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A representation formula for solutions of Hamilton-Jacobi equations

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1 Introduction

The main purpose in this report is to describe briefly some results in [6], which have recently been obtained jointly with Prof. H. Ishii, on representation of solutions of Hamilton-Jacobi equations.

In connection with weak KAM theory, Fathi, Siconolfi, and others (see for instance [2,4]) have recently investigated Hamilton-Jacobi equations on compact manifolds without boundary and established a fairly general representation formula for their solutions. A novel idea in this formula is in its crucial use of the Aubry set, which may be more properly referred as the projected Aubry set. Indeed, as we will explain more precisely later on, if u is the solution of $H(x, Du) = 0$, then the formula has roughly the form of

$$u(x) = \inf\{d(x,y) + \psi(y) \mid y \in \mathcal{A}\},$$

where $\mathcal{A}$ is the Aubry set for H, ψ is a given data, and d is the "Green function" for $H(x, Du) = 0$ in terms of the max-plus algebra.

The results in [6] are concerned with the Dirichlet and state constraint problems for Hamilton-Jacobi equations give representation formulas for viscosity solutions of these problems. These formulas are variants or adaptations of the representation formula to the Dirichlet and state constraint problems.

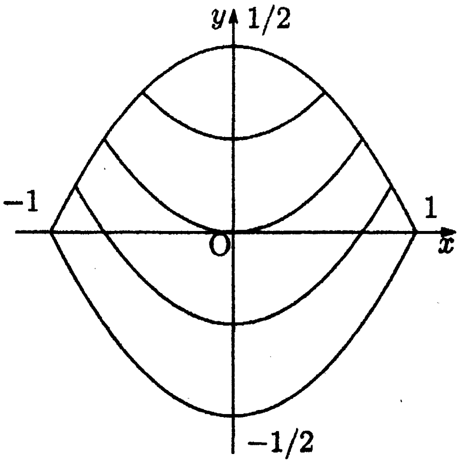
A very primitive form of our formula can be seen in the following well-known formula. If u is a viscosity solution of the one-dimensional Dirichlet problem

$$|Du(x)| = |x| \quad \text{for } x \in (-1,1) \quad \text{and} \quad u(x) = 0 \quad \text{for } x \in \{-1,1\},$$

then

$$\text{(1)} \qquad u(x) = u_a(x) := \min\{\frac{1}{2}(1-|x|^2), \frac{1}{2}|x|^2 + a\} \quad \text{for all } x \in [-1,1]$$

and for some constant $a \in [-1/2, 1/2]$.



In this example the Aubry set $\mathcal{A}_D$ comprises of the origin and all the boundary points -1 and 1. Let $d(\cdot, y)$ denote the maximal viscosity solution of $|Dd(x,y)| \le |x|$ in $(-1,1)$ satisfying $d(y,y) = 0$. Then we have

$$d(x,y) = \left| \int_y^x |t| dt \right|,$$

and, in particular,

$$u_a(x) = \min\{a + d(x,0), d(x,-1), d(x,1)\}.$$

We should remark that a representation formula like (1) has been already obtained in Lions [1] for the multi-dimensional Hamilton-Jacobi equation $|Du| = f(x)$, where $f \ge 0$ and f vanishes only a finite number of points x.

Our approach to establishing the representation formula does not depend on any variational formulas (especially in the treatment of Aubry sets) and therefore is based only on PDE techniques. This PDE approach is hidden or at least is not clearly stated in previous work, but the presentations here and in [6] may hopefully clarifies this point.

We will be dealing only with viscosity solutions of Hamilton-Jacobi equations in this note and thus in this note we mean by "solutions", "subsolutions", and "supersolutions" viscosity solutions, viscosity subsolutions, and viscosity supersolutions, respectively.

This report is organized as follows. In Section 2 we give some preliminaries and our assumptions. Representation formulas for solutions are treated in the case of the Dirichlet problem in Sections 3 and 4 and in the case of the state constraint problem in Section 5.

