# Rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature

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## 1　Introduction

This is a brief report of [16].

In 1992, Bence, Merriman and Osher proposed in [2] an algorithm for computing the motion of a hypersurface by its mean curvature. It is described as follows.

Let $C_0 \subset \mathbb{R}^N$ be a closed set and let $u = u(t, x)$ be the solution of

$$(1.1) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \begin{cases} 1 & x \in C_0, \\ -1 & x \in \mathbb{R}^N \backslash C_0. \end{cases} \end{cases}$$

Fix a time step $h > 0$ and set

$$C_1 = \{ x \in \mathbb{R}^N \mid u(h, x) \geq 0 \}.$$

Next we solve (1.1) with $C_0$ replacing $C_1$ and define a new set $C_2$ with $u$ replaced by the solution of (1.1) with the new initial data. Repeating this procedure, we have a sequence $\{C_k\}_{k=0,1,\ldots}$ of closed sets in $\mathbb{R}^N$. Then we define

$$C_t^h = C_k \text{ if } kh \leq t < (k+1)h, \ k = 0, 1, \ldots$$

for $t \geq 0$. Letting $h \to 0$, we obtain in the limit a flow $\{D_t\}_{t \geq 0}$ of closed subsets in $\mathbb{R}^N$ with $D_0 = C_0$ and then $\partial D_t (:= \Gamma_t)$ moves by its $((N-1)$-times) mean curvature. That is, $\Gamma_t$ satisfies

$$V = \kappa \quad \text{on } \Gamma_t, t > 0.$$

Here $V = V(t, x)$ is the normal velocity of $\Gamma_t$ at $x \in \Gamma_t$ and $\kappa = \kappa(t, x)$ is the mean curvature of $\Gamma_t$ at $x \in \Gamma_t$.

The convergence of the Bence-Merriman-Osher (BMO) algorithm was proved by Mascarenhas [19], Evans [5], Barles-Georgelin [1] and Goto-Ishii-Ogawa [9]. The generalizations of the BMO algorithm were considered by Ishii [13], Ishii-Pires-Souganidis [15], Ishii-Ishii [14], Vivier [22] and Leoni [18]. However, to the author's knowledge, there are a few results on the rate of convergence of the BMO algorithm. In [21] Ruuth gave a time-local error estimate of the BMO algorithm in $\mathbb{R}^2$. Nakamura and the author proved in [17] the Hausdorff distance between the motion by mean curvature $\Gamma_t$ and the approximate

interface $\Gamma_t^h = \partial C_t^h$ is of order $h^{1/2}$. This result is valid before the onset of singularities, but not optimal.

The purposes of this article are to present the rate of convergence of the BMO algorithm globally in time and to show its optimality. In fact, assuming $\{\Gamma_t\}_{0 \leq t < T_0}$ is the motion of a smooth and compact hypersurface by mean curvature, we prove that, for any $T < T_0$,

$$\sup_{t \in [0,T]} d_H(\Gamma_t, \Gamma_t^h) \leq Lh,$$

where $L$ is a constant depending on $T$, but independent of small $h > 0$ and $d_H$ denotes the Hausdorff distance. We can show that this is the optimal rate in the case of a circle evolving by curvature.

Both of the order in $h$ and the optimality are the consequence of the maximum principle and the explicit constructions of sub- and supersolutions of (1.1), which is inspired by the asymptotic analysis of solutions of the Allen-Cahn equation (see, e.g., Fife [6] and de Mottoni-Schatzman [4]). As for the relation between the BMO algorithm and the Allen-Cahn equation, from the viewpoint of the splitting methods in numerical analysis, Vivier [22] first pointed out that we may think the Allen-Cahn equation is an approximation of the BMO algorithm. Leoni [18] and Goto-Ishii-Ogawa [9] gave the proofs of the convergence of the BMO algorithm and a generalized scheme by applying some techniques of the asymptotic analysis for the Allen-Cahn equation. The arguments in this paper also rely on them.

This article is organized as follows. In section 2 we state our main results. Section 3 is devoted to the formal asymptotic expansion of the solution of (1.1). In section 4 we give the outline of the proof of Theorem 2.1 below.

