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# Elliptic Ruijsenaars operators and elliptic hypergeometric integrals

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ABSTRACT

We study a family of mutually commutative difference operators associated with root systems and discuss their simultaneous eigenvector in a special case. For root systems with rank $n$, we construct $3n$ commutative difference operators, which are a generalization of elliptic Ruijsenaars operators. Especially, for the $BC_1$ root system, we construct an explicit simultaneous eigenvector of these operators described in terms of elliptic hypergeometric integrals.

## 1 Introduction

In [10] Ruijsenaars introduced the operators acting on the space of meromorphic functions which are defined by

$$Y_n = \sum_{\substack{I\subset\{1,\dots,l\}\\ |I|=n}} \left(\prod_{\substack{j\in I\\ k\in I^c}} \frac{\sigma(x_j-x_k+\mu;\omega_1,\omega_2)}{\sigma(x_j-x_k;\omega_1,\omega_2)}\right)\prod_{j\in I}\tau_j(\omega_3), \tag{1.1}$$

where $\omega_1,\omega_2\in\mathbb{C}\setminus\{0\}$ are arbitrary such that $\omega_1/\omega_2\notin\mathbb{R}$, and $\omega_3,\mu\in\mathbb{C}\setminus\{0\}$ and the action of $\tau_j(\omega)$ is defined by $(\tau_j(\omega)f)(x_1,\dots,x_j,\dots,x_n)=f(x_1,\dots,x_j-\omega,\dots,x_n)$. He showed that these operators are mutually commutative. The first result of this article is a generalization of the elliptic Ruijsenaars operators. We define

$$Y_n^{(p)} = \sum_{\substack{I\subset\{1,\dots,l\}\\ |I|=n}} \left(\prod_{\substack{j\in I\\ k\in I^c}} e^{\nu_p(x_j-x_k)}\frac{\sigma(x_j-x_k+\mu_p;\omega_q,\omega_r)}{\sigma(x_j-x_k;\omega_q,\omega_r)}\right)\prod_{j\in I}\tau_j(\omega_p), \tag{1.2}$$

*Section 5 and 6 are a jointwork with 神戸大学・大学院自然科学研究科 野海 正俊 (Masatoshi Noumi) *Department of Mathematics, Graduate School of Science and Technology, Kobe University* where $p, q, r \in \mathbb{Z}/3\mathbb{Z}$ are distinct, and for $p \in \mathbb{Z}/3\mathbb{Z}$, $\omega_p \in \mathbb{C} \setminus \{0\}$ are such that $\omega_p/\omega_q \notin \mathbb{R}$ if $p \neq q$. Put $\eta_{pq} = 2\zeta(\omega_p/2; \omega_p, \omega_q)$ with Weierstrass' zeta function $\zeta$ and $a_r = \eta_{pq}\omega_q - \eta_{qp}\omega_p$, then $a_r = \pm 2\pi i$. If $\nu_p, \mu_p$ satisfy three equations $\nu_p\omega_q + \mu_p\eta_{qr} = \nu_q\omega_p + \mu_q\eta_{pr}$ for distinct $p, q, r \in \mathbb{Z}/3\mathbb{Z}$, then all $Y_n^{(p)}$ are shown to be mutually commutative. For instance, these equations are solved by

$$\nu_1 = \frac{a_1(\nu_3\omega_1 + \mu_3\eta_{12}) - (a_2\mu_2 + a_3\mu_3)\eta_{32}}{a_1\omega_3}, \tag{1.3}$$

$$\nu_2 = \frac{\nu_3\omega_2 + \mu_3\eta_{21} - \mu_2\eta_{31}}{\omega_3}, \tag{1.4}$$

$$\mu_1 = \frac{a_2\mu_2 + a_3\mu_3}{a_1}, \tag{1.5}$$

where $\nu_3, \mu_2, \mu_3$ are regarded as free parameters. Although the discussion above is for the $A$-type root system, the construction can be applied to arbitrary root systems.

The second result is a construction of a simultaneous eigenvector of the elliptic Ruijsenaars operators of type $BC_1$. So far, some class of eigenvectors was discussed in [11], where certain transcendental equations should be solved. We first obtain an explicit meromorphic eigenvector described in terms of the elliptic hypergeometric integral.

