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A generalization of a curvature flow of graphs on $\mathbf{R}$

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1. Introduction. Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see [3, 5, 6] and the reference therein). In [5] we proposed and studied the discrete stochastic approximations of nonconvex functions which evolve by a convexified Gauss curvature and the PDE which appears as the continuum limit of discrete stochastic processes (see [6] for the similar results on the convexified Gauss curvature flow of closed hypersurfaces). In this paper we study a class of PDE which gives a generalization of a curvature flow of graphs on $\mathbf{R}$.

We briefly describe [5] to discuss the results in this paper more precisely. Alexandrov-Bakelman's generalized curvature played a crucial role in [5].

Definition 1 *(see e.g. [1, section 9.6]). Let* $R \in L^1(\mathbf{R}^n : [0, \infty), dx)$ *and* $u \in C(\mathbf{R}^n)$. *For* $A \in B(\mathbf{R}^n)$(:=*Borel* σ*-field of* $\mathbf{R}^n$*), put*

$$w(R, u, A) := \int_{\cup_{x \in A} \partial u(x)} R(y)dy, \tag{1}$$

where $\partial u(x) := \{p \in \mathbf{R}^n | u(y) - u(x) \geq < p, y - x > \text{ for all } y \in \mathbf{R}^n\}$ *and* $< \cdot, \cdot >$ *denotes the inner product in* $\mathbf{R}^n$.

(It is known that $w(R,u,\cdot) : B(\mathbf{R}^n) \mapsto [0,\infty)$ is completely additive.)

For $R \in L^1(\mathbf{R}^n : [0,\infty), dx)$, we showed the existence and the uniqueness of a solution $u \in C([0,\infty) \times \mathbf{R}^n)$ to the following equation (see [5, Theorem 1]): for any $\varphi \in C_o(\mathbf{R}^n)$ and any $t \geq 0$,

$$\int_{\mathbf{R}^n} \varphi(x)(u(t,x) - u(0,x))dx = \int_0^t ds \int_{\mathbf{R}^n} \varphi(x) w(R, u(s,\cdot), dx). \qquad (2)$$

In [5, Theorem 2], we proved that a continuous solution u to (2) sweeps in time $t > 0$ a region with volume given by $t \cdot w(R, u(0,\cdot), \mathbf{R}^n)$, and that, for a continuous solution u to (2) with a convex $u(0,\cdot)$, $x \mapsto u(t,\cdot)$ is convex for all $t > 0$.

We also showed that a continuous solution u to (2) is a viscosity solution of the following PDE (see [5, Theorem 3]):

$$\begin{aligned} \partial_t u(t,x) &= \chi(u, Du(t,x), t, x) R(Du(t,x)) \qquad (3) \\ &\quad \times \max(\mathrm{Det}(D^2 u(t,x)), 0) \quad ((0,\infty) \times \mathbf{R}^n), \end{aligned}$$

where $Du(t,x) := (\partial u(t,x)/\partial x_i)_{i=1}^n$, $D^2 u(t,x) := (\partial^2 u(t,x)/\partial x_i \partial x_j)_{i,j=1}^n$ and

$$\chi(u,p,t,x) := \begin{cases} 1 & \text{if } p \in \partial u(t,x), \\ 0 & \text{otherwise.} \end{cases}$$

Here $\partial u(t,x)$ denotes the subdifferential of the function $x \mapsto u(t,x)$. Conversely, we discussed under what conditions a viscosity solution to (3) is a solution to (2).

We briefly discuss what we study in this paper. In (1) we only considered the measure $R(y)dy$ which is absolutely continuous with respect to the Lebesgue measure dy. Otherwise $\omega(R, u, dx)$ is not generally completely additive.

Suppose that $n = 1$. In (1), replace $R(y)dy$ by a continuous Borel probability measure $P(dy)$. Then, using a similar notation, $\omega(P, u, dx)$ turn out to be a measure and (2) has a unique continuous solution u (see Theorem 1 in section 2).