## 2  Main results

First, we state the rate of convergence. For this purpose, we rewrite the BMO algorithm as follows: Let $\Gamma_0 \subset \mathbb{R}^N$ be a smooth and compact hypersurface and $C_0 \subset \mathbb{R}^N$ the compact set such that $\partial C_0 = \Gamma_0$. Fix a time step $h > 0$. Let $u^h = u^h(t, x)$ be the solution of

(2.1)
$$\begin{cases} u_t^h = \Delta u^h \quad \text{in } (kh, (k+1)h) \times \mathbb{R}^N, \\ u^h(kh, x) = \begin{cases} 1, & x \in C_k, \\ -1, & x \in \mathbb{R}^N \backslash C_k, \end{cases} \\ C_k = \begin{cases} \text{the given set } C_0 & \text{for } k = 0, \\ \left\{ x \in \mathbb{R}^N \;\middle|\; \lim_{t \to kh-} u^h(t, x) \geq 0 \right\} & \text{for } k = 1, 2, \ldots \end{cases} \end{cases}$$

Set

(2.2)
$$\Gamma_t^h = \begin{cases} \{ x \in \mathbb{R}^N \mid u^h(t, x) = 0 \} & \text{for } t \neq kh, \\ \partial C_k & \text{for } t = kh. \end{cases}$$

We note that $\Gamma_t^h$ is a smooth and compact hypersurface for each $t \geq 0$, $h > 0$. Using this formulation, we have the following theorem.

**Theorem 2.1** *Let* $\{\Gamma_t\}_{0 \leq t < T_0}$ *be the motion of a smooth and compact hypersurface by mean curvature starting from* $\Gamma_0$. *Let* $\Gamma_t^h$ *be defined by* (2.2). *Then, for any* $T \in (0, T_0)$, *there exist* $h_0 > 0$ *and* $L > 0$ *such that*

$$(2.3) \qquad \sup_{t \in [0,T]} d_H\left(\Gamma_t^h, \Gamma_t\right) \leq Lh$$

*for all* $h \in (0, h_0)$. *Here* $d_H(A, B)$ *denotes the Hausdorff distance between* $A$, $B \subset \mathbb{R}^N$.

**Remark 2.1** (1) As to the motion by mean curvature $\{\Gamma_t\}_{0 \leq t < T_0}$, the following results are well known: Assume that $\Gamma_0$ is the smooth boundary of a bounded domain. If $N = 2$ or $\Gamma_0$ is convex, then $\Gamma_t$ evolves smoothly and it shrinks to a point at a finite time. (see Gage - Hamilton [7], Grayson [10] and Huisken [12]). In other cases the singularities may appear before $\Gamma_t$ shrinks to a point (see, e.g., Grayson [11]). Therefore (2.3) is valid before $\Gamma_t$ shrinks to a point or develops the singularities.
(2) We can formally derive (2.3) by using the result of Ruuth [21]. However, it seems that its mathematical proof has not yet given.

The (2.3) is optimal with respect to the order of $h$. To see this, we consider the following situation: Set $\Gamma_0 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ and $\Gamma_0 = \{x \in \mathbb{R}^2 \mid |x| = \phi(t)\}$ ($\phi(t) = \sqrt{1 - 2t}$). Then it is easily seen that $\Gamma_t$ moves by its curvature and it shrinks to the origin at $T_{max} = 1/2$. For each $h > 0$, let $\{\Gamma_t^h\}_{t \geq 0}$ be the flow defined by (2.2) satisfying $\Gamma_0^h = \Gamma_0$. We note that, for each $t$, $h > 0$, $\Gamma_t^h$ is also a circle centered at origin and thus we can define its radius $R_h(t)$. In this setting, we obtain the following estimate.

**Theorem 2.2** *For each* $T \in (0, T_{max})$, *there exist* $h_0 > 0$ *and* $L > 0$ *such that*

$$(2.4) \qquad \sup_{t \in [0,T]} |R_h(t) - (\phi(t) - h\psi(t))| \leq Lh^{3/2}$$

*for all* $h \in (0, h_0)$. *Here* $\psi(t) = -\dfrac{\log \phi(t)}{3\phi(t)}$.

This theorem implies the optimality of (2.3). In addition, we see that $\Gamma_t^h$ moves faster than $\Gamma_t$.

# 3   Formal asymptotic expansion

To prove Theorem 2.1 and 2.2, we construct suitable sub- and supersolutions of (1.1). For this purpose, we apply the method of asymptotic expansion of solutions of Allen-Cahn equation. As for this method, see Fife [6], de Mottoni-Schatzman [4] etc.

