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A note on demi-eigenvalues for uniformly elliptic Isaacs operators

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1 Introduction and main result

This paper is based on a joint work [8] with Prof. H. Ishii.

We investigate the "eigenvalue problem" for fully nonlinear uniformly elliptic operators.

In 1983, P.-L.Lions [12] studied the eigenvalue problem for the uniformly elliptic Bellman equations

$$\sup_{\alpha\in A}\left\{-\mathrm{tr}\ a_\alpha(x)D^2u(x)+b_\alpha(x)Du(x)+c_\alpha(x)u(x)-f_\alpha(x)\right\}=0 \text{ in } \Omega, \quad (1.1)$$

$$u|_{\partial\Omega}=0, \quad (1.2)$$

where Ω is an open bounded subset of $\mathbb{R}^n$, A is an index set, $a_\alpha, b_\alpha, c_\alpha$ and f_α are Lipschitz functions on $\overline{\Omega}$ with values in $\mathcal{S}^n, \mathbb{R}^n, \mathbb{R}$ and $\mathbb{R}$, respectively, and u is the real-valued unknown function on $\overline{\Omega}$. Here $\mathcal{S}^n$ denotes the space of real $n\times n$ symmetric matrices. He studied a sort of principle eigenvalues and eigenfunctions for nonlinear uniformly elliptic operator $F[\cdot]:\Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n\to\mathbb{R}$, where F is given by

$$F(x,r,p,X)=\sup_{\alpha\in A}\left\{-\mathrm{tr}\ a_\alpha X+b_\alpha(x)\cdot p+c_\alpha r\right\}. \quad (1.3)$$

He called these values *demi-eigenvalues.* He established several interesting properties of demi-eigenvalues including existence of the corresponding demi-eigenfunctions by using stochastic control theory.

Here we investigate the demi-eigenvalue problem for general non-convex fully nonlinear uniformly elliptic operators $F[\cdot]$. We consider fully nonlinear elliptic PDEs

$$F[u](x)=F(x,u(x),Du(x),D^2u(x))=0 \text{ in } \Omega. \quad (1.4)$$

Here F is not assumed to have any convexity because we refer (1.4) as Isaacs operators

$$F(x,r,p,X)=\sup_{\alpha\in A}\inf_{\beta\in B}\left\{-\mathrm{tr}\ a_{\alpha,\beta}(x)X+b_{\alpha,\beta}(x)\cdot p+c_{\alpha,\beta}(x)r-f_{\alpha,\beta}(x)\right\} \quad (1.5)$$

Moreover we adapt the notion of viscosity solution as the solution of (1.4).

We prepare some assumptions on Ω and F. Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and F is a continuous function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$. In addition, we often assume:

(D1) Ω satisfies the uniform exterior sphere condition, i.e., there is a constant $r_1 > 0$ such that for each $z \in \partial\Omega$ there is a point $y \in \mathbb{R}^n$ for which $B(y, r_1) \cap \overline{\Omega} = \{z\}$.

(D2) Ω satisfies the uniform interior sphere condition, i.e., there is a constant $r_2 > 0$ such that for each $z \in \partial\Omega$ there is a point $y \in \Omega$ for which $|z - y| = r_2$ and $B(y, r_2) \subset \overline{\Omega}$.

(F1) F is uniformly elliptic, i.e., there are constants $0 < \theta \leq \Theta < \infty$ such that for $(x, r, p, X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $Y \in \mathcal{S}^n$,

$$\mathcal{P}^-(Y) \leq F(x, r, p, X + Y) - F(x, r, p, X) \leq \mathcal{P}^+(Y),$$

where $\mathcal{P}^\pm$ denote the Pucci extremal operators:

$$\mathcal{P}^-(X) := \inf\{-\mathrm{tr}\, AX \mid A \in \mathcal{S}^n,\ \theta I \leq A \leq \Theta I\},$$
$$\mathcal{P}^+(X) := \sup\{-\mathrm{tr}\, AX \mid A \in \mathcal{S}^n,\ \theta I \leq A \leq \Theta I\}.$$

