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Interior $C^{2,\alpha}$ regularity for fully nonlinear elliptic equations

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1 Introduction

This note is concerned with the $C^{2,\alpha}$ regularity theory for fully nonlinear elliptic equations. First, we briefly present the well established theory for convex equations (see [CC3] and [C] for, respectively, a fully detailed exposition and a survey). Second, we describe a more recent result and method by Cabré and Caffarelli [CC2] on $C^{2,\alpha}$ regularity for a class of nonconvex equations of Isaacs type.

In 1982 Evans [E] and Krylov [K] proved interior $C^{2,\alpha}$ estimates for fully nonlinear elliptic equations $F(D^2u, Du, u, x) = 0$, $x \in \Omega \subset \mathbb{R}^n$, under the assumption that F is either a convex or a concave function of D^2u. These works relied on the Harnack inequality for linear equations in nondivergence form established by Krylov and Safonov in 1979. The Evans–Krylov estimate, together with some extensions due to Caffarelli, Safonov, and Trudinger, led to interior $C^{2,\alpha}$ regularity results for *Bellman's equation*,

$$\sup_{\beta\in\mathcal{B}}\{L_\beta u(x) - f_\beta(x)\} = 0\ , \tag{1.1}$$

associated to a family $L_\beta = a_{ij}^\beta(x)\partial_{ij}$ of linear uniformly elliptic operators (see [CC3], [GT]). Equation (1.1), which is convex in D^2u, is the dynamic programming equation for the optimal cost in some stochastic control problems.

Since then, the validity of interior $C^{2,\alpha}$ estimates for nonconvex fully nonlinear uniformly elliptic equations $F(D^2u) = 0$, in space dimension $n \geq 3$, has been a challenging open question. Examples of such nonconvex equations appear in stochastic control theory and are called *Isaacs equations*. They are of the form

$$\inf_{\gamma\in\mathcal{G}}\sup_{\beta\in\mathcal{B}}\{L_{\beta\gamma}u(x) - f_{\beta\gamma}(x)\} = 0\ , \tag{1.2}$$

where $L_{\beta\gamma} = a_{ij}^{\beta\gamma}(x)\partial_{ij}$ is a family of elliptic operators, all of them with same ellipticity constants. Isaacs equation (1.2) is the dynamic programming equation for the value of some two–player stochastic differential games (see [FS]). At the same time, every uniformly elliptic equation $F(D^2u, x) = 0$ can be written in the form (1.2), for some family $L_{\beta\gamma} = a_{ij}^{\beta\gamma}\partial_{ij}$ of operators with constant coefficients and some functions $f_{\beta\gamma}$ (see Remark 2.1 below).

The best estimates known to be valid for all uniformly elliptic equations $F(D^2u) = 0$ are $C^{1,\alpha}$ and $W^{3,\delta}$ estimates (in particular, also $W^{2,\delta}$), where α and δ are (small) constants that belong to $(0,1)$ and depend on the ellipticity constants of F. To our knowledge, before our work [CC2] described below, no interior $C^{2,\alpha}$ estimates were available for a nonconvex Isaacs operator.

In [CC2] we establish the interior $C^{2,\alpha}$ regularity of viscosity solutions, and in particular the existence of classical solutions, for a class of nonconvex fully nonlinear elliptic equations $F(D^2u, x) = f(x)$. Our assumption is that, for every $x \in B_1 \subset \mathbb{R}^n$, $F(\cdot, x)$ is the minimum of a concave operator and a convex operator of D^2u (where these two operators may depend on the point x). We therefore include the "simplest" nonconvex Isaacs equation

$$F_3(D^2u) := \min\left\{L_1u, \max\{L_2u, L_3u\}\right\} = 0 , \tag{1.3}$$

that we call the 3–operator equation and that motivated our work (see subsection 4.2 below). Here

$$L_ku = a^k_{ij}\partial_{ij}u + c_k , \tag{1.4}$$

where $c_k = L_k0 \in \mathbb{R}$, are three affine elliptic operators with constant coefficients a^k_{ij}. More generally, our results apply to equations of the form

$$F(D^2u) := \min\left\{\inf_{k\in\mathcal{K}} L_ku, \sup_{l\in\mathcal{L}} L_lu\right\} = 0 , \tag{1.5}$$

where $\mathcal{K}$ and $\mathcal{L}$ are arbitrary sets, and L_k, L_l are operators of the form (1.4), all of them with same ellipticity constants and with $\{c_k\}, \{c_l\}$ bounded.