Let $u$ be the solution of (1.1). Set $\widetilde{\Gamma}_t = \{x \in \mathbb{R}^N \mid u(t,x) = 0\}$. Since it follows from Goto-Ishii-Ogawa [9, Proposition 6.1] that $u(t,\cdot) \simeq 1$ or $-1$ away from $\widetilde{\Gamma}_t$ as $t \searrow 0$, we have only to consider the asympotics of $u$ near $\widetilde{\Gamma}_t$. We define the signed distance function $\widetilde{d}$ to $\widetilde{\Gamma}_t$ as follows:

$$\widetilde{d}(t,x) = \begin{cases} \mathrm{dist}(x,\widetilde{\Gamma}_t) & \text{for } x \in \widetilde{P}_t, \\ -\mathrm{dist}(x,\widetilde{\Gamma}_t) & \text{for } x \in \mathbb{R}^N \backslash \widetilde{P}_t, \end{cases}$$

where $\widetilde{P}_t$ is the domain enclosed by $\widetilde{\Gamma}_t$. By Evans [5, Theorem 4.1] we may consider that $\widetilde{\Gamma}_t$ moves by mean curvature. Then we see that, for some $K$, $\delta > 0$, $\widetilde{d}$ satisfies

(3.1) $\qquad |\widetilde{d}_t - \Delta\widetilde{d} - \kappa^s\widetilde{d}| \leq K\widetilde{d}^2 \quad \text{for } 0 < t < h, \ x \in \{y \in \mathbb{R}^N \mid |\widetilde{d}(t,y)| < \delta\},$

where $\kappa^s = \kappa^s(t,x)$ is the sum of squares of all principal curvatures of $\widetilde{\Gamma}_t$ at $x \in \widetilde{\Gamma}_t$ (see, e.g., Chen [3], Gilbarg-Trudinger [8] and Paolini-Verdi [20]).

Assume that $u$ can be expanded near $\widetilde{\Gamma}_t$ as follows:

$$u(t,x) = V_0\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + \sqrt{t}V_1\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + tV_2\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + \cdots.$$

We introduce a new variable $\rho = \widetilde{d}(t,x)/2\sqrt{t}$. Substituting the right-hand side of this equality into (1.1), we get the following:

$$\begin{aligned}
0 = \ & -\frac{1}{4t}(V_{0,\rho\rho} + 2\rho V_{0,\rho}) \\
& - \frac{1}{\sqrt{t}}\left\{V_{0,\rho}\langle DV_0, Dd\rangle + \left(\frac{1}{4}V_{0,\rho\rho} + \frac{\rho}{2}V_{1,\rho} - \frac{1}{2}V_1\right)\right\} \\
& + \left\{V_{0,t} - \Delta V_0 + \kappa^s\rho V_{0,\rho} + \frac{V_{1,\rho}}{2}(d_t - \Delta d - 2\langle DV_1, Dd\rangle)\right. \\
& \left. - \left(\frac{1}{4}V_{2,\rho\rho} + \frac{\rho}{2}V_{2,\rho} - V_2\right)\right\} \\
& + \cdots,
\end{aligned}$$

where $DV_j = (\partial V_j/\partial x_1, \ldots, \partial V_j/\partial x_N)$, $V_{j,\rho} = \partial V_j/\partial\rho$, $V_{j,\rho\rho} = \partial^2 V_j/\partial\rho^2$ $(j = 0,1,2,\ldots)$ and we have used (3.1).

We compare the coefficients of $t^{j/2}$ $(j = -2,-1,0,1,\ldots)$. In the case of the $t^{-1}$-term, we have

$$V_{0,\rho\rho} + 2\rho V_{0,\rho} = 0 \quad \text{on } \mathbb{R}.$$

Taking into the account Goto-Ishii-Ogawa [9, Proposition 6.1], we impose the following condition on $V_0$:

$$V_0(t,x,\rho) \longrightarrow \begin{cases} 1, & (\rho \to +\infty), \\ -1, & (\rho \to -\infty). \end{cases}$$

Hence we obtain $V_0(t,x,\rho) = V_0(\rho) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^\rho e^{-s^2} ds$. As for the $t^{-1/2}$-term, we get

$$\frac{1}{4}V_{0,\rho\rho} + \frac{\rho}{2}V_{1,\rho} - \frac{1}{2}V_1 = 0 \quad \text{on } \mathbb{R}.$$

Since $u - V_0$ converges to 0 as $\rho \to \pm\infty$ exponentially, we see that $V_1$ satisfies

$$V_1(t,x,\rho) \longrightarrow 0 \quad (\rho \to \pm\infty).$$

Then $V_1 \equiv 0$. By a similar way, we observe that $V_2$ satisfies

$$\frac{1}{4}V_{2,\rho\rho} + \frac{\rho}{2}V_{2,\rho} - V_2 = \kappa^s \rho V_{0,\rho} \quad \text{on } \mathbb{R}, \quad V_2(t,x,\rho) \longrightarrow 0 \quad (\rho \to \pm\infty).$$

Thus we obtain $V_2 = -\dfrac{\kappa^s \rho}{2}V_{0,\rho}$.