Since $P(dy)$ is not generally absolutely continuous with respect to dy, we can not consider the PDE for u. For $(t, x) \in [0, \infty) \times \mathbf{R}$, put

$$U(t,x) := \int_{-\infty}^{x} (u(t,y) - u(0,y))dy + \int_{0}^{x} u(0,y)dy + tF(a), \tag{4}$$

where

$$F(x) := P((-\infty, x]), \quad a := \inf\{\cup_{x\in\mathbf{R}} \partial u(0,x)\}.$$

Then from (2),

$$U(t,x) - U(0,x) = \int_0^t F(D_+(\hat{DU}(s,x)))ds. \tag{5}$$

Here $\hat{u}$ denotes a convex envellope of u and for φ, $D_+\varphi(t,x)$ denotes the right derivative of $x \mapsto \varphi(t,x)$.

When $DU(0,x)$ is convex, we show that $U(t,x)$ is a unique continuous viscosity solution in $(0,\infty) \times \mathbf{R}$ of the following (see Theorems 2 and 3 in section 2):

$$\partial_t U(t,x) = F(D^2U(t,x)). \tag{6}$$

Definition 2 (Viscosity solution) *(1) Let $\Omega = (0,\infty) \times \mathbf{R}$.*
(i). A function $U \in USC(\Omega)$ is called a viscosity subsolution of (6) in Ω if whenever $\varphi \in C^{1,2}(\Omega)$, $(s,y) \in \Omega$, and $U - \varphi$ attains a local maximum at (s,y), then $\partial_t\varphi(s,y) \leq F(D^2\varphi(s,y))$.

(ii). A function $U \in LSC(\Omega)$ is called a viscosity supersolution of (6) in Ω if whenever $\varphi \in C^{1,2}(\Omega)$, $(s,y) \in \Omega$, and $U - \varphi$ attains a local minimum at (s,y), then $\quad \partial_t\varphi(s,y) \geq F(D^2\varphi(s,y))$.

(iii). A function $U \in C(\Omega)$ is called a viscosity solution of (6) in Ω if it is both a viscosity subsolution and a viscosity supersolution of (6) in Ω.

(2) Let $t_2 > t_1 > 0$, O be an open subset of $\mathbf{R}$ and $Q := (t_1, t_2] \times O$. A function $U \in C(\overline{Q})$ is called a viscosity solution of (6) in Q if (1,i)-(1,ii) with Ω replaced by Q hold (see [4, p. 66]). Here $\overline{Q}$ denotes the closure of Q.

2. Main results. We state assumptions under which we can generalize [5] when $n = 1$.

(A.1). P is a continuous Borel probability measure on $\mathbf{R}$.

(A.2). $h \in C(\mathbf{R})$ and the set $\partial h(\mathbf{R})$ has a positive Lesbegue measure.

(A.3). For any $p \notin \partial h(\mathbf{R})$ and $C \in \mathbf{R}$,

$$\int_{\mathbf{R}} \max(px + C - h(x), 0)dx = \infty.$$

Theorem 1 *Suppose that (A.1)-(A.3) hold. Then there exists a unique continuous solution u to (2) with $u(0,\cdot) = h$.*

The following assumption implies (A.2)-(A.3).

(A.2)'. h is convex and is not a constant.

Theorem 2 *Suppose that (A.1) and (A.2)' hold. Then the function U defined, by (4), from u in Theorem 1 is a continuous viscosity solution of (6) in $(0,\infty) \times \mathbf{R}$.*

As a regularity result, we have

Proposition 1 *Suppose that (A.1) holds. Then for a continuous viscosity solution v of (6) in $(0,\infty) \times \mathbf{R}$,*

$$0 \leq v(t,x) - v(s,x) \leq t - s \quad (0 \leq s < t, x \in \mathbf{R}). \tag{7}$$

In particular, $t \mapsto v(t,x)$ is absolutely continuous for all $x \in \mathbf{R}$.

We state an additional assumption and an asymptotic behavior of a viscosity solution of (6).

(A.4). (i) $v_0 : \mathbf{R} \mapsto \mathbf{R}$ is twice continuously differentiable.

(ii) $\lim_{x \to -\infty} D^2 v_0(x)$ and $\lim_{x \to \infty} D^2 v_0(x)$ exist and

$$a := \inf_{x \in \mathbf{R}} D^2 v_0(x) = \lim_{x \to -\infty} D^2 v_0(x),$$

$$b := \sup_{x \in \mathbf{R}} D^2 v_0(x) = \lim_{x \to \infty} D^2 v_0(x).$$

Proposition 2 *Suppose that (A.1) and (A.4,i) hold. Then for a continuous viscosity solution v of (6) with $v(0,\cdot) = v_0(\cdot)$ in $(0,\infty) \times \mathbf{R}$, the following holds: for any $t \geq 0$ and $x \in \mathbf{R}$,*

$$F(a)t \leq v(t,x) - v(0,x) \leq F(b)t. \tag{8}$$

Suppose in addition that (A.4,ii) holds. Then for any $T \geq 0$,

$$\lim_{x \to -\infty} \left(\sup_{0 \leq t \leq T} |v(t,x) - v(0,x) - F(a)t| \right) = 0, \tag{9}$$

$$\lim_{x \to \infty} \left(\sup_{0 \leq t \leq T} |v(t,x) - v(0,x) - F(b)t| \right) = 0. \tag{10}$$

