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NOTE ON THE MODICA-MORTOLA FUNCTIONAL

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In this note I describe a few questions I encountered on the Modica-Mortola functional. They concern basically the same question in disguise: How close is the Modica-Mortola functional to the area functional as the thickness of the interface approaches to 0?

1. Upper bound of discrepancy measure for general critical points

Suppose we have a smooth function $u : \Omega \to \mathbf{R}$ defined on a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$, with the properties that

$$-\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} = 0$$

and

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\, dx \leq C \tag{1}$$

where ε is a small positive number and $W(u) = (1-u^2)^2/4$ is the standard double-well potential with equal minima at ± 1. The functional $E(\cdot)$ is the usual Modica-Mortola functional [3]. The equation (1) is the Euler-Lagrange equation for E. We may consider more general equations with non-zero right-hand side, but here we consider the simplest case to clarify the point. It is well-known that $E_\varepsilon(\cdot)$ Γ-converge to the area functional as $\varepsilon \to 0$ [5, 6]. For this problem it is also known that energy concentrate on stationary integral varifold as $\varepsilon \to 0$ [2]. In the latter work, it was crucial to obtain a good estimate on the so-called discrepancy measure

$$\xi = \frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon} \tag{2}$$

which is the difference of two terms in the energy E_ε. Somewhat surprisingly, only with above two assumptions (plus, say, $|u| \leq 2$), one may conclude that there exists some constant c such that, for all small ε,

$$\sup_{\tilde{\Omega}} \xi \leq c$$

where c depends only on $\mathrm{dist}(\partial\Omega, \tilde{\Omega})$ and n [2, Lemma 3.6]. Note that it is only the upper bound of ξ which is bounded. Though the stated bound in [2] is only in this form, a closer inspection of the proof shows that the estimate can be improved so that

$$\sup_{\bar{\Omega}} \xi \le \varepsilon^{\alpha}$$

for any $\alpha < 1$ for all sufficiently small ε. The proof is by repeating the maximum principle argument already repeated twice in the proof.

It is unclear why it cannot be improved to $\alpha = 1$, or even something better. Is it false, or can it be improved? Since the problem itself is very simple, I wish that there is a simple answer. This question relates to the investigation on how close is the transition profile to the standard hyperbolic tangent function in ε-scale. Or one can simply regard the problem as an independent one. I should point out that if $-\Delta u + W'(u) = 0$ on $\mathbf{R}^n$ and u is bounded, then $\xi \le 0$ (with $\varepsilon = 1$ in the definition of ξ) and that if ξ vanishes at any point, u is 'one-dimensional hyperbolic tangent function' [4].

Similar good estimate (2) up to the boundary is not known except for some special case: in case the right-hand side of the equation (1) may be replaced by a (uniformly bounded) constant, u satisfies homogeneous Neumann boundary condition and the domain is convex [8]. To my knowledge this is the only result. Since the energy monotonicity formula is proved by establishing a good upper bound on ξ, so far we only have up to the boundary energy monotonicity formula only for this case. It is not clear if (2) holds up to the boundary for general case.

2. Stable critical points

In addition to the setting in the previous section, suppose the critical point is stable. Namely, in addition to (1), assume that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u + t\phi) \ge 0 \tag{3}$$

for any $\phi \in C_c^1(\Omega)$. With a suitable substitution of the test function [9], one can show

$$\int_{\Omega} \varepsilon \left\{ \sum_{i,j=1}^{n} (u_{ij})^2 - \frac{1}{|\nabla u|^2} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i u_{ij} \right)^2 \right\} \phi^2$$
$$\le \varepsilon \int_{\Omega} |\nabla \phi|^2 |\nabla u|^2$$

for any $\phi \in C_c^1(\Omega)$. Since the right-hand side of (3) is bounded by the energy bound, this means that the second fundamental form of the level set has some L^2 norm control, though we still need some lower bound for the gradient $|\nabla u|$. For later use, define

$$|A|^2 = \left\{ \sum_{i,j=1}^{n} (u_{ij})^2 - \frac{1}{|\nabla u|^2} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i u_{ij} \right)^2 \right\} / |\nabla u|^2.$$

