数理解析研究所講究録
1481巻 2006年 115-118

Some results on Hessian measures for non-commuting vector fields.[1]

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In this talk we present some extensions of the theory of Hessian measures developed in [4,5,6] to more general vector fields. Details of proofs are given in [8].

Let $X=(X_1, \cdots, X_m)$ be a system of vector fields in Euclidean space $\mathbb{R}^n$, $m\leq n$, given by

$$X_i=\sigma^{ij}D_j, \qquad i=1, \cdots, m, \tag{1}$$

where $\sigma^{ij}\in C^{\infty}(\mathbb{R}^n)$, $i=1, \cdots, m$, $j=1, \cdots, m$. For Ω a domain in $\mathbb{R}^n$ and $u\in C^2(\Omega)$, the Hessian and symmetrized Hessian of u, with respect to X, are defined respectively by

$$\begin{aligned} X^2u &= [X_iX_ju], \\ X_s^2u &= [\frac{1}{2}(X_iX_j+X_jX_i)u]_{i,j=1,\cdots,m}. \end{aligned} \tag{2}$$

For a matrix $r\in \mathbb{R}^m\times \mathbb{R}^m$, $k=1, \cdots, m$, we let $S_k(r)$ denote the sums of its $k\times k$ principal minors and define the corresponding operators $\mathcal{F}_k$ by

$$\mathcal{F}_k[u]=S_k(X_s^2u) \tag{3}$$

A function $u\in C^2(\Omega)$ is called $k-$ *convex*, with respect to X, if $\mathcal{F}_j[u]\geq 0$ in Ω, for all $j=1, \cdots k$. A function $u\in L_{loc}^1(\Omega)$ is called $k-$ *convex*, with respect to X, if for each domain $\Omega'\subset\subset\Omega$, there exists a sequence of $k-$convex functions $\{u_\ell\}\subset C^2(\Omega')$ such that $u_\ell\rightarrow u$ as $\ell\rightarrow\infty$, in $L^1(\Omega')$. We denote the class of $k-$convex functions in Ω by $\phi_X^k(\Omega)$ or simply $\phi^k(\Omega)$, when X is understood. The following properties of $k-$convex functions in the Euclidean case, $X_i=D_i$, $i=1, \cdots, k$, are proved in [4,5].

[1]Research supported by Australian Research Council grant

Theorem 1. *For any $u \in \phi^k(\Omega)$ we have $Xu \in L^p_{loc}(\Omega)$ for any $p < nk(n-k)$ and there exists a Borel measure $\mu_k[u]$, extending $\mathcal{F}_k[u]\mathrm{dx}$ for $u \in C^2(\Omega)$, such that if $u_\ell \to u$ a.e. (Ω), then $\mu_k[u_\ell] \to \mu_k[u]$ weakly.*

In [8], these results are extended, in part, to anti-self adjoint systems X, ($X_i^* = -X_i$, $i = 1, \cdots, m$), satisfying the Hörmander condition that at each point of Ω, the Lie algebra generated by X spans $\mathbb{R}^n$. In particular we prove,

Theorem 2. *Let $u \in \phi^k_X(\Omega)$ where X satisfies the above conditions. Then $Xu \in L^p_{loc}(\Omega)$, for any $p < Qk(m-1)/Q(m-k)$, where Q denotes the homogeneous dimension of X. If $k = 2$ and X is of step 2, then the commutators $[X_i, X_j]u \in L^2_{loc}(\Omega), i, j = 1, \cdots, m$ and there exists a Borel measure $\mu_2[u]$, extending $\mathcal{F}_2[u]\mathrm{dx}$ for $u \in C^2(\Omega)$, such that if $u_\ell \to u$ a.e.(Ω), then $\mu_2[u_\ell] \to \mu_2[u]$ weakly in Ω.*

A more general theory for quasilinear operators extending the case $k = 1$ is developed in [7]. The restriction to Step 2 may be weakened [8], but so far we are unaware of any extensions of the commutator regularity and weak continuity to the cases $k > 2$. The proof of Theorem 2 draws upon our techniques in [4,5,7] and stems from an interesting identity, discovered in the special case of the Heisenberg group $\mathcal{H}^1$ in [1]. Namely, if we define the function G on $\mathbb{R}^m \times \mathbb{R}^m$ by

$$G(r) = S_2(r) + \frac{1}{2}\sum_{i<j}(r_{ij} - r_{ji})^2, \tag{4}$$

then, for any $u \in C^2(\Omega)$,

$$Y_j u := X_i\left[\frac{\partial G}{\partial r_{ij}}(X^2 u)\right] = \Big[X_i, [X_i, X_j]\Big]u, \tag{5}$$

that is Y_j, $j = 1, \cdots, m$, are *vector fields*, (vanishing when X is Step 2).