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## 2 Affine Root Systems

We summarize some facts about affine root systems and affine Weyl groups [1, 2, 4, 5]. In this article, we will omit $A_{2l}^{(2)}$-type root system because of simplicity, though it is straightforward. The notation and symbols are a little different from those in the previous papers [6, 7] in order to generalize the results.

Let $\Delta$ be the irreducible reduced finite root system of type $X_l$ in a complex vector space $V$ with $\dim V = l$ and the inner product $\langle\cdot,\cdot\rangle$, $I = \{1,\ldots,l\}$ a set of indices, $\Pi = \{\alpha_i \mid i \in I\} \subset V$ the set of simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset V$ the set of simple coroots, $Q$ and $Q^\vee$ the root and coroot lattices, $P$ and $P^\vee$ the weight and coweight lattices, $\{\Lambda_i \mid i \in I\}$ and $\{\Lambda_i^\vee \mid i \in I\}$ the fundamental weights and fundamental coweights such that $\langle\alpha_i, \Lambda_j^\vee\rangle = \langle\Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$. Then we have

$$Q = \bigoplus_{i\in I} \mathbb{Z}\,\alpha_i \subset P = \bigoplus_{i\in I} \mathbb{Z}\,\Lambda_i \subset V, \tag{2.1}$$

$$Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\,\alpha_i^\vee \subset P^\vee = \bigoplus_{i\in I} \mathbb{Z}\,\Lambda_i^\vee \subset V. \tag{2.2}$$

The inner product $\langle\cdot,\cdot\rangle$ is normalized such that $\langle\alpha,\alpha\rangle = 2$ for the longer roots $\alpha$. Let $\Delta_+$ and $\Delta_-$ be the set of positive roots and negative roots respectively.

Let $\Delta_s \subset \Delta$ be the set of the shorter roots and $\Delta_l \subset \Delta$ the set of the longer roots. Let $r$ be the ratio of the square lengths of the longer roots and the shorter roots. Fix parameters $\gamma_\alpha$ for $\alpha \in \Delta$, such that in nontwisted cases all $\gamma_\alpha = 1$, and in twisted cases, $\gamma_\alpha = r$ if $\alpha \in \Delta_l$, $\gamma_\alpha = 1$ otherwise. Let $\widehat{V} = V \oplus \mathbb{C}\delta$ with $\langle\alpha_i, \delta\rangle = \langle\delta,\delta\rangle = 0$ and its linear extension. Then the associated affine root system $\widehat{\Delta} \subset \widehat{V}$ is written as

$$\widehat{\Delta} = \{\alpha + n\gamma_\alpha\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}. \tag{2.3}$$

Let $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$ be the set of the positive affine roots and the negative affine roots respectively. We denote by $\overline{v}$ for $v \in \widehat{V}$ the natural projection on $V$.

For $\alpha \in \widehat{\Delta}$, let $s_\alpha$ be a reflection defined by

$$s_\alpha(v) := v - \langle\alpha, v\rangle\alpha^\vee, \qquad v \in V, \tag{2.4}$$

where $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. The Weyl group $W$ is generated by the fundamental reflections $\{s_i := s_{\alpha_i} \mid i \in I\}$ on $V$ and the affine Weyl group $\widehat{W}$ is generated by $\{s_i \mid i \in \widehat{I}\}$, where $\widehat{I} = I \cup \{0\}$ and $\alpha_0 = \delta - \theta$ with $\theta$ the highest root in nontwisted cases and the highest short root in twisted cases.

The defining relations are given by $s_i^2 = id$ and the Coxeter relations:

$$(s_i\, s_j)^{m_{ij}} = id, \qquad \text{for } i \neq j \in \widehat{I}, \tag{2.5}$$

where $m_{ij} = 2$ if $\alpha_i$ and $\alpha_j$ are disconnected in the Dynkin diagram and $m_{ij} = 3, 4, 6$ if 1,2,3 lines respectively connect $\alpha_i$ and $\alpha_j$. For $\mu \in V$, we define endomorphisms $\tau_\mu$ of the vector space $V$ by

$$\tau_\mu(\lambda) := \lambda - \langle \lambda, \mu \rangle \delta. \tag{2.6}$$