Continuing the above processes, we can obtain the formal asymptotic expansion of $u$.

# 4 Outline of the proof of Theorem 2.1

We mention only the outline of the proof of Theorem 2.1 because the proof of Theorem 2.2 is similar to that of Theorem 2.1, but it is more complicated.

Let $\{\Gamma_t\}_{0 \le t < T_0}$ be the motion of a smooth and compact hypersurface by mean curvature. We define the signed distance function $d(t,x)$ to $\Gamma_t$ by

$$d(t,x) = \begin{cases} \text{dist}(x,\Gamma_t) & \text{for } x \in P_t, \\ -\text{dist}(x,\Gamma_t) & \text{for } x \in \mathbb{R}^N \backslash P_t, \end{cases}$$

where $P_t$ is the bounded domain enclosed by $\Gamma_t$. Fix $T \in (0, T_0)$ and $h > 0$. Set $m = [T/h]$. Let $u^h$ be the solution of (2.1) and let $\{\Gamma_t^h\}_{t,h}$ be defined by (2.2). Then, for each $t$, $h > 0$, $\Gamma_t^h$ is a smooth and compact hypersurface.

According to the formal asymptotic expansion, we define

$$v^{0,\pm}(t,x) = V_0\left(\frac{d(t,x)}{2\sqrt{t}}\right) + tV_2\left(t,x,\frac{d(t,x)}{2\sqrt{t}}\right) \pm t^{3/2}V_3.$$

Here $V_0$ and $V_2$ is the functions obtained in the previous section. At $t = 0$, we set

$$v^{0,-}(0,x) = v^{0,+}(0,x) = u^h(0,x).$$

Taking $V_3 > 0$ sufficiently large and independent of $h > 0$, we can show that $v^{0,-}$, $v^{0,+}$ are, respectively, a subsolution and a supersolution of (1.1) in $(0,h) \times \mathbb{R}^N$. Hence, by the comparison principle for the heat equation, we get

(4.1) $$v^{0,-} \le u^h \le v^{0,+} \quad \text{on } [0,h] \times \mathbb{R}^N.$$

In addition, it is easily seen from (4.1) and the definitions of $v^{0,\pm}$ that

(4.2) $\qquad v^{0,-}(t,x) \leq 0 \leq v^{0,+}(t,x) \quad \text{for } t \in (0,h], \ x \in \Gamma_t^h,$

(4.3) $\qquad v^{0,+}(t,x) = t^{3/2}V_3, \ v^{0,-}(t,x) = -t^{3/2}V_3 \quad \text{for } t \in (0,h], \ x \in \Gamma_t,$

(4.4) $\qquad v^{0,+}(t,x) < 0 \text{ for } t \in (0,h], \ x \in \{y \in \mathbb{R}^N \mid d(t,y) < -\sqrt{t}\},$

(4.5) $\qquad v^{0,-}(t,x) > 0 \text{ for } t \in (0,h], \ x \in \{y \in \mathbb{R}^N \mid d(t,y) > \sqrt{t}\},$

(4.6) $\qquad D_n v^{0,\pm}(t,x) \geq \dfrac{K}{\sqrt{t}} \quad \text{for } t \in (0,h], \ x \in \Theta_t^h,$

where $D_n v(t,x)$ is the derivative of $v$ with respect to the inner normal direction to $\Gamma_t^h$ at $x \in \Gamma_t^h$ and $K > 0$ is a constant independent of $h > 0$. Put

$$\Sigma_t^h := \{x \in \mathbb{R}^N \mid v^{0,-}(t,x) \leq 0 \leq v^{0,+}(t,x)\}, \quad \Theta_t^h := \{x \in \mathbb{R}^N \mid |d(t,x)| \leq \sqrt{t}\},$$

$$P_t^h = \text{the bounded domain enclosed by } \Gamma_t^h.$$

Then, from (4.2) - (4.5), we have

(4.7) $\qquad \Gamma_t \cup \Gamma_t^h \subset \Sigma_t^h \subset \Theta_t^h \quad \text{for } t \in [0,h].$