Since F is nondecreasing, (6) is a degenerate elliptic PDE and we can use the maximum principle for this equation in a bounded domain (see [4, p. 244, Theorem 8.1] and also [2]). From Prop. 2, we immediately obtain

Theorem 3 *Suppose that (A.1) and (A.4) hold. Then the viscosity solution* v *of (6) with* $v(0,\cdot) = v_0(\cdot)$ *is unique in* $C([0,\infty) \times \mathbf{R})$.

From Theorems 2 and 3, we also obtain

Corollary 1 *Suppose that (A.1) and (A.2)' hold and that* h *is continuously differentiable. Then* U *in Theorem 2 is the unique continuous viscosity solution of (6) with* $U(0,x) = \int_0^x h(y)dy$ *in* $(0,\infty) \times \mathbf{R}$.

In particular, we have

Corollary 2 *Suppose that (A.1) and (A.4, i) hold and that* Dv_0 *is convex. Then (6) with* $v(0,\cdot) = v_0(\cdot)$ *has the unique viscosity solution in* $C([0,\infty) \times \mathbf{R})$.

(Proof of Theorem 1). The proof can be done almost in the same way as in [5, Theorem 1] (In [5, (A.3)], "$x \in \mathbf{R}^d$" should be "$(x, \hat{h}(x)) \in \mathbf{R}^{d+1}$"). The only thing we have to prove is the following (i)-(ii):

(i) For a convex $u \in C(\mathbf{R})$, $w(P, u, dx)$ is completely additive,

(ii) For $u \in C(\mathbf{R})$ for which $\partial u(\mathbf{R}) \neq \emptyset$, $w(P, u, dx) = w(P, \hat{u}, dx)$.

We first prove (i). The set

$$S(u) := \{p \in \mathbf{R} | \{x \in \mathbf{R} | p \in \partial u(x)\} \text{ contains more than one point}\}$$

contains at most countably many points. Indeed, if $p \in S(u)$, then for x_p for which $p \in \partial u(x_p)$, the graph of $y = u(x)$ and the straight line $y = p(x - x_p) + u(x_p)$ contains a line segment with a positive length and the interiors of such line segments are disjoint. Hence for each n, $m \geq 1$, on the set $S(u) \cap [D_- u(-n), D_+ u(n)]$, such line segments with the length $\geq 1/m$ are finitely many. Here $D_- u(x)$ denotes the left derivative of $x \mapsto u(x)$. Since a continuous measure does not have a point mass, we obtain (i) (see [1, p. 118]).

Next we prove (ii). If $u(x) = \hat{u}(x)$, then $\partial u(x) = \partial \hat{u}(x)$. If $u(x) \neq \hat{u}(x)$, then $\partial u(x) = \emptyset$ and $\partial \hat{u}(x) = D\hat{u}(x) \in S(\hat{u})$. Since $P(S(\hat{u})) = 0$ as we explained above, we obtain (ii). □

(Proof of Theorem 2). (A.2)' implies that $x \mapsto DU(t,x)$ is convex for all $t \geq 0$ (see [5, Theorem 2]). We first prove that U is a viscosity subsolution to (6) in $(0,\infty) \times \mathbf{R}$. Suppose that $\varphi \in C^{1,2}((0,\infty) \times \mathbf{R})$, $(s,y) \in (0,\infty) \times \mathbf{R}$, and $U - \varphi$ attains a local maximum at (s,y). Then for x and $y \in \mathbf{R}$ for which $x - y$ is positive,

$$\begin{aligned} \partial_t \varphi(s,y) &\leq F\Big(\frac{U(s,x) - U(s,y) - DU(s,y)(x-y)}{(x-y)^2/2}\Big) \\ &\leq F\Big(\frac{\varphi(s,x) - \varphi(s,y) - D\varphi(s,y)(x-y)}{(x-y)^2/2}\Big) \\ &\to F(D^2\varphi(s,y)) \quad (x \downarrow y). \end{aligned} \tag{11}$$