2 Fully nonlinear elliptic operators

Throughout this note and [CC2], we follow the terminology and notation of [CC3]. We say that an operator $F : \mathcal{S} \times \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a domain, is *uniformly elliptic* if there exist constants $0 < \lambda \leq \Lambda$ (called ellipticity constants) such that

$$\lambda\|N\| \leq F(M+N, x) - F(M, x) \leq \Lambda\|N\| \qquad \forall M \in \mathcal{S} \quad \forall N \geq 0 \quad \forall x \in \Omega . \tag{2.1}$$

Here, $\mathcal{S}$ is the space of $n \times n$ symmetric matrices, $N \geq 0$ means that $N \in \mathcal{S}$ is nonnegative definite and, for $M \in \mathcal{S}$, $\|M\| := \sup_{|z|\leq 1} |Mz|$. We say that a constant C is *universal* when it depends only on n, λ and Λ.

The simplest examples of uniformly elliptic operators are the affine operators $Lu = a_{ij}\partial_{ij}u + c$ as in (1.4). The coefficients could also depend on x (i.e., $a_{ij} = a_{ij}(x)$), in which case uniform ellipticity is guaranteed by having uniform lower and upper positive bounds in Ω for the eigenvalues of the symmetric matrices $a_{ij}(x)$.

Another useful class is given by Pucci's extremal operators. Pucci's maximal operator is defined by

$$\mathcal{M}^+(M) = \mathcal{M}^+(M, \lambda, \Lambda) := \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} L_AM = \max_{A\in\mathcal{A}_{\lambda,\Lambda}} L_AM ,$$

where $e_i = e_i(M)$ are the eigenvalues of $M \in \mathcal{S}$, $A \in \mathcal{A}_{\lambda,\Lambda}$ means that A is a symmetric matrix whose eigenvalues belong to $[\lambda, \Lambda]$, and $L_AM = a_{ij}m_{ij} = \text{trace}(AM)$ (see Section 2.2 of [CC3]).

Later we will use the class $\underline{S}$ of subsolutions. We recall that $\underline{S} = \underline{S}(\lambda, \Lambda)$ in B_1 is formed by those continuous functions u in B_1 such that $\mathcal{M}^+(D^2u, \lambda, \Lambda) \geq 0$ in the viscosity sense in B_1 (see Section 2.1 of [CC3] for the definition of the viscosity sense). Similarly, one defines the class $\overline{S}$ of supersolutions through the inequality $\mathcal{M}^-(D^2u) \leq 0$, where $\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$ is Pucci's minimal operator. The class S of viscosity solutions is defined by $S = \underline{S} \cap \overline{S}$.

More generally, given a continuous function f in B_1, the class $\underline{S}(f) = \underline{S}(\lambda, \Lambda, f)$ contains those continuous functions u such that $\mathcal{M}^+(D^2u, \lambda, \Lambda) \geq f(x)$ in the viscosity sense in B_1. Similarly, one defines $\overline{S}(f)$ and $S(f)$.

Finally, we recall that Isaacs equations (1.2) cover all possible fully nonlinear elliptic equations.

Remark 2.1. Let $F(\cdot, x)$ be uniformly elliptic, with ellipticity constants $0 < \lambda \leq \Lambda$. Then, for M and N in $\mathcal{S}$,

$$\begin{aligned} F(M,x) - F(N,x) &\leq \Lambda \|(M-N)^+\| - \lambda \|(M-N)^-\| \\ &\leq \mathcal{M}^+(M-N, \lambda/n, \Lambda) = \max_{A \in \mathcal{A}} L_A(M-N) \ , \end{aligned}$$

where $\mathcal{A} = \mathcal{A}_{\lambda/n,\Lambda}$ (see Chapter 2 of [CC3]). Since there is equality when $N = M$ we deduce that, for all M and x,

$$\begin{aligned} F(M,x) &= \min_{N \in \mathcal{S}} \max_{A \in \mathcal{A}} \{L_A(M-N) + F(N,x)\} \\ &= \min_{N \in \mathcal{S}} \max_{A \in \mathcal{A}} \{L_A M + (F(N,x) - L_A N)\} \ . \end{aligned}$$

This is an operator of Isaacs type (1.2) associated to a family $\{L_A\}$ of linear operators with constant coefficients.