(F2) For each $(x, X) \in \overline{\Omega} \times \mathcal{S}^n$, the function: $(r, p) \mapsto F(x, r, p, X)$ is Lipschitz continuous on $\mathbb{R} \times \mathbb{R}^n$. More precisely, there is a constant $L > 0$ such that for $(x, X) \in \overline{\Omega} \times \mathcal{S}^n$ and $(r, p),\ (t, q) \in \mathbb{R} \times \mathbb{R}^n$,

$$|F(x, r, p, X) - F(x, t, q, X)| \leq L(|r - t| + |p - q|).$$

(F3) For each $R > 0$, there are a constant $\gamma \in (\frac{1}{2}, 1]$ and a function ω_R, satisfying $\sup_{t \geq 0} \frac{\omega_R(t)}{t+1} < \infty$ such that for all $x, y \in \overline{\Omega}$ and $(r, p, X) \in [-R, R] \times B(0, R) \times \mathcal{S}^n$,

$$|F(x, r, p, X) - F(y, r, p, X)| \leq \omega_R(|x - y|^\gamma(1 + \|X\|)),$$

where ω_R is so called a *modulus*, i.e., it is assumed that $\omega_R \in C([0, \infty])$ is non-decreasing in $[0, \infty)$ and $\omega_R(0) = 0$. Here the norm $\|X\|$ on $\mathcal{S}^n$ is $\|X\| = \sup\{|X\xi| \mid \xi \in \mathbb{R}^n,\ |\xi| = 1\}$.

(F4) For all $x \in \overline{\Omega}$, $\xi \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $s \geq 0$,

$$F(x, s\xi) = sF(x, \xi).$$

From assumption (F3) we get the following condition on F:

(F5) There are constants $\gamma \in (\frac{1}{2}, 1]$ and $C_0 > 0$, $C_1 > 0$ such that for $x, y \in \overline{\Omega}$ and $X \in \mathcal{S}^n$,

$$|F(x, 0, 0, X) - F(y, 0, 0, X)| \leq C_0 + C_1|x - y|^\gamma \|X\|.$$

We define the function Δ_F on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ by

$$\Delta_F(x, \xi) = \inf\{F(x, \xi + \eta) - F(x, \eta) \mid \eta \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n\}.$$

Our main results is the following:

Theorem 1.1. *Assume that (D1), (D2) and (F1)-(F4) hold. Then:*

(i) There exists a unique number $\lambda^+ \in \mathbb{R}$ for which there is a viscosity solution $\phi \in \mathrm{Lip}(\overline{\Omega})$ of

$$\begin{cases} F[\phi] = \lambda^+\phi \text{ in } \Omega, \\ \phi > 0 \text{ in } \Omega, \quad \phi|_{\partial\Omega} = 0. \end{cases}$$

(ii) For any $\lambda < \lambda^+$ and $f \in C(\overline{\Omega})$ such that $f \geq 0$ in Ω, there exists a unique viscosity solution $u \in \mathrm{Lip}(\overline{\Omega})$ of

$$\begin{cases} F[u] = \lambda u + f \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \end{cases}$$

(iii) The number(demi-eigenvalue) λ^+ is characterized by:

$$\lambda^+ = \sup\{\lambda \in \mathbb{R} \mid \text{There is a viscosity supersolution } u \in C(\overline{\Omega}) \text{ of } F[u] = \lambda u + 1 \text{ in } \Omega,\ u \geq 0 \text{ in } \Omega,\ u|_{\partial\Omega} = 0\}.$$

(iv) Define the number λ^+_Δ by

$$\lambda^+_\Delta = \sup\{\lambda \in \mathbb{R} \mid \text{There is a viscosity supersolution } v \in C(\overline{\Omega}) \text{ of } \Delta_F[v] = \lambda v + 1 \text{ in } \Omega,\ v \geq 0 \text{ in } \Omega,\ v|_{\partial\Omega}\}.$$

Then for any $\lambda < \lambda^+_\Delta$ and $f \in C(\overline{\Omega})$, there exists a unique viscosity solution $u \in \mathrm{Lip}(\overline{\Omega})$ of

$$\begin{cases} F[u] = \lambda u + f \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

2 Strong comparison principles

The next comparison theorem is from the theory of viscosity solutions.(See [7])

Theorem 2.1. *Assume (F1)-(F3) hold and that there is a constant $\sigma > 0$ such that for each $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n \times S^n$ the function: $r \mapsto F(x, r, \xi) - \sigma r$ is non-decreasing in $\mathbb{R}$. Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω, respectively, and assume that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

The following theorem is the adaption of the classical strong maximum principle to viscosity solutions. (See [8] for the details.)