Indeed, from (5), for t and $s \geq 0$ for which $s - t$ is positive and is sufficiently small,

$$\varphi(s,y) - \varphi(t,y) \leq U(s,y) - U(t,y) = \int_t^s F(D_+(DU(\alpha,y)))d\alpha,$$

$$\begin{aligned} & U(\alpha,x) - U(\alpha,y) - DU(\alpha,y)(x-y) \\ = & \int_y^x (DU(\alpha,z) - DU(\alpha,y))dz \\ \geq & \int_y^x D_+(DU(\alpha,y))(z-y)dz = D_+(DU(\alpha,y))(x-y)^2/2. \end{aligned}$$

Since U and $DU \in C([0,\infty) \times \mathbf{R})$ from Theorem 1, we obtain the first ineqaulity in (11). Since $DU(s,y) = D\varphi(s,y)$ and F is nondecreasing, the second inequality of (11) holds.

Next we prove that U is a viscosity supersolution to (6) in $(0,\infty)\times\mathbf{R}$. Suppose that $\varphi\in C^{1,2}((0,\infty)\times\mathbf{R})$, $(s,y)\in(0,\infty)\times\mathbf{R}$, and $U-\varphi$ attains a local minimum at (s,y). Then in the same way as in (11), for x and $y\in\mathbf{R}$ for which $y-x$ is positive,

$$\begin{aligned}\partial_t\varphi(s,y) &\geq F\Big(\frac{U(s,x)-U(s,y)-DU(s,y)(x-y)}{(x-y)^2/2}\Big)\\ &\geq F\Big(\frac{\varphi(s,x)-\varphi(s,y)-D\varphi(s,y)(x-y)}{(x-y)^2/2}\Big)\\ &\to F(D^2\varphi(s,y))\quad (x\uparrow y).\square\end{aligned}$$

(Proof of Proposition 1) Without loss of generality, we can put $s=0$. We first prove the first inequality of (7). Suppose that there exists $(t_0,x_0)\in(0,\infty)\times\mathbf{R}$ such that

$$v(t_0,x_0)-v(0,x_0)<0. \tag{12}$$

Put

$$\begin{aligned}C_0 &:= \min\{v(t,x)-v(0,x)|0\leq t\leq t_0,|x-x_0|\leq 1\}(<0), \quad (13)\\ \underline{v}(t,x) &:= v(0,x)+C_0(x-x_0)^2.\end{aligned}$$

Then it is easy to see that $\underline{v}$ is a viscosity subsolution of (6) in $(0,t_0]\times(x_0-1,x_0+1)$ since $F\geq 0$. By the maximum principle (see [4, p.244, Th. 8.1]),

$$\begin{aligned}&\min\{v(t,x)-\underline{v}(t,x)|0\leq t\leq t_0,|x-x_0|\leq 1\} \quad (14)\\ =\ &\min\{v(t,x)-\underline{v}(t,x)|0\leq t\leq t_0,|x-x_0|=1 \text{ or } t=0,|x-x_0|\leq 1\}\\ \geq\ &0,\end{aligned}$$

from (13). This contradicts (12).

Next we prove the second inequality of (7). Suppose that there exists $(\bar{t}_0, \bar{x}_0) \in (0, \infty) \times \mathbf{R}$ such that

$$v(\bar{t}_0, \bar{x}_0) - v(0, \bar{x}_0) - \bar{t}_0 > 0. \tag{15}$$

Put

$$\begin{aligned} \overline{C}_0 &:= \max\{v(t,x) - v(0,x) - t \,|\, 0 \le t \le \bar{t}_0, |x - \bar{x}_0| \le 1\}(> 0), \quad (16) \\ \overline{v}(t,x) &:= v(0,x) + t + \overline{C}_0(x - \bar{x}_0)^2. \end{aligned}$$