3 Regularity theory for convex equations

For a solution of a second order elliptic equation one expects, in general, to control the second derivatives of the solution by the oscillation of the solution itself. More precisely, the following $C^{2,\alpha}$ and $W^{2,p}$ interior a priori estimates hold. Let u be a solution of a linear uniformly elliptic equation of the form

$$a_{ij}(x)\partial_{ij}u = f(x) \quad \text{in } B_1 \subset \mathbb{R}^n \ .$$

Then we have:

(a) *Schauder's estimates*: if a_{ij} and f belong to $C^\alpha(\overline{B}_1)$, for some $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)})$, where C depends on the ellipticity constants and the $C^\alpha(\overline{B}_1)$-norm of a_{ij}; see Chapter 6 of [GT].

(b) *Calderón-Zygmund estimates*: if $a_{ij} \in C(\overline{B}_1)$ and $f \in L^p(B_1)$, for some $1 < p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and $\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)})$, where C depends on the ellipticity constants and the modulus of continuity of the coefficients a_{ij}; see Chapter 9 of [GT].

These statements should be understood as regularity results for appropriate linear small perturbations of the Laplacian. Indeed, these estimates are proven by regarding the equation $a_{ij}(x)\partial_{ij}u = f(x)$ as

$$a_{ij}(x_0)\partial_{ij}u = [a_{ij}(x_0) - a_{ij}(x)]\,\partial_{ij}u + f(x) \ .$$

One then applies to this equation the corresponding estimates for the constant coefficients operator $a_{ij}(x_0)\partial_{ij}$ (that one can think of as the Laplacian), observing that the factor in the right hand side $a_{ij}(x_0) - a_{ij}(x)$ is small (locally around x_0) in some appropriate norm, due to the regularity assumptions made on a_{ij}. Thus, the key point is to prove $C^{2,\alpha}$ and $W^{2,p}$ estimates for Poisson's equation $\Delta u = f(x)$.

The goal is to extend these regularity theories to fully nonlinear elliptic equations of the form $F(D^2u, x) = f(x)$. The previous discussion shows that one should start considering the case of equations with constant "coefficients" $F(D^2u) = f(x)$ (here, we think of $F(D^2u)$ as being equal to $F(D^2u(x), x_0)$ for a fixed x_0). In fact, the key ideas already appear by considering the simpler equation

$$F(D^2u) = 0 \, .$$

Assume that $F \in C^1$ and that $u \in C^3(\overline{B}_1)$ satisfies $F(D^2u) = 0$. Differentiate this equation with respect to a direction x_k. Writing $u_k = \partial_k u$, we have

$$F_{ij}(D^2u(x))\, \partial_{ij} u_k = 0 \quad \text{in } B_1 \, ,$$

where F_{ij} denotes the first partial derivative of F with respect to its ij-th entry. This can be regarded as a linear equation $Lu_k = 0$ for the function u_k, where $L = a_{ij}(x)\partial_{ij}$ and $a_{ij}(x) = F_{ij}(D^2u(x))$. The ellipticity hypothesis (2.1) leads to the uniform ellipticity of L. Note that a regularity hypothesis on the coefficients $a_{ij}(x)$ would mean to make a regularity assumption on the second derivatives of u —which is our goal and hence we need to avoid. The tool that one uses is the Krylov-Safonov Harnack inequality and its corollary on Hölder continuity of solutions of uniformly elliptic equations in nondivergence form with measurable coefficients (see [CC3]). The key point is that the Krylov-Safonov theory makes no assumption on the regularity of the functions a_{ij}. This theory applied to the equation $Lu_k = 0$ leads to $\|u_k\|_{C^\alpha(\overline{B}_{1/2})} \le C\|u_k\|_{L^\infty(B_1)}$, where $0 < \alpha < 1$ and C are universal constants. Thus, we have the $C^{1,\alpha}$ estimate for u:

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \le C\|u\|_{C^1(\overline{B}_1)}. \tag{3.1}$$

This a priori estimate may be improved in the following way. Let F be uniformly elliptic and $u \in C(B_1)$ be a viscosity solution of $F(D^2u) = 0$ in B_1. Then there exist universal constants $0 < \alpha < 1$ and C such that $u \in C^{1,\alpha}(B_1)$ and

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \le C\{\|u\|_{L^\infty(B_1)} + |F(0)|\} \, .$$

A direct proof of this result, which does not rely on existence results and which applies to viscosity solutions and to nondifferentiable functionals F (recall that Pucci's, Bellman's, and Isaacs' equations are not differentiable in general), was found by the author and Caffarelli in [CC1]. This paper also contains a direct proof of the $C^{1,1}$ regularity of viscosity solutions when the operator F is convex —a case that we discuss next.