2 Assumptions and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$. We consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0 \text{ in } \Omega.$$

We give a list of assumptions on the Hamiltonian H:

Assumptions.

(H0) *The function* $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *is continuous in* $\Omega \times \mathbb{R}^n$.

(H1) *There is a subsolution* $\phi \in C(\Omega)$ *such that*

$$H[\phi](x) \leq 0 \textit{ in } \Omega.$$

For notational simplicity, we write $H[\phi](x)$ *for* $H(x, D\phi(x))$.

(H2) *The function* $p \mapsto H(x,p)$ *is convex for each* $x \in \Omega$.

(H3) *For any* $x \in \Omega$, *there is* $M > 0$ *such that*

$$\{p \in \mathbb{R}^n \mid H(x,p) \leq 0\} \subset B(0, M).$$

Remark 1. $H(x,p) := |p| - |x|$ *satisfies* (H0)-(H3). *As we observes in the introduction, in general, the uniqueness of the Dirichlet problem for* $H = 0$ *in* Ω *does not hold under* (H0)-(H3).

Hereinafter we give the preliminaries to define the Aubry set.

We define $d_H : \Omega \times \Omega \to \mathbb{R}$ by

$$d_H(x,y) := \sup\{v(x) \in C(\Omega) \mid H[v] \leq 0 \text{ in } \Omega, v(y) \leq 0\}.$$

We note the following properties of d_H.

1. $H[d_H(\cdot, y)](x) \leq 0$ in Ω for any $y \in \Omega$.
2. $H[d_H(\cdot, y)](x) = 0$ in $\Omega \setminus \{y\}$ for any $y \in \Omega$.

Property 1 is easy to be verified by using the stability in viscosity theory. We can verify property 2 by using the Perron method.

We consider the Dirichlet problem

$$\begin{cases} H(x, Du(x)) &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases} \tag{DP}$$

Here H and g are given functions on $\Omega \times \mathbb{R}^n$ and $\partial\Omega$, respectively. We assume that H satisfies (H0)-(H3), and g is continuous function on $\partial\Omega$. Moreover, $\Omega \subset \mathbb{R}^n$ is assumed to satisfy the following assumption.

(D) The function $d_E : \Omega \times \Omega \to \mathbb{R}$ defined by

$$d_E(x,y) := \inf\{\int_0^T |\dot{X}(t)| dt \mid T > 0, X \in C(x,y,T)\},$$

where

$$C(x,y,T) := \{X \in \mathrm{AC}([0,T]) \mid X(0) = x, X(T) = y, X(t) \in \Omega \ (0 \leq t \leq T)\},$$

is uniformly continuous in $\Omega \times \Omega$.

Remark 2. *A sufficient condition for a domain Ω to satisfy* (D) *is that Ω is bounded and $\partial\Omega$ is Lipschitz.*

Proposition 1. *Let Ω satisfy* (D). *Then*

$$d_H(x, y) \leq M d_E(x, y),$$

where M is given by (H3), *for any $x, y \in \Omega$.*

Proof. Let v be a subsolution of $H[v] \leq 0$ satisfying $v(y) \leq 0$. Then v is a solution of $|Dv| \leq M$ by (H3). Set $v_\epsilon(x) := v * \rho_\epsilon(x)$, where $\epsilon > 0$ and ρ_ϵ is a standard mollifier kernel. We have $|Dv_\epsilon| \leq M$. Fix any $T > 0$ and any $X \in \mathcal{C}(x, y, T)$. Then we have

$$v_\epsilon(x) - v_\epsilon(y) = \int_0^T Dv_\epsilon(X(t)) \cdot \dot{X}(t)dt \leq \int_0^T |Dv_\epsilon(X(t))||\dot{X}(t)|dt \leq M \int_0^T |\dot{X}(t)|dt.$$