Then it is easy to see that $\overline{v}$ is a viscosity supersolution of (6) in $(0, \bar{t}_0] \times (\bar{x}_0 - 1, \bar{x}_0 + 1)$ since $F \le 1$. By the maximum principle,

$$\begin{aligned} & \max\{v(t,x) - \overline{v}(t,x) \,|\, 0 \le t \le \bar{t}_0, |x - \bar{x}_0| \le 1\} \quad (17) \\ = & \max\{v(t,x) - \overline{v}(t,x) \,|\, 0 \le t \le \bar{t}_0, |x - \bar{x}_0| = 1 \text{ or } t = 0, |x - \bar{x}_0| \le 1\} \\ \le & \ 0, \end{aligned}$$

from (16). This contradicts (15). □

(Proof of Proposition 2) First of all, we prove the first inequality in (8). Suppose that there exists $(t_0, x_0) \in (0, \infty) \times \mathbf{R}$ for which $F(a)t_0 > v(t_0, x_0) - v(0, x_0)$. Take $\varepsilon_0 > 0$ so that

$$v(t_0, x_0) - (v(0, x_0) + F(a)t_0 - \varepsilon_0 t_0) < 0. \tag{18}$$

For $n \ge 1$, put

$$\begin{aligned} C_n &:= \min\{v(t,x) - (v(0,x) + F(a)t) \,|\, 0 \le t \le t_0, |x - x_0| \le n\}, \quad (19) \\ \psi_n(t,x) &:= v(t,x) - \Big(v(0,x) + F(a)t - \varepsilon_0 t + \frac{C_n(x - x_0)^2}{n^2}\Big). \end{aligned}$$

Then from (18)-(19),

$$\begin{aligned} & \min\{\psi_n(t,x) | 0 \le t \le t_0, |x - x_0| \le n\} \\ = \; & \min\{\psi_n(t,x) | 0 < t \le t_0, |x - x_0| < n\}. \qquad (20) \end{aligned}$$

Take $(t_n, x_n) \in (0, t_0] \times (x_0 - n, x_0 + n)$ which attains the minimum in (20). Since v is a viscosity supersolution of (6) and since $|C_n| \le t_0$ from Prop.1,

$$F(a) - \varepsilon_0 \ge F\Big(D^2 v(0, x_n) + \frac{2C_n}{n^2}\Big) \ge F\Big(a + \frac{2C_n}{n^2}\Big) \to F(a), \qquad (21)$$

as $n \to \infty$, which is a contradiction.

Next we prove the second inequality in (8). Suppose that there exists $(\bar{t}_0, \bar{x}_0) \in (0,\infty) \times \mathbf{R}$ for which $F(b)\bar{t}_0 < v(\bar{t}_0, \bar{x}_0) - v(0, \bar{x}_0)$. Take $\bar{\varepsilon}_0 > 0$ so that

$$v(\bar{t}_0, \bar{x}_0) - (v(0, \bar{x}_0) + F(b)\bar{t}_0 + \bar{\varepsilon}_0 \bar{t}_0) > 0. \qquad (22)$$

For $n \ge 1$, put

$$\begin{aligned} \overline{C}_n &:= \max\{v(t,x) - (v(0,x) + F(b)t) | 0 \le t \le \bar{t}_0, |x - \bar{x}_0| \le n\}, \qquad (23) \\ \overline{\psi}_n(t,x) &:= v(t,x) - \Big(v(0,x) + F(b)t + \bar{\varepsilon}_0 t + \frac{\overline{C}_n (x - \bar{x}_0)^2}{n^2}\Big). \end{aligned}$$

Then from (22)-(23),

$$\begin{aligned} & \max\{\overline{\psi}_n(t,x) | 0 \le t \le \bar{t}_0, |x - \bar{x}_0| \le n\} \\ = \; & \max\{\overline{\psi}_n(t,x) | 0 < t \le \bar{t}_0, |x - \bar{x}_0| < n\}. \qquad (24) \end{aligned}$$

Take $(\bar{t}_n, \bar{x}_n) \in (0, \bar{t}_0] \times (\bar{x}_0 - n, \bar{x}_0 + n)$ which attains the maximum in (24). Since v is a viscosity subsolution of (6) and since $|\overline{C}_n| \le \bar{t}_0$ from Prop.1,

$$F(b)+\bar{\varepsilon}_0 \le F\Big(D^2v(0,\bar{x}_n)+\frac{2\overline{C}_n}{n^2}\Big) \le F\Big(b+\frac{2\overline{C}_n}{n^2}\Big) \to F(b), \tag{25}$$

as $n \to \infty$, which is a contradiction.