When the operator F is concave or convex, Evans [E] and Krylov [K] established in 1982 that classical solutions of $F(D^2u) = 0$ satisfy the $C^{2,\alpha}$ estimate

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C\left\{\|u\|_{L^\infty(B_1)} + |F(0)|\right\} ,$$

where $0 < \alpha < 1$ and C are universal constants. Recall that Pucci's equations are either convex or concave, and that Bellman's equations are convex. Recall that convex elliptic equations $F(D^2u) = 0$ get transformed into concave ones by writing them as $-F(-D^2v) = 0$, where $v = -u$.

The proof of this $C^{2,\alpha}$ estimate is based on a delicate application of the Krylov-Safonov weak Harnack inequality to $C - u_{kk}$, where u_{kk} denotes a pure second derivative of u. Assuming that F is concave and differentiating $F(D^2u) = 0$ twice with respect to x_k, we have

$$\begin{aligned} 0 &= F_{ij}(D^2u(x))\partial_{ij}u_{kk} + F_{ij,rs}(D^2u(x))(\partial_{ij}u_k)(\partial_{rs}u_k) \\ &\leq F_{ij}(D^2u(x))\partial_{ij}u_{kk} \end{aligned}$$

(by the concavity of F), and hence every u_{kk} is a subsolution of a linear equation. Roughly speaking, this allows to control D^2u by above. Once this is acomplished, the ellipticity of equation $F(D^2u) = 0$ controls D^2u by below.

As said, the Evans–Krylov theory establishes interior $C^{2,\alpha}$ estimates for $F(D^2u) = 0$ when F is either convex or concave. More generally, the same proofs of the theory apply when $\{M \in \mathcal{S} : F(M) = 0\}$ is a convex hypersurface in the space $\mathcal{S}$ of $n \times n$ symmetric matrices —that is, when $\{M \in \mathcal{S} : F(M) = 0\}$ is the boundary of a convex open set. Note that this does not hold for our simplest model, the 3–operator (1.3).

Under no convexity or concavity assumption, the work [Cf] by Caffarelli (see also [CC3]) established interior $C^{2,\alpha}$ estimates and $C^{2,\alpha}$ regularity for viscosity solutions of equations of the form $F(D^2u, x) = f(x)$ assuming that the dependence of F and f on x is C^α and that, for every fixed x_0, the Dirichlet problem for $F(D^2u(x), x_0) = f(x_0)$ has classical solutions and interior $C^{2,\bar{\alpha}}$ estimates, where $0 < \alpha < \bar{\alpha}$. [Cf] also establishes a similar $W^{2,p}$ regularity result. These are fully nonlinear extensions of the linear Schauder and Calderón-Zygmund theories described at the beginning of this section. By means of Caffarelli's theory, we can reduce our study to operators $F(M, x) = F(M)$ with constant coefficients —such as (1.3) and (1.5) defined by operators of the form (1.4).

4 Regularity for a class of nonconvex equations

By the comments in the previous paragraph, regularity for equations $F(D^2u, x) = f(x)$ follows once it has been established for those of the form $F(D^2u) = c$, with c a constant, that we can write as $F(D^2u) = 0$ after subtracting a constant to F.

4.1 The class of operators and the main results

In [CC2], we consider the class of operators F of the following form:

$$\begin{cases} F(M) = \min\{F^\cap(M), F^\cup(M)\} \text{ for all } M \in \mathcal{S}, \\ F(0) = 0, \; F^\cap \text{ and } F^\cup \text{ are uniformly elliptic,} \\ F^\cap \text{ is concave and } F^\cup \text{ is convex.} \end{cases} \tag{4.1}$$

Since (2.1) holds for both $F^\cap$ and $F^\cup$, it also holds for F. Hence, F is uniformly elliptic. We assume $F(0) = 0$ only for convenience. Indeed, after an appropriate translation in $\mathcal{S}$ (which amounts to subtract a quadratic polynomial to u), every operator F can be assumed to satisfy $F(0) = 0$ (see Remark 1 in Section 6.2 of [CC3]). Moreover, the concavity of $F^\cap$ and the convexity of $F^\cup$ are preserved under translations in $\mathcal{S}$.

We do not require $F^\cap$ and $F^\cup$ to be of class C^1. In this way, our results apply to the equations of Isaacs type described above. Note also that the class (4.1) of operators F includes all concave operators. Indeed, if $F^\cap$ is concave then there is an affine, uniformly elliptic operator L with constant coefficients such that $F^\cap \le L$ in $\mathcal{S}$. Take then $F^\cup = L$, so that $F = F^\cap$.