Let $M := \mathbb{Z}(W \cdot \theta^\vee) \subset V$. For an arbitrary lattice $L$, we denote by $T_L$ the corresponding group of translations of $L$. Then one sees that $\widehat{W}$ is the semidirect product $\widehat{W} = W \ltimes T_M$. Let $\widetilde{M} := \{\lambda \in V \mid \alpha \in \Delta, \langle \alpha, \lambda \rangle \in \gamma_\alpha \mathbb{Z}\}$. The extended affine Weyl group $\widetilde{W}$ is defined by the semidirect product $\widetilde{W} := W \ltimes T_{\widetilde{M}}$. Let $\Omega$ be the subgroup of $\widetilde{W}$ which stabilizes the affine Weyl chamber $C$. Then one sees that $\widetilde{W}$ is isomorphic to the semidirect product $\widehat{W} \rtimes \Omega$. Here are the explicit description of $\widetilde{M}$ and its canonical basis $\{\lambda_i \mid i \in I\}$:

$$\widetilde{M} = \begin{cases} P^\vee, & \text{nontwisted case,} \\ P, & \text{twisted case,} \end{cases} \qquad \lambda_i = \begin{cases} \Lambda_i^\vee, & \text{nontwisted case,} \\ \Lambda_i, & \text{twisted case.} \end{cases} \tag{2.7}$$

We also use $\widetilde{M}_- := \oplus_{i \in I} \mathbb{Z}_{\leq 0} \lambda_i$.

The length $\ell(w)$ of $w \in \widetilde{W}$ is defined by the length $\ell$ of a reduced decomposition:

$$w = s_{i_1} \dots s_{i_\ell} \omega, \qquad i_k \in \widehat{I}, \omega \in \Omega. \tag{2.8}$$

It is equivalent to the number of the negative roots made positive by $\hat{w}$:

$$\ell(\hat{w}) := |\Delta_{\hat{w}}|, \qquad \Delta_{\hat{w}} := \widehat{\Delta}_+ \cap \hat{w} \widehat{\Delta}_-. \tag{2.9}$$

The set $\Delta_{\hat{w}}$ is explicitly described as $\Delta_{\hat{w}} = \{\alpha^{(1)} = \alpha_{i_1}, \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \dots, \alpha^{(\ell)} = w s_{i_\ell}(\alpha_{i_\ell})\}$. By definition, $\Delta_{\hat{w}}$ is independent of reduced expressions. One sees that $\Omega = \{\omega \in \widetilde{W} \mid \ell(\omega) = 0\}$. A weight $\lambda \in \widetilde{M}$ is said to be minuscule if $\Delta_{\tau_{-\lambda}} \subset \Delta_+$.

In the following, we use the constants

$$\rho_x := \sum_{i \in I} x_{\alpha_i} \Lambda_i = \frac{1}{2} \sum_{\alpha \in \Delta_+} x_\alpha \alpha, \tag{2.10}$$

$$h_x^\vee := \langle \rho_x, \theta \rangle + x_{\alpha_0}, \tag{2.11}$$

where $x_\alpha \in \mathbb{C}$ which depends only on the length of roots,

We shall define the root algebras after Cherednik [2, 7].

**Definition 2.1.** *Root algebra $\mathcal{R}$ is generated by independent variables $\{R_\alpha \mid \alpha \in \widehat{\Delta}\}$ and $\{\tau_\lambda \mid \lambda \in \widetilde{M}\}$ with the following defining relations:*

$$\underbrace{R_{w\alpha_i}R_{ws_i\alpha_j}R_{ws_is_j\alpha_i}\cdots}_{m_{ij}\ factors} = \underbrace{R_{w\alpha_j}R_{ws_j\alpha_i}R_{ws_js_i\alpha_j}\cdots}_{m_{ij}\ factors}, \quad \text{for } w \in \widetilde{W} \tag{2.12}$$

$$\tau_\lambda R_\alpha = R_{\tau_\lambda \alpha}\tau_\lambda, \tag{2.13}$$

$$\tau_\lambda \tau_{\lambda'} = \tau_{\lambda+\lambda'}. \tag{2.14}$$

**Theorem 2.2 (Cherednik).** *1. There exists a unique set $\{R_w \mid w \in \widetilde{W}\} \subset \mathcal{R}$ satisfying the relations:*

$$R_{vw} = R_v\, {}^vR_w, \qquad R_{s_i} = R_{\alpha_i} \quad (i \in \widehat{I}), \qquad R_\omega = 1, \tag{2.15}$$

