数理解析研究所講究録
1481 巻 2006 年 125-133

THE MAXIMUM PRINCIPLE FOR VECTOR FIELDS

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ABSTRACT. We discuss an extension of Jensen's uniqueness theorem for viscosity solutions of second order partial differential equations to the case of equations generated by vector fields.

1. INTRODUCTION

The comparison principle between sub and super-solutions of elliptic partial differential equations is a basic result that allows for the application of techniques from potential theory (Perron's method) and implies that the notion of viscosity solution is a genuine generalization of the character of solution for functions that lack the necessary smoothness to be plugged into the equation.

For second order elliptic equations R. Jensen established in a celebrated theorem [J] the comparison principle of viscosity solutions of fully non-linear second order partial differential equations in $\mathbb{R}^n$. These equations are of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0,$$

where x is in some domain $\Omega \subset \mathbb{R}^n$, the function $u\colon \Omega \mapsto \mathbb{R}$ is real valued, the gradient Du is the vector $(\partial_{x_1}u, \partial_{x_2}u, \ldots, \partial_{x_n}u)$, and the second derivatives D^2u is the $n \times n$ symmetric matrix with entries $\partial^2_{x_i x_j}u$. Jensen's theorem was later crafted in the language of jets and extended in [CIL]. In this latter reference, Jensen's theorem follows from the *Maximum Principle for Semi-continuous Functions*

In this talk we present and extension of the Crandall-Ishii-Lions maximum principle for semi-continuous functions following [BBM] and investigate the analogue of Jensen's theorem when the vector fields $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ are replaced by an arbitrary collection of vector fields or *frame*

$$\mathfrak{X} = \{X_1, X_2, \ldots, X_m\}.$$

The natural gradient is the vector

$$\mathfrak{X}u = (X_1(u), X_2(u), \ldots, X_m(u))$$

Date: December 29, 2005.

The author wishes to express his appreciation to the organizers of this conference Professors Shigeaki Koike, Hitoshi Ishii, and Yoshikazu Giga for their gracious invitation to participate.

Research supported in part by NSF award DMS-0500983.

and the natural second derivative is the $m \times m$ *not necessarily symmetric* matrix $\mathfrak{X}^2u$ with entries $X_i(X_j(u))$. Two important examples are:

(i) when $m = n$ and the frame $\mathfrak{X}$ is the orthonormal frame determined by a Riemannian metric, and

(ii) when $m < n$ and the frame $\mathfrak{X}$ satisfies the Hörmander condition

$$\dim(\text{Lie Algebra span}\{X_1, X_2, \ldots, X_m\}(x)) = n. \tag{1.1}$$

Our main result, see Theorem 1 below, extends the maximum principle for semi-continuous functions to the case (i). In case (ii) an extension of Jensen's theorem has been recently found by Wang [W] when the frame $\mathfrak{X}$ is the horizontal subspace of the graded Lie algebra of a Carnot group. Wang extended a previous result of Bieske [Bi1], who considered the Heisenberg group. To the best of my knowledge the general case of Hörmander vector fields without group structure remains open, except in the case of the Grušin plane, where Bieske has obtained several results [Bi2], [Bi3].

2. Taylor Formula for Vector Fields

In order to define point-wise generalized derivatives or jets, we need to express the regular derivatives in a convenient form. This is done by using a Taylor formula adapted for our frame $\mathfrak{X} = \{X_1, X_2, \ldots, X_n\}$ in $\mathbb{R}^n$,consisting of n linearly independent smooth vector fields as in [NSW]. Write $X_i(x) = \sum_{j=1}^n a_{ij}(x)\partial_{x_j}$ for smooth functions $a_{ij}(x)$. Denote by $\mathbb{A}(x)$ the matrix whose (i,j)-entry is $a_{ij}(x)$. We always assume that $\det(\mathbb{A}(x)) \neq 0$ in $\mathbb{R}^n$.

Fix a point $p \in \mathbb{R}^n$ and let $t = (t_1, t_2, \ldots, t_n)$ denote a vector close to zero. We define the (flow) exponential based at p of t, denoted by $\Theta_p(t)$, as follows: Let γ be the unique solution to the system of ordinary differential equations

$$\gamma'(s) = \sum_{i=1}^n t_i X_i(\gamma(s))$$

satisfying the initial condition $\gamma(0) = p$. We set $\Theta_p(t) = \gamma(1)$ and note this is defined in a a neighborhood of zero. Note that the flow exponential is different from the Riemannian exponential defined via geodesics.