Hence we have

$$v_\epsilon(x) - v_\epsilon(y) \leq M d_E(x, y).$$

Sending $\epsilon \to 0$ yields

$$v(x) - v(y) \leq M d_E(x, y).$$

By the arbitrariness of v, we have consequently

$$d_H(x, y) \leq M d_E(x, y).$$

$\square$

Remark 3. *Proposition 1 means that if Ω satisfies* (D), *$d_H : \Omega \times \Omega \to \mathbb{R}$ is uniformly continuous in $\Omega \times \Omega$. Thus we may extend uniquely the domain of definition of d_H to $\overline{\Omega} \times \overline{\Omega}$ by continuity. Hereafter we denote the resulting function defined on $\overline{\Omega} \times \overline{\Omega}$ again by d_H.*

Remark 4. *By* (H3) *and* (D), *any viscosity subsolutions of $H = 0$ are Lipschitz continuous on $\overline{\Omega}$.*

3 Main theorems for the Dirichlet problem

In view of the properties of d_H stated above, we define the Aubry set as follows.

Definition 1. *Define the set $\mathcal{A}_D$ as*

$$\begin{aligned} \mathcal{A}_D &:= \{y \in \Omega \mid H[d_H(\cdot, y)] = 0 \text{ in } \Omega\} \cup \partial\Omega \\ &= \{y \in \overline{\Omega} \mid H[d_H(\cdot, y)] = 0 \text{ in } \Omega\}. \end{aligned}$$

We call $\mathcal{A}_D$ the Aubry set for the Dirichlet problem.

We show the main properties of the Aubry set $\mathcal{A}_D$ in the following propositions.

Theorem 2. *Let $u, v \in C(\overline{\Omega})$ be viscosity solutions of $H = 0$. Then*

$$u(x) = v(x) \quad on\ \mathcal{A}_D \Rightarrow u(x) = v(x)\ on\ \overline{\Omega}.$$

Theorem 3. *Let $g : \mathcal{A}_D \to \mathbb{R}$ be bounded and satisfy the compatibility condition, i.e.*

$$g(x) - g(y) \le d_H(x, y) \text{ for } x, y \in \mathcal{A}_D.$$

We define $u_g : \overline{\Omega} \to \mathbb{R}$ by

$$u_g(x) := \inf\{g(y) + d_H(x, y) \mid y \in \mathcal{A}_D\}.$$

Then $u_g \in C(\overline{\Omega})$ and $u_g(x) = g(x)$ for *any $x \in \mathcal{A}_D$. Moreover, $H[u_g](x) = 0$ in Ω.*

Corollary 4 (Representation formula for the Dirichlet problem). *Let $g : \mathcal{A}_D \to \mathbb{R}$ be bounded and satisfy compatibility condition and let $u : \overline{\Omega} \to \mathbb{R}$ be a solution of*

$$\begin{cases} H(x, Du(x)) &= 0 \quad in \quad \Omega, \\ u &= g \quad on \quad \mathcal{A}_D. \end{cases}$$

Then $u(x) = u_g(x)$.

4 Sketch of proof

Lemma 5. *For $a, b \in \mathbb{R}$, let $u, v \in C(\overline{\Omega})$ be $H[u] \le a, H[v] \le b$ in Ω, respectively. Set $w(x) := \lambda u(x) + (1 - \lambda)v(x)$ for $\lambda \in (0, 1)$. Then*

$$H[w] \le \lambda a + (1 - \lambda)b \text{ in } \Omega.$$

Proof. Fix $\hat{x} \in \Omega$. We choose a test function $\phi \in C^1(\overline{\Omega})$ satisfying

$$w(\hat{x}) = \phi(\hat{x}), \quad u(x) \le \phi(x) \text{ in } \Omega.$$

Here for $\alpha > 0$, we set $\Phi_\alpha : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ as

$$\Phi_\alpha(x, y) := \lambda u(x) + (1 - \lambda)v(y) - \phi(x) - \alpha|x - y|^2.$$