More generally, we can define *subharmonic* functions along the lines of [5,6]. In particular we define an upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ to be subharmonic with respect to the operator $\mathcal{F}_k$ if $\mathcal{F}_k[u] \geq 0$ in the *viscosity* sense, that is for any quadratic polynomial q for which the difference $u - q$ has a finite local maximum at a point $y \in \Omega$, we have $\mathcal{F}_k[q](y) \geq 0$. A $k-$convex function is then equivalent to a subharmonic function. Furthermore it follows from [3,9], that if X generates the Lie algebra of a Carnot group, then a proper subharmonic function, ($\not\equiv -\infty$ on a set of positive measure), is $k-$convex. The equivalence of various notions of convexity in the case $k = m$, is treated in [2,9], where other references are also given. In this case, Theorem 2 can be improved to $Xu \in L^{\infty}_{loc}(\Omega)$ if $u \in \phi^k(\Omega)$.

The extension of Theorem 2 to arbitrary step can be expressed in terms of the vector fields $Y = (Y_1, \cdots, Y_m)$ defined by (??). For example, if $m = 2$, then the commutator $[X_1, X_2]u \in L^2_{loc}(\Omega)$ if $Yu \in L^1_{loc}(\Omega)$, while $\mu_2[u_\ell] \to \mu_2[u]$ weakly if also $\{Yu_\ell\}$ is uniformly bounded in $L^1_{loc}(\Omega)$.

Finally we remark that from (??), we have a monotonicity property for arbitrary anti-self adjoint systems of vector fields X,which extends the Euclidean case in [4] and the case of the Heisenberg group in [1]. Namely defining, for $u \in C^2(\Omega)$,

$$\mathcal{G}[u] = G(X^2u) + \frac{1}{2}Xu.Yu, \qquad (6)$$

we obtain

$$\int_\Omega \mathcal{G}[u] \geq \int_\Omega \mathcal{G}[v]), \qquad (7)$$

for any functions $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfying $u \leq v$ in Ω, $u = v$ on $\partial\Omega$ with the operator $\mathcal{F}_2$ degenerate elliptic with respect to their sum $u + v$. In certain cases, including the Heisenberg group in [1], Theorem 2 may be derived from (??),using the approach in [4], rather than that through integral estimates in [5].

References

[1] C.E. Gutiérrez and A. Montanari. Maximum and comparison principles for convex functions on the Heisenberg group. *Comm. Part. Diff. Eqns.*, 29:1305–1334, 2004.

[2] P. Juutinen, G. Lu, J.J. Manfredi, and B. Stroffolini. Convex functions on Carnot groups, (preprint).

[3] V. Magnani. Lipschitz continuity, Aleksandrov theorem and characterizations for H-convex functions. *Math. Ann. to appear.*

[4] N.S. Trudinger and X.J. Wang. Hessian Measures I. *Topol. Methods Nonlinear Anal.*, 10:225–239, 1997.

[5] N.S. Trudinger and X.J. Wang. Hessian Measures II. *Ann. Math*, 150:579–604, 1999.

[6] N.S. Trudinger and X.J. Wang. Hessian Measures III. *J. Funct. Anal.*, 193:1–23, 2002.

[7] N.S. Trudinger and X.J. Wang. On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math*, 124:369–410, 2002.

[8] N.S. Trudinger. On Hessian measures for non-commuting vector fields. *Pure App. Math. Quarterly*, 2, 149-163, 2006.

[9] C. Y. Wang. Viscosity convex functions on Carnot groups, *Proc. Amer. Math. Soc.* 133, 1274-1253, 2004.

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