*where $\omega \in \Omega$, $v, w \in \widetilde{W}$ and $\ell(vw) = \ell(v) + \ell(w)$, and ${}^v(R_{\alpha_1}\dots R_{\alpha_i}) = R_{v\alpha_1}\dots R_{v\alpha_i}$*

*2. We have the R-matrix for $w \in \widetilde{W}$ and its arbitrary reduced decomposition $w = s_{i_1}\dots s_{i_\ell}\omega$ as*

$$\begin{aligned} &R_w = R_{\alpha^{(1)}}\dots R_{\alpha^{(\ell)}},\\ &\alpha^{(1)} = \alpha_{i_1}, \quad \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \alpha^{(\ell)} = ws_{i_\ell}(\alpha_{i_\ell}) \in \Delta_w. \end{aligned} \tag{2.16}$$

**Theorem 2.3.** *The subalgebra $\mathcal{S} \subset \mathcal{R}$ generated by $\{Y^\lambda := R_{\tau_\lambda}\tau_\lambda \mid \lambda \in \widetilde{M}_-\}$ forms a commutative algebra and is generated by $\{Y^{-\lambda_i} \mid i \in I\}$.*

# 3 Representation and Difference Operators

Let $\gamma_\alpha^{(1)} = \gamma_\alpha$ and $\gamma_\alpha^{(2)}$, $\gamma_\alpha^{(3)}$ be taken similarly as $\gamma_\alpha^{(1)}$. Accordingly let $\widetilde{M}^{(1)} = \widetilde{M}$ and $\widetilde{M}^{(2)}$, $\widetilde{M}^{(3)}$ be taken similarly. Here $\gamma_\alpha^{(i)}$ and $\gamma_\alpha^{(j)}$ may differ if $i \neq j$. Let $\mathcal{M}$ be the set of meromorphic functions on $V$. To define the action of $\widetilde{W}$ on $\mathcal{M}$, it is sufficient to specify the action of $s_i$ for $i \in I$ and $\tau_\lambda$ for $\lambda \in \widetilde{M}^{(1)}$. For $f \in \mathcal{M}$, we define

$$s_i(f)(v) = f(s_i(v)), \qquad \tau_\lambda(f)(v) = \tau_\lambda^{(1)}(f)(v) = f(v - \omega_1\lambda). \tag{3.1}$$

Fix $\xi^{(1)}$, $\zeta^{(1)} \in V$ and let $\mu_\alpha^{(1)}$, $\nu_\alpha^{(1)}$ be constants depending only on the length of roots. Then one can check that (3.1) and

$$(R_\alpha f)(v) := H_\alpha(\mu_\alpha^{(1)}, \nu_\alpha^{(1)}) f(v) - H_\alpha(\langle \xi^{(1)}, \alpha^\vee \rangle, \langle \zeta^{(1)}, \alpha^\vee \rangle) f(s_\alpha v), \tag{3.2}$$

satisfy the defining relations of the root algebra [8], where $H_\alpha(\eta, \kappa)$ is the meromorphic function defined by

$$H_\alpha(\eta, \kappa)(v) := e^{\kappa\alpha(v)} \frac{\sigma(\mu_\alpha^{(1)}; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}{\sigma(\eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)} \frac{\sigma(\alpha(v) + \eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}{\sigma(\alpha(v); \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}, \tag{3.3}$$

and a root $\alpha$ acts on $V$ as an affine linear functional $\alpha(v) = \langle \alpha, v \rangle + n\omega_1$ for $v \in V$ and $\alpha = \alpha' + n\delta$, $\alpha' \in \Delta$. Then we have the following theorems [6, 7]

**Theorem 3.1.** *Let $\mathcal{V} := \mathcal{M}^W$, the $W$-invariant subspace of $\mathcal{M}$ and let $\xi^{(1)} = -\rho_{\mu^{(1)}}$, $\zeta^{(1)} = -\rho_{\nu^{(1)}}$. Then $Y_\lambda^{(1)} := Y^\lambda \in \operatorname{End}_{\mathbb{C}} \mathcal{V}$.*

**Theorem 3.2.** *Let $(-\lambda) \in \widetilde{M}^{(1)}$ be minuscule. Then we have*

$$Y^\lambda|_{\mathcal{V}} = \frac{1}{|W_\lambda|} \sum_{w \in W} w \left( \prod_{\substack{\alpha \in \Delta_+ \\ \langle \lambda, \alpha \rangle = -\gamma_\alpha}} H_\alpha(\mu_\alpha^{(1)}, \nu_\alpha^{(1)}) \, \tau_\lambda^{(1)} \right) \Bigg|_{\mathcal{V}}, \tag{3.4}$$