We choose $(x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega}$ such that

$$\Phi(x_\alpha, y_\alpha) := \max_{\overline{\Omega} \times \overline{\Omega}} \Phi_\alpha.$$

Then

$$\Phi_\alpha(x_\alpha, y_\alpha) \ge \Phi_\alpha(y_\alpha, y_\alpha).$$

Therefore

$$\lambda(u(x_\alpha) - u(y_\alpha)) - (\phi(x_\alpha) - \phi(y_\alpha)) \ge \alpha|x_\alpha - y_\alpha|^2.$$

Moreover since u is Lipschitz continuous and ϕ is C^1-class on $\overline{\Omega}$, we gain

$$\alpha|x_\alpha - y_\alpha| \leq C,$$

where $C := \lambda \mathrm{Lip}(u) + \max_{\overline{\Omega}} |D\phi|$. Taking subsequence, we get

$$\alpha(x_\alpha - y_\alpha) \to p \quad (\alpha \to \infty).$$

Moreover

$$x_\alpha, y_\alpha \to \hat{x} \quad (\alpha \to \infty).$$

Thus we may assume $x_\alpha, y_\alpha \in \Omega$, if $\alpha > 0$ is enough large. The maps:

$$\begin{aligned} x &\mapsto u(x) - (\frac{1}{\lambda}\phi(x) + \frac{\alpha}{\lambda}|x - y_\alpha|^2 - \frac{1-\lambda}{\lambda}v(y_\alpha)) \\ y &\mapsto u(x) - (\frac{\alpha}{1-\lambda}|x_\alpha - y|^2 + \frac{1}{1-\lambda}\phi(x_\alpha) - \frac{\lambda}{1-\lambda}u(x_\alpha)) \end{aligned}$$

take maximum at x_α, y_α and u, v are a subsolution of $H = a, H = b$, respectively. Therefore

$$\begin{aligned} H(x_\alpha, \frac{1}{\lambda}(D\phi(x_\alpha) + 2\alpha(x_\alpha - y_\alpha)) &\leq a, \\ H(y_\alpha, \frac{1}{1-\lambda}(2\alpha(y_\alpha - x_\alpha)) &\leq b. \end{aligned}$$

Sending $\alpha \to \infty$ yields

$$\begin{aligned} H(\hat{x}, \frac{1}{\lambda}(D\phi(\hat{x}) + 2p))) &\leq a, \\ H(\hat{x}, \frac{1}{1-\lambda}(-2p))) &\leq b, \end{aligned}$$

by the continuity of H. Noting the convexity of Hamiltonian, we find that

$$\begin{aligned} H(\hat{x}, D\phi(\hat{x})) &= H(\hat{x}, \lambda(\frac{1}{\lambda}(D\phi(\hat{x}) + 2p)) + (1-\lambda)(\frac{1}{1-\lambda}(-2p))) \\ &\leq \lambda H(\hat{x}, \frac{1}{\lambda}(D\phi(\hat{x}) + 2p)) + (1-\lambda)H(\hat{x}, \frac{1}{1-\lambda}(-2p)) \\ &\leq \lambda a + (1-\lambda)b. \end{aligned}$$

Consequently we get $H[w] \leq \lambda a + (1-\lambda)b$. □

This comparison result is well known.

Lemma 6. *For $a > 0$, let $u, v \in C(\overline{\Omega})$ be $H[u] \leq -a, H[v] \geq 0$ in Ω, respectively. Then*

$$u(x) \leq v(x) \text{ on } \partial\Omega \Rightarrow u(x) \leq v(x) \text{ on } \overline{\Omega}.$$

The next lemma overcomes difficulties in the proof of Theorem 2. Here we show a sketch of the proof of this lemma. Theorem 2 is a corollary of this lemma.