Applying the one-dimensional Taylor's formula to $u(\gamma(s))$ we get

Lemma 1. *([NSW]) Let u be a smooth function in a neighborhood of p. We have:*

$$u(\Theta_p(t)) = u(p) + \langle \mathfrak{X}u(p), t\rangle + \frac{1}{2}\langle (\mathfrak{X}^2u(p))^* t, t\rangle + o(|t|^2)$$

as $t \to 0$.

Note that the quadratic form determined by $\mathfrak{X}^2u$ is the same as the quadratic form form determined by the symmetrized second derivative

$$(\mathfrak{X}^2u)^* = \frac{1}{2}\left(\mathfrak{X}^2u + (\mathfrak{X}^2u)^t\right).$$

Applying lemma (1) to the coordinates functions we obtain a relation between $\mathbb{A}(p)$ and $\Theta_p(0)$:

Lemma 2. *Write* $\Theta_p(t) = \left(\Theta_p^1(t), \Theta_p^2(t), \dots, \Theta_p^n(t)\right)$. *Note that we can think of* $X_i(x)$ *as the i-th row of* $\mathbb{A}(x)$. *Similarly* $D\Theta_p^k(0)$ *is the k-column of* $\mathbb{A}(p)$ *so that*

$$D\Theta_p(0) = \mathbb{A}(p).$$

For the second derivative we get

$$\langle D^2\Theta_p^k(0)h, h\rangle = \langle \mathbb{A}^t(p)h, D(\mathbb{A}^t(p)h)_k\rangle$$

for all vectors $h \in \mathbb{R}^n$.

In the next lemma we denote the gradient relative to the canonical frame by Du and the second derivative matrix by D^2u.

Lemma 3. *For smooth functions u we have*

$$\mathfrak{X}u = \mathbb{A} \cdot Du,$$

and for all $t \in \mathbb{R}^n$

$$\langle (\mathfrak{X}^2u)^* \cdot t, t\rangle = \langle \mathbb{A} \cdot D^2u \cdot \mathbb{A}^t \cdot t, t\rangle + \sum_{k=1}^{n} \langle \mathbb{A}^t \cdot t, \nabla \left(\mathbb{A}^t \cdot t\right)_k\rangle \frac{\partial u}{\partial x_k}.$$

A comparison principle for smooth functions follows right away.

Lemma 4. *Let u and v be smooth functions such that u − v has an interior local maximum at p. Then we have*

$$\mathfrak{X}u(p) = \mathfrak{X}v(p) \tag{2.1}$$

and

$$\left(\mathfrak{X}^2u(p)\right)^* \le \left(\mathfrak{X}^2v(p)\right)^*. \tag{2.2}$$

Let us consider some examples:

Example 1. *The canonical frame*

This is just $\{\partial_{x_1}, \partial_{x_2}, \dots \partial_{x_n}\}$. The first and second derivatives are just the usual ones and the exponential mapping is just addition

$$\Theta_p(t) = p + t.$$

Example 2. *The Heisenberg group*

We consider the Riemannian frame which is given by the left invariant vector fields $\{X_1, X_2, X_3\}$ in $\mathbb{R}^3$. For $p = (x, y, z)$ the matrix $\mathbb{A}$ is just

$$\mathbb{A}(p) = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation shows that

$$\langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_k\rangle = 0$$

not only for $k = 1$ and $k = 2$, but also for $k = 3$. That is, although $\mathbb{A}$ is not constant, we have that Lemma 3 simplifies to

$$\langle (D^2_{\mathfrak{X}}u)^{*} \cdot t, t\rangle = \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot t, t\rangle. \tag{2.3}$$

The exponential mapping is just the group multiplication

$$\Theta_p(t) = p \cdot \Theta_0(t) = (x + t_1, y + t_2, z + t_3 + (1/2)(xt_2 - yt_1)).$$

From Lemma (3), we see that the additional simplification of (2.3) occurs whenever $D^2\Theta^k_p(0) = 0$. In particular this is true for all step 2 groups as it can be seen from the Campbell-Hausdorff formula. However this is not true for groups of rank 3 of higher. See [BBM] for a explicit example.