*where $W_\lambda$ is the stabilizer of $\lambda$ in $W$.*

# 4 Commutativity in Case of Minuscule Weights

Let $Y_\lambda^{(2)}$ be the operator obtained by changing the role of the indices 1 and 2 in the construction of $Y_\lambda^{(1)}$, and $Y_\lambda^{(3)}$ be obtained in the same manner. Then one sees that $Y_\lambda^{(j)}$ for $\lambda \in \widetilde{M}^{(j)}$ are commutative [7]. In the following, we assume that $\mu_\alpha^{(j)} \eta_{kl} + \nu_\alpha^{(j)} \omega_k = \mu_\alpha^{(k)} \eta_{jl} + \nu_\alpha^{(k)} \omega_j$ where $j \neq k \neq l \neq j$. We demonstrate that $Y_\lambda^{(j)}$ and $Y_\nu^{(k)}$ are also commutative if $-\lambda$ and $-\nu$ are minuscule.

**Lemma 4.1 (Proposition 7.7 in [7]).** *Let $x_\alpha$ be constants dependent only on the length of roots. Then we have*

$$-\sum_{\alpha \in \Delta_{\tau_\lambda}} x_\alpha \overline{\alpha} = h_x^\vee \lambda. \tag{4.1}$$

**Theorem 4.2.** *Let $j, k \in \mathbb{Z}/3\mathbb{Z}$ and $-\lambda \in \widetilde{M}^{(j)}, -\nu \in \widetilde{M}^{(k)}$ be minuscule and Then $Y_\lambda^{(j)}$ and $Y_\nu^{(k)}$ are commutative.*

*Proof.* If $j = k$, then the statement follows from Theorem 6.3 in [7]. So it is sufficient to show the case $j \neq k$ and the commutativity of each term. Let

$$F_{j,\lambda} := \prod_{\substack{\alpha\in\Delta_+ \\ \langle\lambda,\alpha\rangle=-\gamma_\alpha}} e^{\nu_\alpha^{(j)}\langle\alpha,v\rangle} \frac{\sigma(\langle\alpha, v\rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}{\sigma(\langle\alpha, v\rangle; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}, \tag{4.2}$$

where $j \neq l \neq k$. Since for $u, w \in W$,

$$\begin{aligned} &\tau_{u\nu}^{(k)}\left(e^{\nu_\alpha^{(j)}\langle w\alpha,v\rangle} \frac{\sigma(\langle w\alpha, v\rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}{\sigma(\langle w\alpha, v\rangle; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}\right) && (4.3)\\ &= e^{\nu_\alpha^{(j)}\langle w\alpha,v-\omega_k u\nu\rangle} \frac{\sigma(\langle w\alpha, v - \omega_k u\nu\rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}{\sigma(\langle w\alpha, v - \omega_k u\nu\rangle; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)} \\ &= e^{-(\mu_\alpha^{(j)}\eta_{kl}+\nu_\alpha^{(j)}\omega_k)\langle w\alpha,u\nu\rangle} e^{\nu_\alpha^{(j)}\langle w\alpha,v\rangle} \frac{\sigma(\langle w\alpha, v\rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}{\sigma(\langle w\alpha, v\rangle; \gamma_\alpha^{(k)}\omega_k, \gamma_\alpha^{(l)}\omega_l)}, && (4.4) \end{aligned}$$

we have

$$\tau_{u\nu}^{(k)}(F_{j,w\lambda}) = e^{A\langle w\lambda,u\nu\rangle}(F_{j,w\lambda}), \tag{4.5}$$

where by Lemma 4.1

$$-\sum_{\substack{\alpha\in\Delta_+ \\ \langle\lambda,\alpha\rangle=-\gamma_\alpha}} (\mu_\alpha^{(j)}\eta_{kl} + \nu_\alpha^{(j)}\omega_k)\alpha = A\lambda. \tag{4.6}$$

Thus

$$\begin{aligned} F_{k,u\nu}\tau_{u\nu}^{(k)}F_{j,w\lambda}\tau_{w\lambda}^{(j)} &= e^{A\langle w\lambda,u\nu\rangle}F_{k,u\nu}F_{j,w\lambda}\tau_{u\nu}^{(k)}\tau_{w\lambda}^{(j)} \\ &= F_{j,w\lambda}\tau_{w\lambda}^{(j)}F_{k,u\nu}\tau_{u\nu}^{(k)}. \end{aligned} \tag{4.7}$$

□

For minuscule weights $-\lambda$, the periodicity of the coefficients is easily obtained since the explicit forms of the operators $Y^\lambda$ are calculated. However $Y^\lambda$ for general $\lambda$ is complicated and the proof of the commutativity requires a further investigation and is omitted here.