Lemma 7. *Let $K \subset \overline{\Omega} \setminus \mathcal{A}_D$ be compact set. Then there exist $\delta_K > 0, w_K \in C(\Omega)$ such that*

$$H(x, Dw_K(x)) \le -\delta_K \text{ on } K,$$
$$H(x, Dw_K(x)) \le 0 \text{ in } \Omega.$$

Proof. Fix $z \in \overline{\Omega} \setminus \mathcal{A}_D$. Noting that $d_H(x, z)$ is not a supersolution at $\{z\}$, we may choose a test function $\phi \in C^1(\Omega)$ such that

$$d_H(x,z) - \phi(x) \ge 0 \quad \text{in } \Omega, \quad d_H(z,z) = \phi(z),$$
$$H(z, D\phi(z)) < 0.$$

If we choose $\delta_z > 0$ well,

$$H(x, D\phi(x)) < -\delta_z \quad \text{for } \forall x \in B(z, \delta_z).$$

We set

$$\psi_z(x) := \begin{cases} \max\{\phi(x) + \epsilon_z, d_H(x,z)\} & \text{for } x \in B(z, r_z), \\ d_H(x,z) & \text{otherwise.} \end{cases}$$

We may choose $0 < r_z < \delta_z, \epsilon_z > 0$ such that ψ_z is continuous in Ω. By the properties of ϕ and d_H, we find that ψ_z satisfies

$$H[\psi_z] \le 0 \quad \text{in } \Omega, \quad H[\psi_z] \le -\delta_z \quad \text{in } B(z, r_z).$$

Next we fix $K \subset \overline{\Omega} \setminus \mathcal{A}_D$ such that K is compact. Then $\{B(z, r_z)\}_{z \in \overline{\Omega} \setminus \mathcal{A}_D}$ (Here r_z is chosen as before.) is an open covering of K. Because K is compact, there are $z_1, \dots, z_N \in \overline{\Omega} \setminus \mathcal{A}_D$ such that $K \subset \bigcup_{i=1}^N B(z_i, r_{z_i})$. Set

$$\delta_K := \min_{i=1,\dots,N} r_i, \quad w_K(x) := \sum_{i=1}^N \psi_{z_i}(x).$$

By lemma 5, we verify that w_K, δ_K satisfy claims of this lemma. $\square$

By the definition of d_H, we have the following proposition.

Proposition 8. *Let $u \in C(\overline{\Omega})$ be a solution of $H[u] \le 0$ in Ω. Then*

$$u(x) - u(y) \le d_H(x,y) \text{ for } \forall x, y \in \Omega.$$

Here we recall the viscosity theory due to Barron and Jensen. The definition of viscosity solution in this theory is the following.

Definition 2. *$u \in C(\Omega)$ is a BJ viscosity solution of $H(x, Du(x)) = 0$ in Ω if*

$$H(x,p) = 0 \quad \text{for any } x \in \Omega, \quad \text{any } p \in D^- u(x).$$

If Hamiltonian satisfies (H0)-(H3), the above definition of viscosity solution is equivalent to the usual definition of viscosity solution due to Crandall and Lions. (see Theorem 2.3. in [5]) Using this equivalence, we can verify that the function u_g defined in Theorem 3 is a solution of $H = 0$ in Ω.

A sketch of proof of Theorem 3. We show the continuity of u_g. For $y_n \in \mathcal{A}_D$, $\{g(y_n) + d_H(x, y_n)\}_{n\in\mathbf{N}}$ is uniformly bounded and equi-Lipschitz continuous. Thus we may take a subsequence such that

$$g(y_{n_i}) + d_H(x, y_{n_i}) \to u_g \quad \text{uniformly as} \quad i \to \infty$$

by the Ascoli-Arzela theorem. Consequently u_g is continuous on $\overline{\Omega}$.
We can verify that $u_g = g$ on $\mathcal{A}_D$, noting the definition of d_H and the compatibility condition. Hereafter we will prove u_g is a solution of $H = 0$. We may choose a test function $\phi \in C^1(\Omega)$ such that

$$\begin{aligned} u_g(x) \geq \phi(x) \ \text{ in } \Omega, \quad u_g(\hat{x}) = \phi(\hat{x}), \\ u_g(x) - \phi(x) \geq |x - \hat{x}|^2 \ \text{ in } \Omega. \end{aligned}$$