Theorem 2.2. *Assume that (F1) and (F2) hold and that $F(x, 0) \leq 0$ for all $x \in \Omega$. Let $u \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity supersolution of $F = 0$ in Ω and satisfy $u \geq 0$ in Ω. Then either $u(x) > 0$ for all $x \in \Omega$ or $u(x) = 0$ for all $x \in \Omega$.*

Theorem 2.3. *Assume that (D2), (F1) and (F2) hold and that $F(x,0) \leq 0$ for all $x \in \Omega$. Let $u \in \mathrm{LSC}(\Omega)$ be a viscosity supersolution of $F = 0$ in Ω and satisfy $u(x) > 0$ for all $x \in \Omega$. Then there is a constant $\delta > 0$ such that $u(x) \geq \delta \mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$.*

The next theorem is the strong comparison principle adapted to viscosity solutions for which we refer to [8].

Theorem 2.4. *Assume that (F1)-(F3) hold. Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω, respectively. Assume that $u(x) \leq v(x)$ for all $x \in \Omega$. Then either $u(x) < v(x)$ for all $x \in \Omega$ or $u(x) = v(x)$ for all $x \in \Omega$.*

Theorem 2.5. *Assume that (D2) and (F1)-(F3) hold. Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω, respectively. Assume that $u(x) < v(x)$. Then there is a constant $\varepsilon > 0$ such that*

$$u(x) + \varepsilon \mathrm{dist}(x, \partial\Omega) \leq v(x) \text{ for all } x \in \Omega.$$

Let us introduce the function $(\Delta_F)_*$ which is the lower semi-continuous envelope of Δ_F defined by

$$(\Delta_F)_*(\xi) = \lim_{r \searrow 0} \inf \left\{ \Delta_F(\eta) \mid \eta \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n,\ |\eta - \xi| < r \right\}.$$

To prove Theorems 2.4 and 2.5 we use the following proposition.

Proposition 2.6. *Assume that (F1)-(F3) hold. Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω, respectively. Set $w = u - v$. Then $w \in \mathrm{LSC}(\Omega)$ is a viscosity subsolution of $(\Delta_F)_*[w] = 0$ in Ω.*

Proof. Suppose by contradiction that there are $\varphi \in C^2(\Omega)$ and $\hat{x} \in \Omega$ for which $w - \varphi$ attains its maximum at $\hat{x}$ and

$$(\Delta_F)_*(\hat{x}, w(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) > 0.$$

We may assume that $w(\hat{x}) = \varphi(\hat{x})$ and $x(x) < \varphi(x)$ for all $x \in \Omega \backslash \{\hat{x}\}$. By using the lower semi-continuity of $(\Delta_F)_*$ and continuity of φ, we deduce that there is a constant $\delta > 0$ such that $B(\hat{x}, \delta) \subset \Omega$ and

$$(\Delta_F)_*(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \geq 2\delta \text{ for } x \in B(\hat{x}, \delta) \tag{2.1}$$

where $B(\hat{x}, \delta) = \{y \mid |y - \hat{x}| \leq \delta\}$. Define the function $v_\varphi \in \mathrm{LSC}(\Omega)$ by $v_\varphi = v + \varphi$. Let $x \in B(\hat{x}, \delta)$ and $(p, X) \in J^{2,-}v_\varphi(x)$. Then we see that $(p - D\varphi(x), X - D^2\varphi(x)) \in J^{2,-}v(x)$. Using (2.1) and that v is a supersolution of $F = 0$ in Ω, we deduce that

$$\begin{aligned} F(x, v_\varphi(x), p, X) \geq\ & F(x, v(x), p - D\varphi(x), X - D^2\varphi(x)) \\ & + \Delta_F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \geq 2\delta. \end{aligned}$$

This shows that v_φ is a supersolution of $F - 2\delta = 0$ in $\mathrm{int}B(\hat{x}, \delta)$.

We intend to apply Theorem 2.1. Define $F_L(x, r, p, X) = F(x, r, p, X) + (L+1)r$. We observe that $z := v_\varphi, u$ are a supersolution of $F_L[z] - (L+1)v_\varphi(x) - 2\delta = 0$ and a subsolution of $F_L[z] - (L+1)u(x) = 0$ in $\mathrm{int}B(\hat{x}, \delta)$, respectively.