3. Jets

To define second order superjets[1] of an upper-semicontinuous function u, let us consider smooth functions φ touching u from above at a point p.

$$K^{2,+}(u,p) = \Big\{ \varphi \in C^2 \text{ in a neighborhood of } p,\ \varphi(p) = u(p),$$
$$\varphi(q) \geq u(q),\ q \neq p \text{ in a neighborhood of } p \Big\}$$

Each function $\varphi \in K^{2,+}(u,p)$ determines a pair (η, X) by

$$\begin{aligned} \eta &= (X_1\varphi(p), X_2\varphi(p), \ldots, X_n\varphi(p)) \\ A_{ij} &= \tfrac{1}{2}\big(X_i(X_j(\varphi))(p) + X_j(X_i(\varphi))(p)\big). \end{aligned} \tag{3.1}$$

This representation clearly depends on the frame $\mathfrak{X}$. Using the Taylor theorem for φ and the fact that φ touches u from above at p we get

$$u(\Theta_p(t)) \leq u(p) + \langle \eta, t\rangle + \frac{1}{2}\langle Xt, t\rangle + o(|t|^2). \tag{3.2}$$

We may also consider $J^{2,+}_{\mathfrak{X}}(u,p)$ defined as the collections of pairs (η, X) such that (3.2) holds. Using the identification given by (3.1) it is clear that

$$K^{2,+}(u,p) \subset J^{2,+}_{\mathfrak{X}}(u,p).$$

In fact, we have equality. This is the analogue of the Crandall-Ishii Lemma of [C] that follows from [C] and Lemma 3.

Lemma 5.

$$K^{2,+}(u,p) = J^{2,+}_{\mathfrak{X}}(u,p).$$

Before stating the final version of the comparison principle, we need to take care of a technicality. We need to consider the closures of the second order sub and superjets, $\bar{J}^{2,+}_{\mathfrak{X}}(u, p_\tau)$ and $\bar{J}^{2,-}_{\mathfrak{X}}(v, p_\tau)$. These are defined by taking pointwise limits as follows: A pair $(\eta, X) \in \bar{J}^{2,+}_{\mathfrak{X}}(u,p)$ if there exist

[1] Note that superjets are used in the definition of subsolution and subjets in the definition of supersolution.

sequences of points $p_m \to p$, vectors $\eta_m \to \eta$ and matrices $X_m \to X$ as $m \to \infty$ such that $u(p_m) \to u(p)$ and $(\eta_m, X_m) \in J^{2,+}_{\mathfrak{X}}(u, p_m)$.

Theorem 1. THE COMPARISON PRINCIPLE FOR SEMICONTINUOUS FUNCTIONS *Let u be upper semi-continuous in a bounded domain $\Omega \subset \mathbb{R}^n$. Let v be lower semi-continuous in Ω. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y\to x} u(y) \le \liminf_{y\to x} v(y),$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. If $u - v$ has a positive interior local maximum

$$\sup_{\Omega}(u - v) > 0$$

then we have:

For $\tau > 0$ we can find points $p_\tau, q_\tau \in \mathbb{R}^n$ such that

i) $\lim_{\tau\to\infty} \tau\psi(p_\tau, q_\tau) = 0$, *where* $\psi(p, q) = |p - q|^2$,

ii) *there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by (i)) and* $\sup_\Omega(u - v) = u(\hat{p}) - v(\hat{p}) > 0$,

iii) *there exist symmetric matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau$ and vectors η^+_τ, η^-_τ so that*

iv)
$$(\eta^+_\tau, \mathcal{X}_\tau) \in \overline{J}^{2,+}_{\mathfrak{X}}(u, p_\tau),$$

v)
$$(\eta^-_\tau, \mathcal{Y}_\tau) \in \overline{J}^{2,-}_{\mathfrak{X}}(v, q_\tau),$$

vi)
$$\eta^+_\tau - \eta^-_\tau = o(1)$$

and

vi)
$$\mathcal{X}_\tau \leqq \mathcal{Y}_\tau + o(1)$$

as $\tau \to \infty$.

Note that the first generalized derivatives η^+_τ and η^-_τ do not agree but the error term vanishes as $\tau \to \infty$. Similarly we don't have the usual order of the generalized second derivatives $\mathcal{X}_\tau$ and $\mathcal{Y}_\tau$ but the error term is also $o(1)$ as $\tau \to \infty$.

Proof. The idea of the proof is to use the Euclidean theorem to get the jets and then twist them into position. As in the Euclidean case we get points p_τ and q_τ so that (i) and (ii) hold. We apply now the **Euclidean maximum principle** for semicontinuous functions of Crandall-Ishii-Lions [CIL]. There exist $n \times n$ symmetric matrices X_τ, Y_τ so that

$$(\tau D_p(\psi(p_\tau, q_\tau)), X_\tau) \in \overline{J}^{2,+}_{\text{eucl.}}(u, p_\tau)$$

and

$$(-\tau D_q(\psi(p_\tau, q_\tau)), Y_\tau) \in \overline{J}^{2,-}_{\text{eucl.}}(v, q_\tau)$$

with the property

$$\langle X_\tau\gamma, \gamma\rangle - \langle Y_\tau\chi, \chi\rangle \leqq \langle C\gamma \oplus \chi,\ \gamma \oplus \chi\rangle \tag{3.3}$$

where the vectors $\gamma, \chi \in \mathbb{R}^n$, and

$$C = \tau(A^2 + A)$$

and

$$A = D^2_{p,q}(\psi(p_\tau, q_\tau))$$

are $2n \times 2n$ matrices.

