psi^{NS}(z)$ and $\Psi^R(z)$ act on $\mathcal{F}_{M;m,\ell,n}$ as

$$\Psi^{NS,R}(z) \ : \ \mathcal{F}_{M;m,\ell,n} \to \mathcal{F}_{M;m,k-\ell,n}. \tag{5.3}$$

Let us consider the space $\mathcal{F}_{m,\ell}(n) = \bigoplus_{\substack{M=0\\ \mathrm{mod}\ 4}}^{3} \mathcal{F}_{M;m,\ell,n}$. From (4.8) or an independent calculation by using the fermion realization (5.2), we have

$$(x^4)^{-\frac{c_{Vir}}{24}} \mathrm{tr}_{\mathcal{F}_{m,\ell}(n)} (x^4)^{-\hat{d}} = \sum_{\substack{M=0,1,2,3\\ \mathrm{mod}\ 2}} c^{\lambda_\ell}_{\lambda_M}(\bar{\tau})\ x^{\frac{1}{2rr^*}(mr-(M+m+4n)r^*)^2}. \tag{5.4}$$

Here $r^* = r-2$, $c_{Vir} = \frac{3}{2}\left(1-\frac{8}{rr^*}\right)$, and $c^{\lambda_\ell}_{\lambda_M}(\bar{\tau})$ denotes the level-2 string function

$$\begin{aligned} c^{\lambda_\ell}_{\lambda_\ell}(\bar{\tau}) &= \frac{x^{-\frac{1}{12}}}{2\eta(\bar{\tau})}\left[(-x^2;x^4)_\infty + (x^2;x^4)_\infty\right] \qquad (\ell = 0,2), \\ c^{\lambda_\ell}_{\lambda_{2-\ell}}(\bar{\tau}) &= \frac{x^{-\frac{1}{12}}}{2\eta(\bar{\tau})}\left[(-x^2;x^4)_\infty - (x^2;x^4)_\infty\right] \qquad (\ell = 0,2), \\ c^{\lambda_1}_{\lambda_1}(\bar{\tau}) &= \frac{\eta(2\bar{\tau})}{\eta(\bar{\tau})^2} = \frac{(-x^4;x^4)_\infty}{(x^4;x^4)_\infty}. \end{aligned}$$

Setting $a = M + m + 4n \equiv \ell + m \mod 2$, we have

$$(x^4)^{-\frac{c_{Vir}}{24}} \mathrm{tr}_{\mathcal{F}_{m,\ell(n)}}(x^4)^{-\hat{d}} = \begin{cases} \frac{(-x^2;x^4)_\infty}{(x^4;x^4)_\infty}(x^4)^{-\frac{1}{16}} x^{\frac{1}{2rr^*}(mr-ar^*)^2} & \text{for } \ell = 0,2 \\ 2\frac{(-x^4;x^4)_\infty}{(x^4;x^4)_\infty} x^{\frac{1}{2rr^*}(mr-ar^*)^2} & \text{for } \ell = 1 \end{cases}$$

This coincides with the character of the irreducible super-Virasoro module $Vir_{m,a}$ ($a \equiv \ell + m \mod 2$) associated with the coset $(\widehat{\mathfrak{sl}}_2)_2 \oplus (\widehat{\mathfrak{sl}}_2)_{r-4}/(\widehat{\mathfrak{sl}}_2)_{r-2}$ with the central charge $c_{Vir} = \frac{3}{2}\left(1-\frac{8}{rr^*}\right)$ and the highest weight $h_{m,a} = \frac{\ell(2-\ell)}{16} + \frac{(mr-ar^*)^2-4}{8rr^*}$.

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