In the following I related this bound to the mean curvature of the varifold which can be defined naturally from u. For u define a varifold V by

$$V(\phi) = \int_\Omega \phi(x, \frac{\nabla u}{|\nabla u|}) \frac{\varepsilon}{2} |\nabla u|^2 \, dx$$

for $\phi \in C(\Omega \times G(n, n-1))$, where $G(n, n-1)$ is the space of $n-1$-dimensional subspace and that we identify the unit vector $\frac{\nabla u(x)}{|\nabla u(x)|}$ with the normal subspace in $G(n, n-1)$. Define the Radon measure $||V||$ on $\mathbf{R}^n$ by the projection of V on $\mathbf{R}^n$, which really means $||V|| = \frac{\varepsilon}{2}|\nabla u|^2 \, dx$. Note that $||V||$ is a measure concentrated mostly around the transition region $\{u \approx 0\}$. One may define the generalized mean curvature [1] for V as follows. For any $g \in C_c^1(\Omega; \mathbf{R}^n)$, define

$$\delta V(g) = \int_{\Omega \times G(n,n-1)} S \cdot Dg(x) \, dV(x, S).$$

Here $S \in G(n, n-1)$ is identified with n by n matrix representing the orthogonal projection onto the subspace S, Dg is the n by n matrix of first derivatives, and $S \cdot Dg(x) = \sum_{i,j} S_{ij} D_i g^j$. In our setting, one finds that $\delta V(g)$ amounts to

$$\delta V(g) = \int_\Omega \left(\operatorname{div} g - \sum_{i,j} \frac{u_i}{|\nabla u|} \frac{u_j}{|\nabla u|} g_i^j \right) \frac{\varepsilon}{2} |\nabla u|^2 \, dx.$$

At the same time, by multiplying $\nabla u \cdot g$ to the equation (1) and after performing integration by parts, one finds that

$$(4) \qquad \delta V(g) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right) \operatorname{div} g \, dx = \int_\Omega \xi \operatorname{div} g \, dx.$$

Define

$$H = \left(\varepsilon |\nabla u|^2 \right)^{-1} \nabla \xi$$

for $|\nabla u| \neq 0$ and $H = 0$ otherwise. Another integration by parts of (4) then gives

$$\delta V(g) = -\int_\Omega 2H \cdot g \, d||V||.$$

Thus above $2H$ is precisely the generalized mean curvature vector of V according to [1]. If u is stable, one can check that

$$|H|^2 \, d||V|| \leq c(n) |A|^2 \, d||V||$$

and the integral on the right-hand side is locally bounded in terms of energy bound. Thus V has uniform L^2 mean curvature bound in terms of energy. From [2] we know that ξ converges (as $\varepsilon \to 0$) to 0 in $L^1_{loc}(\Omega)$, thus H converges to 0 weakly in the following sense: for any $g \in C^1_c(\Omega; \mathbf{R}^n)$,

$$-\int_\Omega g \cdot 2H \, d||V|| = \int_\Omega \xi \mathrm{div} g \, dx \to 0$$

as $\varepsilon \to 0$. One may wonder if H converges to 0 strongly, for example,

$$\int_{\tilde{\Omega}} |H|^2 \, d||V|| \to 0$$

as $\varepsilon \to 0$ for any $\tilde{\Omega} \subset\subset \Omega$? Or is anything of this nature true? It is interesting to point out that the following formula holds:

$$\nabla \cdot H = -|H|^2 + |A|^2 \tag{5}$$

for $n \geq 3$ and $\nabla \cdot H = 0$ for $n = 2$. In closer inspection one finds that the right-hand side is in fact equal precisely to the Gauss curvature of the level set of u for $n = 3$. I don't know if this is useful but at least it is an interesting relation.

In dimension 3, I point out that the gradient of u cannot be zero around transition region when the L^2-norm of the second fundamental form is sufficiently small.

Theorem 1. *Given $0 < s < 1$ there exists $\epsilon_0 > 0$ and $c_1 > 0$ such that*

$$\inf_{\{|u| \leq 1-s\} \cap \tilde{\Omega}} |\nabla u| \geq \frac{c_1}{\varepsilon}$$

if $\int_\Omega |A|^2 \, d||V|| < \epsilon_0$.

This was proved for $n = 2$ in [9] without the smallness assumption on $|A|$. Here I give the proof for $n = 3$.