Noting that $u \le v_\varphi$ and $u, -v_\varphi \in \mathrm{USC}(\Omega)$, we infer that there is a function $g \in \mathrm{Lip}(B(\hat{x}, \delta))$ such that $(L+1)u(x) \le g(x) \le (L+1)v_\varphi(x) + \delta$ for $x \in B(\hat{x}, \delta)$. Fix such a function $g \in \mathrm{Lip}(B(\hat{x}, \delta))$ and observe that $z := v_\varphi, u$ are a supersolution of $F_L[z] - g(x) - \delta = 0$ and a subsolution of $F_L[z] - g(x) = 0$ in $\mathrm{int}B(\hat{x}, \delta)$, respectively. Choose a constant $\varepsilon > 0$ so that $\max_{\partial B(\hat{x}, \delta)}(u - v_\varphi) < -\varepsilon$ and $2L\varepsilon \ge \delta$, and set $v_{\varphi,\varepsilon} = v_\varphi - \varepsilon$. We observe that $v_{\varphi,\varepsilon}(x) \ge u(x)$ for all $x \in \partial B(\hat{x}, \delta)$ and $v_{\varphi,\varepsilon}(\hat{x}) < u(\hat{x})$. Also, since $F_L(x, r-\varepsilon, p, X) \ge F_L(x, r, p, X) - 2L\varepsilon$ for $(x, r, p, X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n$ by (F2), we see that $z = v_{\varphi,\varepsilon}$ is a supersolution of $F_L[z] - g(x) = 0$ in $\mathrm{int}B(\hat{x}, \delta)$. We now apply Theorem 2.1, to conclude that $v_{\varphi,\varepsilon}(x) \ge u(x)$ for all $x \in B(\hat{x}, \delta)$. In particular, we have $v_{\varphi,\varepsilon}(\hat{x}) \ge u(\hat{x})$, which is contradiction. □

3 Sketch of proof

In this paper we will prove (i)-(iii) of Theorem 1.1. To prove Theorem 1.1, we list the following theorems which we get from the theory of viscosity solutions.

We obtain an estimate of the Lipschitz continuity of viscosity solution of $F = 0$. That proof is based on [7]. The Lipschitz constant depends on the norm of u and F especially.

Theorem 3.1. *Assume that (D1) and (F1)-(F3) hold. Let $u \in C(\overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} F[u] = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then there is a constant $C > 0$, depending only on $n, \gamma, \theta, \Theta, r_1, L, C_1$, and $\mathrm{diam}(\Omega)$, *such that for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$,*

$$|u(x) - u(y)| \le C(\|u\|_{L^\infty(\Omega)} + \max_{x \in \overline{\Omega}} |F(x, 0)| + C_0)|x - y|.$$

The next theorem is concerned with a viscosity solution of $F = 0$. Here we do not assume the strict monotonicity of the function: $r \mapsto F(x, r, p, X)$.

Theorem 3.2. *Assume that (D1) and (F1)-(F3) hold and that there are a viscosity subsolution $f \in C(\overline{\Omega})$ and a viscosity supersolution $g \in C(\overline{\Omega})$ of $F = 0$ in Ω which satisfy $f \le g$ in Ω and $f = g = 0$ on $\partial\Omega$. Then there is a viscosity solution $u \in \mathrm{Lip}(\overline{\Omega})$ of $F = 0$ in Ω which satisfies $f \le u \le g$ in $\overline{\Omega}$ (and hence $u = 0$ on $\partial\Omega$).*

Sketch of proof. We solve the Dirichlet problem

$$\begin{cases} F[u] = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

by using a recursive formula.

We define the sequence $\{u_k\}_{k\in\mathbb{N}} \subset C(\overline{\Omega})$ by setting $u_1 = f$ and then by solving inductively the problem

$$\begin{cases} F(x, u_{k+1}, Du_{k+1}, D^2u_{k+1}) + (L+1)u_{k+1} = (L+1)u_k \text{ in } \Omega, \\ u_{k+1}|_{\partial\Omega} = 0. \end{cases}$$

Then by Perron's method(See e.g., [5]), we see that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is well-defined and $f \le u_1 \le u_2 \le \cdots \le u_k \le \cdots \le g$. In the other hand, Theorem 3.1 shows that $\{u_k\}_{k\ge 2}$ is equi-Lipschitz continuous in $\overline{\Omega}$.