Choose $r > 0$ such that $B(\hat{x}, r) \subset \Omega$. By the definition of u_g, for $n \in \mathbf{N}$ there exists $y_n \in \mathcal{A}_D$ such that

$$u_g(\hat{x}) + \frac{1}{n} \geq g(y_n) + d_H(\hat{x}, y_n).$$

Set $f_n(x) := g(y_n) + d_H(x, y_n)$ and choose $x_n \in B(\hat{x}, r)$ such that

$$(f_n - \phi)(x_n) = \min_{x\in B(\hat{x},r)} (f_n - \phi)(x)$$

By the way of choice of the test function, $|x_n - \hat{x}|^2 \leq 1/n$. Thus we get $x_n \to \hat{x}$. Moreover by the definition of the Aubry set, f_n is a solution of $H = 0$. In view of the equivalence to a BJ viscosity solution and get

$$H(x_n, D\phi(x_n)) = 0.$$

Sending $n \to \infty$ here, we get

$$H(\hat{x}, D\phi(\hat{x})) = 0.$$

□

Corollary 4 is a direct consequence of Proposition 3 and Theorem 2.

5 State Constraint Problem

Now we consider the state constraint problem.

$$\begin{cases} H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{SC}$$

Here H and Ω satisfy the same assumptions in the case of the Dirichlet problem. As before we define d_H by

$$d_H(x,y) := \sup\{v(x) \in C(\Omega) \mid H[v] \leq 0 \text{ in } \Omega, v(y) \leq 0\}.$$

We have the following lemma.

Lemma 9. *$d(\cdot,y)$ is a solution of* (SC) *on $\overline{\Omega} \setminus \{y\}$ for any $y \in \overline{\Omega}$.*

Proof. We can show easily by the stability in viscosity theory that

$$H(x, Dd_H(x,y)) \leq 0 \quad \text{in } \Omega.$$

What we need to proove is:

$$H(x, Dd_H(x,y)) \geq 0 \quad \text{on } \overline{\Omega} \setminus \{y\}.$$

We will show this by contradiction. Assume that the above statement were not true. Then we may choose the test function $\phi \in C^1(\overline{\Omega})$ such that for $z \in \overline{\Omega} \setminus \{y\}$,

$$d_H(x,y) - \phi(x) \geq 0 \quad \text{on } \overline{\Omega} \setminus \{y\}, \quad d_H(z,y) = \phi(z),$$

$$H(z, D\phi(z)) < 0.$$

Thus we may choose $\delta_z > 0$ such that

$$H(x, D\phi(x)) < 0 \quad \text{in } B(z, \delta_z).$$

Here if $\delta_z > 0$ is enough small, we may assume that $B(z,\delta_z) \subset \overline{\Omega} \setminus \{y\}$. Define $w_z : \overline{\Omega} \to \mathbb{R}$ by

$$w_z(x) := \begin{cases} \max\{\phi(x) + \epsilon_z, d_H(x,y)\} & \text{for } x \in B(z, r_z), \\ d_H(x,y) & \text{otherwise.} \end{cases}$$

If $0 < r_z < \delta_z, \epsilon_z > 0$ is enough small, w_z is continuous on $\overline{\Omega}$. Noting the properties of ϕ and d_H, we find that w_z is a subsolution of $H = 0$. Moreover

$$w_z(y) = d_H(y,y) = 0, \quad w_z(z) = \phi(z) + \epsilon_z > d_H(z,y).$$

But this is the contradiction of the definition of d_H. □

Now we define the Aubry set for the state constraint problem as follows.

Definition 3. *Define the set $\mathcal{A}_{SC}$ as*

$$\mathcal{A}_{SC} := \{y \in \overline{\Omega} \mid d_H(\cdot,y) \textit{is a solution of } (\mathrm{SC})\}.$$

We call $\mathcal{A}_{SC}$ the Aubry set for the state constraint problem.