# 5 $BC_1$-type Operators and Eigenvector

Generally, it is very difficult to construct explicit eigenvectors of the elliptic Ruijsenaars operators. However we can construct a simultaneous eigenvector in the $BC_1$ root system. In this root system, there are three mutually commutative operators.

First we give the explicit forms of these operators. Let $\omega_1, \omega_2, \mu_0, \ldots, \mu_6 \in \mathbb{C}$ such that $\Im\omega_1, \omega_2 > 0$ and $2\mu_0 + \mu_1 + \cdots + \mu_6 + \omega_1 + \omega_2 = 0$. Then the commutative operators are given by

$$
\begin{aligned}
Y^{(1)} &= e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_1; \omega_2) \prod_{j=0}^{6} \vartheta(x - \mu_j; \omega_2)}{\vartheta(2x; \omega_2) \vartheta(2x + \omega_1; \omega_2)} (\tau(\omega_1) - 1), \\
&+ (x \leftrightarrow -x)
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
Y^{(2)} &= e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_2; \omega_1) \prod_{j=0}^{6} \vartheta(x - \mu_j; \omega_1)}{\vartheta(2x; \omega_1) \vartheta(2x + \omega_2; \omega_1)} (\tau(\omega_2) - 1), \\
&+ (x \leftrightarrow -x)
\end{aligned} \tag{5.2}
$$

$$
Y^{(3)} = \tau(1) + \tau(-1) - 2, \tag{5.3}
$$

where $\vartheta$ denotes the Jacobi odd theta function. For these commutative difference operators we can construct an explicit simultaneous eigenvector:

**Theorem 5.1.** *Let*

$$
\begin{aligned}
\phi(c) = &\frac{\prod_{i=1}^{7} \Gamma(pqc_0/c_i; p, q)}{\Gamma(pqc_0; p, q) \prod_{1 \le i < j \le 7} \Gamma(pqc_0/c_i c_j; p, q)} \\
&\times \int_C \frac{\Gamma(pq/(pqc_0)^{1/2} z^{\pm 1}; p, q) \prod_{j=1}^{7} \Gamma((pqc_0)^{1/2}/c_j z^{\pm 1}; p, q)}{\Gamma(z^2, z^{-2}; p, q)} \frac{dz}{z},
\end{aligned} \tag{5.4}
$$

*with*

$$c_0 = e^{2\pi i(\mu_0-\mu_1-\omega_1-\omega_2)}, \quad c_1 = e^{2\pi i(\mu_0+\mu_2)}, \quad c_2 = e^{2\pi i(\mu_0+\mu_3)},$$
$$c_3 = e^{2\pi i(\mu_0+\mu_4)}, \quad c_4 = e^{2\pi i(\mu_0+\mu_5)}, \quad c_5 = e^{2\pi i(\mu_0+\mu_6)},$$
$$c_6 = e^{2\pi i(-x-\mu_1)}, \quad c_7 = e^{2\pi i(x-\mu_1)},$$
$$p = e^{2\pi i\omega_1}, \quad q = e^{2\pi i\omega_2},$$

*and $C$ a closed circle taken appropriately, and $\Gamma(z;p,q)$ elliptic Gamma function defined by*

$$\Gamma(z;p,q) = \frac{(pqz^{-1};p,q)_\infty}{(z;p,q)_\infty}.$$

*Then $\phi(c)$ is a simultaneous eigenvector of $Y^{(j)}$ with the eigenvalues*

$$E^{(1)} = e^{-2\pi i\mu_0}\prod_{i=1}^{6}\vartheta(\mu_0+\mu_i;\omega_2), \quad E^{(3)} = 0,$$
$$E^{(2)} = e^{-2\pi i\mu_0}\prod_{i=1}^{6}\vartheta(\mu_0+\mu_i;\omega_1).$$