Let us now twist the jets according to lemma 3. Call $\xi^+_\tau = \tau D_p(\psi(p_\tau, q_\tau))$ and $\xi^-_\tau = -\tau D_q(\psi(p_\tau, q_\tau))$. By our choice of ψ we get $\xi^+_\tau = \xi^-_\tau$. Set

$$\eta^+_\tau = \mathbb{A}(p_\tau) \cdot \xi^+_\tau$$

and

$$\eta^-_\tau = \mathbb{A}(q_\tau) \cdot \xi^-_\tau.$$

We see that

$$\begin{aligned} |\eta^+_\tau - \eta^-_\tau| &= |\mathbb{A}(p_\tau) - \mathbb{A}(q_\tau)||\xi^+_\tau| \\ &\leq C\tau|p_\tau - q_\tau||D_p(\psi(p_\tau, q_\tau))| \\ &\leq C\tau\psi(p_\tau, q_\tau) \\ &= o(1), \end{aligned}$$

where we have used the fact that $|p-q||D_p\psi(p,q)| \leq C\psi(p,q)$, property (i) and the smoothness, in the form of a Lipschitz condition, of $\mathbb{A}(p)$.

The second order parts of the jets are given by

$$\langle \mathcal{X}_\tau \cdot t, t\rangle = \langle \mathbb{A}(p_\tau) X_\tau \mathbb{A}^t(p_\tau) \cdot t, t\rangle + \sum_{k=1,n} \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle (\xi^+_\tau)_k$$

and

$$\langle \mathcal{Y}_\tau \cdot t, t\rangle = \langle \mathbb{A}(q_\tau) Y_\tau \mathbb{A}^t(q_\tau) \cdot t, t\rangle + \sum_{k=1,n} \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle (\xi^-_\tau)_k.$$

In order to estimate their difference we write

$$\begin{aligned} \langle \mathcal{X}_\tau \cdot t, t\rangle - \langle \mathcal{Y}_\tau \cdot t, t\rangle &= \langle X_\tau \mathbb{A}^t(p_\tau)\cdot t, \mathbb{A}^t(p_\tau)\cdot t\rangle - \langle Y_\tau \mathbb{A}^t(q_\tau)\cdot t, \mathbb{A}^t(q_\tau)\cdot t\rangle \\ &\quad + \sum_{k=1}^n \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle (\xi^+_\tau)_k \\ &\quad - \sum_{k=1}^n \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle (\xi^-_\tau)_k. \end{aligned}$$

Using inequality 3.3, we get

$$\begin{aligned} \langle \mathcal{X}_\tau \cdot t, t\rangle - \langle \mathcal{Y}_\tau \cdot t, t\rangle &\leq \langle C\left(\mathbb{A}(p_\tau)\cdot t \oplus \mathbb{A}(q_\tau)\cdot t\right), \mathbb{A}(p_\tau)\cdot t \oplus \mathbb{A}(q_\tau)\cdot t\rangle \\ &\quad + \tau\left[\sum_{k=1}^n \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle \frac{\partial\psi}{\partial p_k}(p_\tau, q_\tau)\right] \\ &\quad - \tau\left[\sum_{k=1}^n \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle \frac{\partial\psi}{\partial p_k}(p_\tau, q_\tau)\right] \end{aligned}$$

To estimate the first term in the right hand side we note that symmetries of ψ give a block structure to $D^2_{p,q}\psi$ so that we have

$$\langle C(\gamma \oplus \delta), \gamma \oplus \delta\rangle \leq C\tau|\gamma - \delta|^2.$$

Replacing γ by $\mathbb{A}(p_\tau)\cdot t$ and δ by $\mathbb{A}(q_\tau)\cdot t$, using the smoothness of $\mathbb{A}$, and property (i) we get that this first term is $o(1)$. The second and third term together are also $o(1)$ since their difference is estimated by a constant times $\tau|p_\tau - q_\tau||D_p\psi(p_\tau, q_\tau)|$.