Next we prove (9)-(10). From (8), we only have to prove the following:

$$\limsup_{x\to-\infty}\Big(\sup_{0\le t\le T}\{v(t,x)-v(0,x)-F(a)t\}\Big) \le 0, \tag{26}$$

$$\liminf_{x\to\infty}\Big(\inf_{0\le t\le T}\{v(t,x)-v(0,x)-F(b)t\}\Big) \ge 0. \tag{27}$$

We first prove (26). Suppose that (26) does not hold. Then there exists $\varepsilon_1 > 0$ so that

$$\limsup_{x\to-\infty}\Big(\sup_{0\le t\le T}\{v(t,x)-v(0,x)-F(a)t-\varepsilon_1 t\}\Big) > 0. \tag{28}$$

In particular, there exists $(s_n, y_n) \in (0,T] \times (-\infty, -n^2)$ for which

$$v(s_n,y_n)-v(0,y_n)-F(a)s_n-\varepsilon_1 s_n > 0 \quad (n \ge 1). \tag{29}$$

For $n \ge 1$, put

$$\begin{aligned} \gamma_n &:= \max\{v(t,x)-(v(0,x)+F(a)t)|0\le t\le T, |x-y_n|\le n\}, \quad (30)\\ \phi_n(t,x) &:= v(t,x)-\Big(v(0,x)+F(a)t+\varepsilon_1 t+\frac{\gamma_n(x-y_n)^2}{n^2}\Big). \end{aligned}$$

Then from (29)-(30).

$$\begin{aligned} &\max\{\phi_n(t,x)|0\le t\le T, |x-y_n|\le n\}\\ =\ &\max\{\phi_n(t,x)|0< t\le T, |x-y_n|< n\}. \end{aligned} \tag{31}$$

Take $(r_n, z_n) \in (0,T] \times (y_n - n, y_n + n)$ which attains the maximum in (31). Since v is a viscosity subsolution of (6), $z_n < -n^2 + n \to -\infty$ as $n \to \infty$ and $|\gamma_n| \le T$ from Prop. 1,

$$F(a) + \varepsilon_1 \le F\Big(D^2 v(0, z_n) + \frac{2\gamma_n}{n^2}\Big) \to F(a) \quad (n \to \infty), \tag{32}$$

which is a contradiction.

Next we prove (27). Suppose that (27) does not hold. Then there exists $\overline{\varepsilon}_1 > 0$ so that

$$\liminf_{x\to\infty}\Big(\inf_{0\le t\le T}\{v(t,x) - v(0,x) - F(b)t + \overline{\varepsilon}_1 t\}\Big) < 0. \tag{33}$$

In particular, there exists $(\overline{s}_n, \overline{y}_n) \in (0,T] \times (n^2, \infty)$ for which

$$v(\overline{s}_n, \overline{y}_n) - v(0, \overline{y}_n) - F(b)\overline{s}_n + \overline{\varepsilon}_1 \overline{s}_n < 0 \quad (n \ge 1). \tag{34}$$

Put

$$\begin{aligned} \overline{\gamma}_n &:= \min\{v(t,x) - (v(0,x) + F(b)t) | 0 \le t \le T, |x - \overline{y}_n| \le n\}, \quad (35) \\ \overline{\phi}_n(t,x) &:= v(t,x) - \Big(v(0,x) + F(b)t - \overline{\varepsilon}_1 t + \frac{\overline{\gamma}_n (x - \overline{y}_n)^2}{n^2}\Big). \end{aligned}$$

Then from (34)-(35),

$$\begin{aligned} &\min\{\overline{\phi}_n(t,x) | 0 \le t \le T, |x - \overline{y}_n| \le n\} \\ =\ &\min\{\overline{\phi}_n(t,x) | 0 < t \le T, |x - \overline{y}_n| < n\}. \end{aligned} \tag{36}$$

Take $(\overline{r}_n, \overline{z}_n) \in (0,T] \times (\overline{y}_n - n, \overline{y}_n + n)$ which attains the minimum in (36). Since v is a viscosity supersolution of (6), $\overline{z}_n > n^2 - n \to \infty$ as $n \to \infty$ and $|\overline{\gamma}_n| \le T$ from Prop. 1,

$$F(b) - \bar{\varepsilon}_1 \geq F\Big(D^2 v(0, \bar{z}_n) + \frac{2\overline{\gamma}_n}{n^2}\Big) \to F(b) \quad (n \to \infty), \tag{37}$$

which is a contradiction.□

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