Our main result is the following interior $C^{2,\alpha}$ a priori estimate for classical solutions of $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^n$, where $0 < \alpha < 1$ is a (small) exponent depending only on n and on the ellipticity constants λ and Λ.

Theorem 4.1 **([CC2]).** *Let $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in $B_1 \subset \mathbb{R}^n$, where F is of the form* (4.1). *Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C\|u\|_{L^\infty(B_1)} , \tag{4.2}$$

where $0 < \alpha < 1$ and C are universal constants.

The proof of Theorem 4.1 requires $u \in C^2$ and does not apply to viscosity solutions. We need $u \in C^2$ to make sense of Proposition 4.4 below, which states that $F^\cup(D^2u)$ is in the class of viscosity subsolutions. It would be interesting to adapt the proof to viscosity solutions u —for instance, by approximating $F^\cup(D^2u)$ in the spirit of the regularity theory for convex operators developed by the author and Caffarelli in [CC1] (see also Section 6.2 of [CC3]).

Recall that the Dirichlet problem associated to every uniformly elliptic operator F always admits a unique viscosity solution. However, the $C^{2,\alpha}$ estimate of Theorem 4.1 requires the solution to be C^2. Hence, to complete our theory we need to show that $F(D^2u) = 0$ admits C^2 solutions whenever F is of the form (4.1). This is given by the following:

Theorem 4.2 **([CC2]).** *Let F be of the form* (4.1). *Then, there exists a universal constant $\overline{\alpha} \in (0,1)$ such that for every $\alpha \in (0,\overline{\alpha})$, $f \in C^\alpha(\overline{B}_1)$ and $\varphi \in C(\partial B_1)$, the problem*

$$\begin{cases} F(D^2u) = f(x) & \text{in } B_1 \\ u = \varphi(x) & \text{on } \partial B_1 \end{cases}$$

admits a unique solution $u \in C^{2,\alpha}(B_1) \cap C(\overline{B}_1)$. Moreover, we have that

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C_\alpha\Big\{\|f\|_{C^\alpha(\overline{B}_1)} + \|\varphi\|_{L^\infty(\partial B_1)}\Big\} ,$$

for some constant C_α depending only on n, λ, Λ and α.

The existence of classical solutions, Theorem 4.2, and the a priori estimate of Theorem 4.1 lead immediately to the $C^{2,\alpha}$ regularity of every viscosity solution of $F(D^2u) = f(x) \in C^\alpha$, when $0 < \alpha < \overline{\alpha}$. Furthermore, we also have $W^{2,p}$ regularity for $n \le p < \infty$ in case that $f \in L^p$. The precise statement is the following:

Corollary 4.3 **([CC2]).** *Let $u \in C(B_1)$ be a viscosity solution of $F(D^2u) = f(x)$ in B_1, where f is a continuous function in B_1 and F is an operator of the form* (4.1). *Then:*
(i) If $f \in C^\alpha(B_1)$ for some $0 < \alpha < \overline{\alpha}$, where $\overline{\alpha} \in (0,1)$ is a universal constant, then $u \in C^{2,\alpha}(B_1)$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C_\alpha\{\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_{3/4})}\} ,$$

for some constant C_α depending only on n, λ, Λ and α.
(ii) If $f \in L^p(B_1)$ and $n \le p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \le C_p\{\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\} ,$$

for some constant C_p depending only on n, λ, Λ and p.

4.2 Motivation: the 2- and 3-operators

A first hint towards the validity of second derivative estimates for our class of operators came up when we realized that, for the 3–operator (1.3), $H^2 = W^{2,2}$ estimates followed easily from some variational tools used by Brezis and Evans in [BE]. Let us explain these interesting ideas, even that we do not use them in [CC2]. Paper [BE] (written in 1979, that is, before the development of the Evans–Krylov theory) established $C^{2,\alpha}$ estimates for the 2–operator convex equation

$$\max\{L_1 u - f_1(x), L_2 u - f_2(x)\} = 0 \ . \tag{4.3}$$

For simplicity let us take $L_k = a_{ij}^k \partial_{ij}$ to have constant coefficients. The first step in [BE] is to obtain an H^2 estimate using Sobolevsky's inequality, which states that