Proof. Let us work in the re-scaled setting of x replaced by x/ε. Write $\zeta = W(u) - \frac{|\nabla u|^2}{2}$. Write $\bar{\zeta} = \frac{1}{\omega_3 L^3} \int_{B_L} \zeta$ as the average over the ball of radius L centered at a point where $|u| \leq 1 - s$ and suppose $\bar{\zeta} \geq c_1$ for some L and c_1. By the upper bound of the scaled energy (and the energy monotonicity formula),

$$C \geq \frac{1}{\omega_2 L^2} \int_{B_L} \zeta \geq \frac{\omega_3 L c_1}{\omega_2}.$$

Thus if we set $Lc_1 = \frac{2C\omega_2}{\omega_3}$, then we can make sure that $\bar{\zeta} < c_1$. Fix $c_1 = \frac{1}{6} \min_{|u| \leq 1-s} W(u)$. Assume that $\zeta(x_0) \geq 3c_1$ for some point x_0.

Then there exists a neighborhood $B_{c_2}(x_0)$ such that $\zeta(x) \geq 2c_1$ on $B_{c_2}(x_0)$ (where we assume $c_2 < L$). Then by the Poincare inequality,

$$\left(\int_{B_L} |\zeta - \bar{\zeta}|^{1.5}\right)^{1/1.5} \leq c \int_{B_L} |\nabla \zeta|$$

$$\leq c\left(\int_{B_L} |\nabla u|^2\right)^{1/2}\left(\int_{B_L} \left((u_{ij})^2 - \frac{\sum(\sum u_i u_{ij})^2}{|\nabla u|^2}\right)\right)^{1/2}$$

$$\leq cC^{1/2} L \epsilon_0^{1/2}.$$

On the other hand

$$\left(\int_{B_L} |\zeta - \bar{\zeta}|^{1.5}\right)^{1/1.5} \geq cc_1.$$

Thus for ϵ_0 sufficiently small this is a contradiction. This shows $\zeta \leq 3c_1$. By the choice of c_1 this shows $\frac{|\nabla u|^2}{2} \geq c_1$ on $\{|u| \leq 1 - s\}$.

End of proof.

Note that this is a local result so that we may apply this result except for a finite number of points in the interior of the domain. In particular for $n = 3$, as ε approaches to 0, we have a sequence of smooth level sets whose second fundamental form L^2 norm locally uniformly bounded.

Another interesting estimate is the following:

Theorem 2. *For $n = 3$ and for all sufficiently small ε, we have*

$$\int_{\bar{\Omega}} |\xi|^2 \, dx \leq \varepsilon^{\alpha}$$

for $\alpha <$ 1.

Before I prove this, let me point out that it would be interesting to obtain the following result, which I do not know if it is true or not.

$$\int_{\bar{\Omega}} |\xi|^2 \, dx = o(\varepsilon)?$$

If this is true, then, we know how to prove (via (5))

$$\int_{\bar{\Omega}\cap\{|u|\leq 1/2\}} |H|^2 \, d\|V\| \to 0$$

as ε approaches to 0. This in turn is useful to pursue the regularity theory for the limit interface of the stable critical points for $n = 3$ [9]. I expect that the limit interfaces of stable critical points are always smooth for $n = 3$ but so far I do not know how to prove it. For $n = 2$ we know they are straight lines with no junction points.

Proof. Cover the transition region by balls B of size ε^{α}. On this ball, note that ζ is exponentially small away from the transition region, so we can make sure that $(\zeta - \varepsilon^k)_+$, k large, is 0 for a large fraction of the ball. Then we may apply a suitable version of Sobolev inequality to conclude that

$$\int_B \zeta^2 \le |B|^{2/3} \Big(\int_B \zeta^6\Big)^{1/2} \le \varepsilon^{2\alpha} \int |\nabla \zeta|^2.$$

Adding over all balls and noting that the last integral is bounded by c/ε, we may conclude that the L^2-norm of ζ is bounded by $c\varepsilon^{2\alpha-1}$, whose exponent can be made as close to 1 as possible, but not equal to 1. Note that if $\alpha = 1$, then we do not know if ζ is zero for large fraction of the ball B and we cannot apply the Sobolev inequality.
End of proof.

The control of divergence of vector field reminds one of some applications of div-curl lemma, but I do not see how to use it.

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