Define $u \in \mathrm{Lip}(\overline{\Omega})$ by

$$u(x) = \lim_{k\to\infty} u_k(x).$$

Noting that as $k \to \infty$,

$$F_L(x,r,p,X) - (L+1)u_k(x) \to F_L(x,r,p,X) - (L+1)u(x),$$

uniformly on bounded sets of $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n$, by the stability of viscosity solutions under uniform convergence, we find that u is a solution of

$$F[u] = 0 \text{ in } \Omega.$$

It is clear that $u|_{\partial\Omega} = 0$. □

We are ready to prove Theorem 1.1.

For $\lambda \in \mathbb{R}$ we consider the problem

$$\begin{cases} F[u] = \lambda u + 1 \text{ in } \Omega, \\ u \ge 0 \text{ in } \Omega, \ u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

and set

$$J = \{\lambda \in \mathbb{R} \mid (3.1) \text{ has a viscosity supersolution } u \in C(\overline{\Omega})\},$$
$$\lambda^+ = \sup J.$$

We call λ^+ the *demi-eigenvalue* for the operator $F[\cdot]$ or the function F.

We easily see that $J = (-\infty, \lambda^+)$ and that if (D1), (F1) and (F2) hold then we can get $\lambda^+ \in \mathbb{R}$ as in [3]. Moreover when (D1) and (F1)-(F3) hold and $F(x,0) \le 0$ for all $x \in \overline{\Omega}$, we may replace "supersolution" in the above definition of J by "solution." Indeed, fix $\lambda \in \mathbb{R}$ and assume that u is a supersolution of (3.1). The function 0 is a subsolution of (3.1). Applying Theorem 3.2, we find a solution $v \in C(\overline{\Omega})$ of (3.1) such that $0 \le v \le u$ on $\overline{\Omega}$.

Proof of Theorem 1.1(i). We pick a sequence $\{\lambda_k\} \subset J$ so that $\lambda_k \nearrow \lambda^+$. By the definition of λ^+, there is a sequence $\{\psi_k\}$ such that ψ_k is a viscosity solution of

$$\begin{cases} F[\psi_k] = \lambda_k\psi_k + 1 \text{ in } \Omega, \\ \psi_k \ge 0 \text{ in } \Omega, \ \psi_k|_{\partial\Omega} = 0. \end{cases}$$

We show that $\|\psi_k\|_{L^\infty(\Omega)} \to \infty$. For this, we suppose that there were a subsequence $\{\psi_{k_j}\}$ such that $\sup_{j\in\mathbb{N}} \|\psi_{k_j}\|_{L^\infty(\Omega)} < \infty$. Theorem 3.1 shows that $\{\psi_{k_j}\}_{j\in\mathbb{N}}$ is equi-Lipschitz continuous. By the Ascoli-Arzela theorem and the stability of solutions, we find a function $\psi \in C(\overline{\Omega})$ such that $\psi_{k_j} \to \psi$ and

$$\begin{cases} F[\psi] = \lambda^+\psi + 1 \text{ in } \Omega, \\ \psi|_{\partial\Omega} = 0. \end{cases}$$

Choose $\varepsilon > 0$ so that $2\varepsilon\|\psi\|_{L^\infty(\Omega)} \le 1$. Then 2ψ is a viscosity solution of $F[u] = \lambda^+ u + 2$ in Ω and hence a viscosity supersolution of $F[u] = (\lambda^+ + \varepsilon)u + 1$ in Ω. We have $\lambda^+ + \varepsilon \in J$. This is a contradiction since $\lambda^+ = \sup J$. Thus we have shown that $\|\psi_k\|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$.