Using (4.3), (4.6) and (4.7), we observe that

(4.8) $\qquad \Gamma_t \cup \Gamma_t^h \subset \Sigma_t^h \subset \{x \in \mathbb{R}^N \mid |d(t,x)| \leq C_1 t^2\} \quad \text{for } t \in [0,h],$

where $C_1 > 0$ is a large constant independent of $h > 0$. Let $\Gamma_t^{h,+} = \partial(P_t^h \cap P_t)$ and $\Gamma_t^{h,-} = \partial(P_t^h \cup P_t)$. Then $\Gamma_t^{h,+} \cup \Gamma_t^{h,-} = \Gamma_t^h \cup \Gamma_t$ and thus, by (4.8) and some elementary arguments, we obtain

$$\begin{aligned} d_H(\Gamma_t^h, \Gamma_t) &= \max\{d_H(\Gamma_t^{h,+}, \Gamma_t), d_H(\Gamma_t^{h,-}, \Gamma_t)\} \leq \sup_{x:|d(t,x)|=C_1 t^2} \text{dist}(x, \Gamma_t) \\ &= C_1 t^2 \leq C_1 h^2 \quad \text{for } t \in [0,h]. \end{aligned}$$

Set $\alpha_1 = C_1$ and define

$$v^{1,\pm}(t,x) = V_0\left(\frac{d^{1,\pm}(t,x)}{2\sqrt{t}}\right) + tV_2\left(t,x,\frac{d^{1,\pm}(t,x)}{2\sqrt{t}}\right) \pm ((t-h)^{3/2}V_3 + V_4\alpha_1 h^2\sqrt{t-h}),$$

$$v^{1,\pm}(h,x) = \begin{cases} 1 & \text{on } \{x \in \mathbb{R}^N \mid d^{1,\pm}(h,x) \geq 0\}, \\ -1 & \text{in } \{x \in \mathbb{R}^N \mid d^{1,\pm}(h,x) < 0\}, \end{cases}$$

$$d^{1,\pm}(t,x) = d(t,x) \pm \alpha_1 h^2 \quad \text{for } t \in [h, 2h].$$

Here $V_3$ is the same constant as above and $V_4 > 0$ is chosen sufficiently large and independent of $h > 0$. Then we can verify that $v^{1,-}$ and $v^{1,+}$ are, respectively, a subsolution and a supersolution of (2.1) in $(h, 2h) \times \mathbb{R}^N$ satisfying $v^{1,-}(h,x) \leq u^h(h,x) \leq v^{1,+}(h,x)$ on $\mathbb{R}^N$. Hence we get, by the comparison principle for the heat equation,

$$v^{1,-} \leq u^h \leq v^{1,+} \quad \text{on } [h, 2h] \times \mathbb{R}^N.$$

It is observed by a similar argument to the above that

$$d_H(\Gamma_t^h, \Gamma_t) \leq \alpha_2 h^2 \quad \text{for } t \in [h, 2h] \quad (\alpha_2 := C_1 + (1 + C_2 h)\alpha_1)$$

for some $C_2 > 0$ independent of $h > 0$.

We repeat this argument inductively. Set $\alpha_k = C_1 + (1 + C_2 h)\alpha_{k-1}$ $(k \geq 2)$ and define

$$v^{k,\pm}(t, x) = V_0 \left( \frac{d^{k,\pm}(t, x)}{2\sqrt{t}} \right) + tV_2 \left( t, x, \frac{d^{k,\pm}(t, x)}{2\sqrt{t}} \right) \pm ((t - kh)^{3/2} V_3 + V_4 \alpha_k h^2 \sqrt{t - kh}),$$

$$v^{k,\pm}(kh, x) = \begin{cases} 1 & \text{on } \{x \in \mathbb{R}^N \mid d^{k,\pm}(kh, x) \geq 0\}, \\ -1 & \text{in } \{x \in \mathbb{R}^N \mid d^{k,\pm}(kh, x) < 0\}, \end{cases}$$

$$d^{k,\pm}(t, x) = d(t, x) \pm \alpha_k h^2 \quad \text{for } t \in [kh, (k+1)h].$$

We can check that $v^{k,-}$ and $v^{k,+}$ are, respectively, a subsolution and a supersolution of (2.1) in $(kh, (k+1)h) \times \mathbb{R}^N$ satisfying $v^{k,-}(kh, x) \leq u^h(kh, x) \leq v^{k,+}(kh, x)$ on $\mathbb{R}^N$. Applying the comparison principle for the heat equation, we get

$$v^{k,-} \leq u^h \leq v^{k,+} \quad \text{on } [kh, (k+1)h] \times \mathbb{R}^N.$$