□

3.1. **Fully Non-Linear Elliptic Equations.** Consider a continuous function

$$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S(\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$(x, z, \eta, \mathcal{X}) \longrightarrow F(x, z, \eta, \mathcal{X}).$$

We will always assume that F is proper; that is, F is increasing in u and F is decreasing in $\mathcal{X}$.

Definition 1. *A lower semicontinuous function v is a* ***viscosity supersolution*** *of the equation*

$$F(x, u(x), \mathfrak{X}u(x), (\mathfrak{X}^2u(x))^*) = 0$$

if whenever $(\eta, \mathcal{Y}) \in J^{2,-}_{\mathfrak{X}}(v, x_0)$ we have

$$F(x_0, v(x_0), \eta, \mathcal{Y}) \geq 0.$$

Equivalently, if $\varphi \in C^2$ touches v from below at x_0, then we must have

$$F(x_0, v(x_0), \mathfrak{X}\varphi(x_0), (\mathfrak{X}^2\varphi(x_0))^*) \geq 0.$$

Definition 2. *An upper semicontinuous function u is a* ***viscosity subsolution*** *of the equation*

$$F(x, u(x), \mathfrak{X}u(x), (\mathfrak{X}^2u(x))^*) = 0$$

if whenever $(\eta, \mathcal{X}) \in J^{2,+}_{\mathfrak{X}}(u, x_0)$ we have

$$F(x_0, u(x_0), \eta, \mathcal{X}) \leq 0.$$

Equivalently, if $\varphi \in C^2$ touches u from above at x_0, then we must have

$$F(x_0, u(x_0), \mathfrak{X}\varphi(x_0),\ (\mathfrak{X}^2\varphi(x_0))^*) \leqq 0.$$

Note that if u is a viscosity subsolution and $(\eta, \mathcal{X}) \in \bar{J}^{2,+}_{\mathfrak{X}}(u, x_0)$ then, by the continuity of F, we still have

$$F(x_0, u(x_0), \eta, \mathcal{X}) \leq 0.$$

A similar remark applies to viscosity supersolutions and the closure of second order subjets.

A **viscosity solution** is defined as being both a viscosity subsolution and a viscosity supersolution. Observe that since F is proper, it follows easily that if u is a smooth classical solution then u is a viscosity solution.

Examples:

- Uniformly elliptic equations with continuous coefficients:

$$-Lu = -\sum_{j=1}^{n} \alpha_{i,j}(p) X_j X_j u(p) = f(p),$$

where the symmetric matrix $(\alpha_{i,j})$ has eigenvalues in an interval $[\lambda, \Lambda]$, $\lambda > 0$, and f is continuous. When the matrix $(\alpha_{i,j})$ is the identity matrix the operator L is the Hörmander-Kohn Laplacian and it is denoted by $\Delta_{\mathfrak{X}}$.

- The ∞-Laplace equation ([Bi1]) relative to the frame $\mathfrak{X}$:

$$-\Delta_{\mathfrak{X},\infty} u = -\sum_{i,j=1}^{n} (X_i u)(X_j u) X_i X_j u = -\langle (\mathfrak{X}^2 u)^* \mathfrak{X}u, \mathfrak{X}u \rangle$$

- The p-Laplace equation, $2 \leq p < \infty$, relative to the frame $\mathfrak{X}$:

$$\begin{aligned} -\Delta_{\mathfrak{X},p} u &= -\left[|\mathfrak{X}u|^{p-2} \Delta_{\mathfrak{X}} u + (p-2) |\mathfrak{X}u|^{p-4} \Delta_{\mathfrak{X},\infty} u \right] \\ &= -\mathrm{div}_{\mathfrak{X}} \left(|\mathfrak{X}u|^{p-2} \mathfrak{X}u \right) = 0 \end{aligned}$$

Here $\mathrm{div}_{\mathfrak{X}}$ is the natural divergence relative to the frame $\mathfrak{X}$ defined by duality with respect to $\mathfrak{X}u$. See [M] for details. We need $p \geq 2$ for the continuity assumption of the corresponding F.

Once we have the maximum principle (Theorem 1) we get comparison theorems for viscosity solutions of various classes of fully nonlinear equations of the general form

$$F(x, u(x), \mathfrak{X}u(x), (\mathfrak{X}^2 u(x))^*) = 0$$

where F is continuous and proper as it is done in [CIL]. We refer to [M] for concrete examples that include the uniformly elliptic case as well as the p-Laplacian. The infinite Laplacian case has recently been settled by Bieske [Bi4].

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