Define $\phi_k \in C(\overline{\Omega})$ by $\phi_k(x) = \frac{\psi_k(x)}{\|\psi_k\|_{L^\infty(\Omega)}}$. We observe that ϕ_k is a viscosity solution of

$$\begin{cases} F[\phi_k] = \lambda_k\phi_k + \frac{1}{\|\psi_k\|_{L^\infty(\Omega)}} \text{ in } \Omega, \\ \phi_k|_{\partial\Omega} = 0. \end{cases}$$

As before, we find a function $\phi \in C(\overline{\Omega})$ which $\{\phi_k\}_{k\in\mathbb{N}}$ converges to and is a viscosity solution of

$$\begin{cases} F[\phi] = \lambda^+\phi \text{ in } \Omega, \\ \phi|_{\partial\Omega} = 0. \end{cases}$$

Since $\|\phi\|_{L^\infty(\Omega)} = 1$ and $\phi \ge 0$ in Ω, we conclude $\phi(x) > 0$ for all $x \in \Omega$ by the strong maximum principle(Theorem 2.2). Thus we have completed the proof. □

Then we consider Theorem 1.1(ii). The existence of a viscosity solution and its Lipschitz continuity can be proved by Theorems 3.1 and 3.2. To see its uniqueness, we may prove the next proposition.

Proposition 3.3. *Assume that (D1) and (F1)-(F4) hold and $\lambda \in (-\infty, \lambda^+)$. Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be a viscosity subsolution and a viscosity supersolution of $F[w] = \lambda w + f$ in Ω where $f \ge 0$ in Ω, respectively. Assume that $u \ge 0$ and $v \ge 0$ in Ω and $u \le v$ on $\partial\Omega$. Then $u \le v$ in $\overline{\Omega}$.*

For the proof of the above proposition, we need the following lemma, whose proof is omitted.

Lemma 3.4. *Assume (F1)-(F3) hold. Let $f, g \in C(\overline{\Omega})$. If a function $u \in C(\overline{\Omega})$ is both a viscosity subsolution of $F[u] = f$ in Ω and a viscosity supersolution of $F[u] = g$ in Ω. Then $g \le f$ in Ω.*

Proof of Proposition 3.3. We first consider the case where $f \not\equiv 0$. we see that $v \not\equiv 0$. Indeed, if $v = 0$, then we have

$$0 = F(x, 0) \ge \lambda \cdot 0 + f(x) \text{ for all } x \in \Omega,$$

which is a contradiction. Thus we get $v > 0$ in Ω by Theorem 2.2. Suppose by contradiction that $\max_{\overline{\Omega}}(u - v) > 0$. We set

$$a = \sup\{t \geq 0 \mid tu(x) \leq v(x) \text{ for all } x \in \Omega\}.$$

It is seen that $0 \leq a < 1$. Therefore we observe that au is a subsolution of $F[w] = \lambda w + f$ and that $au \leq v$ in Ω. In view of Theorem 2.4, we see that either $au = v$ in Ω or $au < v$ in Ω. Suppose $au < v$ in Ω. By Theorem 2.5, there exists $\varepsilon > 0$ such that

$$au(x) + \varepsilon \mathrm{dist}(x, \partial\Omega) \leq v(x) \text{ for all } x \in \Omega.$$

On the other hand, by Theorem 3.1, we find $C > 0$ so that

$$u(x) \leq C\mathrm{dist}(x, \partial\Omega) \text{ for all } x \in \Omega.$$

Thus for $\delta = \varepsilon/C$,

$$(a + \delta)u(x) \leq au(x) + C\delta \mathrm{dist}(x, \partial\Omega) = au(x) + \varepsilon \mathrm{dist}(x, \partial\Omega) \leq v(x).$$

This is a contradiction. Therefore we get $au = v$ in Ω and $a > 0$. Since $au = v$ is a viscosity subsolution of $F[w] = \lambda w + af$ in Ω as well as a viscosity supersolution of $F[w] = \lambda w + f$ in Ω, we see $f \leq af$ in Ω by Lemma 3.4. This is contradictory to $f > 0$. Thus we conclude that $u \leq v$ in Ω.

Finally we consider the case where $f = 0$. By the definition of λ^+, there is a supersolution $w \in C(\overline{\Omega})$ of $F[w] = \lambda w + 1$ in Ω, $w \geq 0$ in Ω, and $w|_{\partial\Omega} = 0$. It is seen as before that $w > 0$ in Ω. Define the sequence $\{w_k\} \subset C(\overline{\Omega})$ by $w_k(x) = w(x)/k$ and note that, for each $k \in \mathbb{N}$, u and w_k is a subsolution and a supersolution of $F[z] = \lambda z + 1/k$ in Ω, respectively. The observation when $f \neq 0$ guarantees that $u \leq w_k$ in Ω. Sending $k \to \infty$ yields that $u = 0$. It is now clear that $u \leq v$ in Ω, which completes the proof. □

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