*sketch of proof.* One can show the following contiguous relations directly or by use of the result in [9]:

$$\begin{aligned}\frac{\langle c_i\rangle\prod_{k\neq i,j}\langle pqc_0/c_jc_k\rangle}{\langle pqc_0/c_j, p^2q^2c_0/c_j\rangle}\phi^+(c_j^-) - \frac{\langle c_j\rangle\prod_{k\neq i,j}\langle pqc_0/c_ic_k\rangle}{\langle pqc_0/c_i, p^2q^2c_0/c_i\rangle}\phi^+(c_i^-)\\ = \frac{\langle c_i/c_j\rangle\prod_{k\neq i,j}\langle pqc_0/c_k\rangle}{\langle pqc_0, p^2q^2c_0\rangle}\phi,\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\phi - \phi(c_j^-, c_i^+)\\ &= \frac{\langle pqc_i/c_j\rangle\langle pqc_0/c_ic_j\rangle\langle pqc_0\rangle\langle p^2q^2c_0\rangle}{\langle pqc_0/c_j\rangle\langle p^2q^2c_0/c_j\rangle\langle c_0/c_i\rangle\langle pqc_0/c_i\rangle}\prod_{k\neq i,j}\frac{\langle c_k\rangle}{\langle pqc_0/c_k\rangle}\phi^+(c_j^-),\end{aligned} \tag{5.6}$$

where $\langle\cdot\rangle$ is the multiplicative form of $\vartheta(\cdot;p)$, $\phi^+ = \phi(q^2c_0, qc_1, \ldots, qc_7)$ and $\phi(c_j^\pm) = \phi(c_0, c_1, \ldots, q^{\pm1}c_j, \ldots, c_7)$. Combining these relations, we see that $\phi(c)$ is an eigenvector of $Y^{(1)}$ and thus is an eigenvector of $Y^{(2)}$ due to the symmetry of $p$ and $q$, and that by the definition of $c_6$ and $c_7$, $\phi(c)$ is an eigenvector of $Y^{(3)}$. □

# 6 Special Cases

In this section, we clarify the relation between $\phi(c)$ and the elliptic hypergeometric series ${}_{10}E_9$ introduced by Frenkel and Turaev [3].

**Theorem 6.1.** *If* $c_i = p^{-M}q^{-N}$, *then*

$$\phi(c) = \tilde{\phi}(c_0, \ldots, q^{-N}, \ldots, c_7; p) \times \tilde{\phi}(c_0, \ldots, p^{-M}, \ldots, c_7; q), \tag{6.1}$$

*where*

$$\tilde{\phi}(c_0, \ldots, c_7; r) = \sum_{k=0}^{\infty} \frac{\langle (pq)^{2k} c_0; r\rangle}{\langle c_0; r\rangle} \frac{\langle\langle c_0; r\rangle\rangle_k}{\langle\langle pq; r\rangle\rangle_k} \prod_{i=1}^{7} \frac{\langle\langle c_i; r\rangle\rangle_k}{\langle\langle pqc_0/c_i; r\rangle\rangle_k}, \tag{6.2}$$

*and* $\langle\langle u; r\rangle\rangle_k = \langle u; r\rangle \cdots \langle u(pq)^{k-1}; r\rangle$.

*sketch of proof.* We see that both the right hand side of (6.1) and the elliptic hypergeometric integral (5.4) satisfy the contiguous relation (5.6), with the same initial condition. □

Now we state the relation between $\phi(c)$ and ${}_{10}E_9$ which is defined by

$${}_{10}E_9(b_0; b_1, \ldots, b_7; r) = \sum_{k=0}^{\infty} \frac{\langle q^{2k} b_0; r\rangle}{\langle b_0; r\rangle} \frac{\langle b_0; r\rangle_k}{\langle r; r\rangle_k} \prod_{i=1}^{7} \frac{\langle b_i; r\rangle_k}{\langle rb_0/b_i; r\rangle_k},$$

where one of $b_i$ should be $r^{-N}$ and $\langle u; r\rangle_k = \langle u; r\rangle \cdots \langle ur^{k-1}; r\rangle$. The following is shown by a direct calculation.

**Theorem 6.2.**

$$\tilde{\phi}(c_0, \ldots, c_7; p) = {}_{10}E_9(pc_0; c_1, \ldots, pc_k, \ldots, c_7; p),$$

*which is independent of the choice of* $k$.

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