By a similar argument to the case $k = 0$, we have

$$d_H(\Gamma_t^h, \Gamma_t) \leq \alpha_{k+1} h^2 \quad \text{for } t \in [kh, (k+1)h] \quad (\alpha_{k+1} := C_1 + (1 + C_2 h)\alpha_k).$$

Finally we estimate $\{\alpha_k\}_{1 \leq k \leq m}$. It follows from the definitions of $m$ and $\{\alpha_k\}_{1 \leq k \leq m}$ that

$$\alpha_k = C_1 \sum_{l=1}^{k} (1 + C_2 h)^{l-1} \leq C_1 \frac{(1 + C_2 h)^m - 1}{C_2 h} \leq C_1 m (1 + C_2 h)^{m-1} \leq \frac{C_1 T e^{C_2 T}}{h}.$$

Thus we obtain (2.3) with $L = C_1 T e^{C_2 T}$. $\square$

## 参考文献

[1] G. Barles and C. Georgelin. A simple proof of convergence for an approximation scheme for computing motion by mean curvature. *SIAM J. Numer. Anal.*, 32:484–500, 1995.

[2] J. Bence, B. Merriman, and S. Osher. Diffusion generated motion by mean curvature. in "Computational Crystal Growers Workshop", J. Taylor ed. *Selected Lectures in Math., Amer. Math. Soc., Province*, 1992.

[3] X. Chen. Generation and propagation of the interface for reaction-diffusion equations. *J. Differential Equations*, 96:116–141, 1992.

[4] P. de Mottoni and M. Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.*, 347:1533–1589, 1995.

[5] L. C. Evans. Convergence of an algoritm for mean curvature motion. *Indiana Univ. Math. J.*, 42:533–557, 1993.

[6] P. C. Fife. *Dynamics of Internal Layers and Diffusive Interfaces*. SIAM, Philadelphia, 1988.

[7] M. Gage and R. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geometry*, 23:69–95, 1986.

[8] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, 1983.

[9] Y. Goto, K. Ishii, and T. Ogawa. Method of the distance function to the Bence-Merriman-Osher algorithm for motion by mean curvature. *Comm. Pure Appl. Anal.*, 4:311–339, 2005.

[10] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geometry*, 26:285–314, 1987.

[11] M. A. Grayson. A short note on the evolution of surfaces via mean curvature. *Duke Math. J.*, 58:555–558, 1989.

[12] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geometry*, 20:237–266, 1984.

[13] H. Ishii. A generalization of the Bence, Merryman and Osher algorithm for motion by mean curvature. *in Curvature flows and related topics, (ed. A. Damlamian, J. Spruck, A. Visintin), Gakko Tosho, Tokyo*, 00:111–127, 1995.

[14] H. Ishii and K. Ishii. An approximation scheme for motion by mean curvature with right-angle boundary condition. *SIAM J. Math. Anal.*, 33:369–389, 2001.

[15] H. Ishii, G. E. Pires, and P. E. Souganidis. Threshold dynamics type approximation schemes for propagating fronts. *J. Math. Soc. Japan*, 50:267–308, 1999.

[16] K. Ishii. Optimal rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature. SIAM J. Math. Anal. (in press), 2005.

[17] K. Ishii and T. Nakamura. An error estimate to Bence-Merriman-Osher algorithm for motion by mean curvature. *Rev. Kobe Univ. Mercantile Marine*, 51:105–115, 2003.

[18] F. Leoni. Convergence of an approximation scheme for curvature-dependent motion of sets. *SIAM J. Numer. Anal.*, 39:1115–1131, 2001.

[19] P. Mascarenhas. Diffusion generated motion by mean curvature. Campus Report, Math. Dept., University of California, Los Angeles, 1992.

[20] M. Paolini and C. Verdi. Asymptotic and numerical analysis of the mean curvature flow with a space-dependent relaxation parameter. *Asymptotic Anal.*, 5:553–574, 1992.

[21] S. J. Ruuth. Efficient Algorithms for Diffusion-Generated Motion by Mean Curvature. Ph. D. Thesis, The University of British Columbia, 1996.

[22] L. Vivier. Convergence of an approximation scheme for computing motions with curvature dependent velocities. *Differential Integral Equations*, 13:1263–1288, 2000.