We show main theorems about the state constraint problem below.

Theorem 10. *Assume $\mathcal{A}_{SC} \neq \emptyset$. Let $g : \mathcal{A}_{SC} \to \mathbb{R}$ be bounded and satisfy the compatibility condition. Define $u_g : \overline{\Omega}$ by*

$$u_g(x) := \inf\{g(y) + d_H(x,y) \mid y \in \mathcal{A}_{SC}\}.$$

Then u_g is continuous on $\overline{\Omega}$ and the unique solution of

$$\begin{cases} H(x, Du(x)) & \leq 0 & \text{in } \Omega, \\ H(x, Du(x)) & \geq 0 & \text{on } \overline{\Omega}, \\ u & = g & \text{on } \mathcal{A}_{SC}. \end{cases}$$

Corollary 11 (Representation formula for the state constraint problem). *Let $g : \mathcal{A}_{SC} \to \mathbb{R}$ be bounded and satisfy the compatibility condition and $u : \overline{\Omega} \to \mathbb{R}$ be a solution of*

$$\begin{cases} H(x, Du(x)) & \leq 0 & \text{in } \Omega, \\ H(x, Du(x)) & \geq 0 & \text{on } \overline{\Omega}, \\ u & = g & \text{on } \mathcal{A}_{SC}. \end{cases}$$

Then $u(x) = u_g(x)$.

The propositions above can be proved in the same way as those for the Dirichlet problem. The following example examines a simple case:
Example 2: Consider the state constraint problem.

$$\begin{cases} |Du(x)| & \leq f(x) & \text{in } (-2,2), \\ |Du(x)| & \geq f(x) & \text{in } [-2,2], \end{cases}$$

where

$$f(x) := \begin{cases} -x-1 & \text{on } [-2,-1), & x+1 & \text{on } [-1,0), \\ -x+1 & \text{on } [0,1), & x-1 & \text{on } [1,2]. \end{cases}$$

Then we obtain

$$\mathcal{A}_{SC} = \{-1\} \cup \{1\},$$

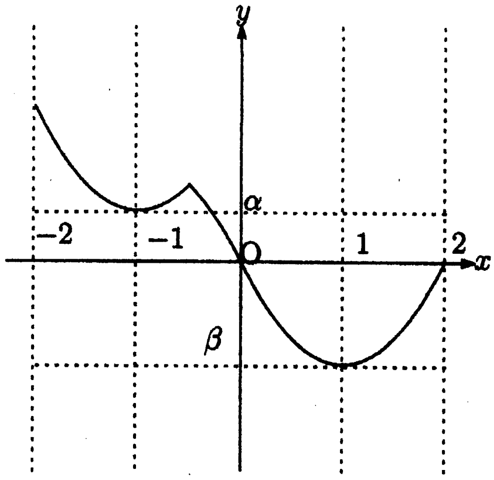
$$d_H(x,-1) = \begin{cases} \frac{1}{2}(x+1)^2 & \text{on } [-2,0), \\ -\frac{1}{2}(x-1)^2+1 & \text{on } [0,1), \\ \frac{1}{2}(x-1)^2+1 & \text{on } [1,2], \end{cases}$$

$$d_H(x,1) = \begin{cases} \frac{1}{2}(x+1)^2+1 & \text{on } [-2,-1), \\ -\frac{1}{2}(x+1)^2+1 & \text{on } [-1,0), \\ \frac{1}{2}(x-1)^2 & \text{on } [0,2]. \end{cases}$$

The solutions of this example are

$$u_{\alpha,\beta}(x) = \min\{d_H(x,-1) + \alpha, d_H(x,1) + \beta\}.$$

Here $|\alpha - \beta| \leq d_H(-1,1) = 1$. The figure of $u_{\alpha,\beta}$ is as follows.



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