$$\|u\|^2_{H^2(B_1)} \le C\left\{\int_{B_1} L_1 u\, L_2 u \, dx + \|u\|^2_{L^2(B_1)}\right\} \tag{4.4}$$

for all $u \in H^2(B_1) \cap H^1_0(B_1)$, where C is a universal constant. Then, for a sufficiently nice solution u of (4.3) in B_1, we have $(L_1 u - f_1)(L_2 u - f_2) \equiv 0$ and hence $L_1 u L_2 u = f_1 L_2 u + f_2 L_1 u - f_1 f_2$. Then, if $u \equiv 0$ on ∂B_1, the previous equality, (4.4) and Cauchy–Schwarz lead to $\|u\|_{H^2} \le C\{\|u\|_{L^2} + \|f_1\|_{L^2} + \|f_2\|_{L^2}\}$.

We realized that the same idea works for the 3–operator equation

$$\min\{L_1 u, \max\{L_2 u, L_3 u\}\} = f(x) \ , \tag{4.5}$$

among other equations. Indeed, we have $L_2 u - f \le \max\{L_2 u - f, L_3 u - f\}$ and, since $L_1 u - f \ge 0$, we deduce $(L_1 u - f)(L_2 u - f) \le (L_1 u - f)\max\{L_2 u - f, L_3 u - f\} \equiv 0$. Hence $L_1 u L_2 u \le f(L_1 u + L_2 u) - f^2$, that combined with Sobolevsky's inequality (4.4) leads to $\|u\|_{H^2} \le C\{\|u\|_{L^2} + \|f\|_{L^2}\}$ for every solution of (4.5) with $u \equiv 0$ on ∂B_1.

We do not use this tool in [CC2]. Instead, the proof of Theorem 4.1 is based in the following fact of nonvariational nature. We observe that if $F(D^2 u) = 0$ in B_1 and F is of the form (4.1), then $F^{\cup}(D^2 u)$ belongs to the class $\underline{S}$ of subsolutions in B_1.

Let us prove the previous assertion in the easiest situation, that is, when u is a classical solution of (1.3):

$$F_3(D^2 u) = \min\{\Delta u, \max\{L_2 u, L_3 u\}\} = 0 \quad \text{in } B_1 \ ,$$

and L_k are second order operators with constant coefficients and where we have taken $L_1 = \Delta$. Then, it is elementary to show that the continuous function

$$F^{\cup}(D^2 u) := \max\{L_2 u, L_3 u\}$$

is subharmonic in B_1. Indeed, note first that $F^{\cup}(D^2 u) \ge 0$ in B_1. Hence, it suffices to show that $F^{\cup}(D^2 u)$ is subharmonic in the open set $\Omega = \{F^{\cup}(D^2 u) > 0\}$. But $\Delta u = 0$ in Ω and, therefore, $L_2 u$ and $L_3 u$ are also harmonic in Ω. It follows that $F^{\cup}(D^2 u) = \max\{L_2 u, L_3 u\}$ is subharmonic in Ω.

4.3 Main lemmas and ideas of proofs

The proof of Theorem 4.1 uses two main ingredients. The first one is stated as follows.

Proposition 4.4. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form* (4.1). *Then*

$$0 \le F^{\cup}(D^2u) \in \underline{S}(\lambda/n, \Lambda) \quad in\ B_1 .$$

It is remarkable that this leads immediately to interior $W^{2,p}$ estimates for every $p < \infty$. Indeed, since $0 \le F^{\cup}(D^2u)$ is a subsolution in B_1, a local version of the ABP estimate gives an interior L^∞ bound for $F^{\cup}(D^2u)$. In particular, $F^{\cup}(D^2u) \in L^p$ in the interior, for all $p < \infty$. Then, since $F^{\cup}$ is a convex operator, the fully nonlinear Calderón–Zygmund theory proved by Caffarelli [Cf] leads to $W^{2,p}$ estimates for u (for all $p < \infty$).

The second important ingredient in the proof of Theorem 4.1 is the following. It applies to more general equations than those of the form (4.1). Its statement assumes that u is a solution of $G(D^2u) = 0$ in B_1, where G is uniformly elliptic and $G(0) = 0$, and that H is a uniformly elliptic operator with $C^{2,\alpha}$ estimates. The conclusion is that if G and H coincide in a ball in $\mathcal{S}$ centered at 0 of sufficiently large radius compared to $\|u\|_{L^\infty(B_1)}$, then $H(D^2u) = 0$ in the smaller ball $B_{1/2}$.

Applied to our class of operators, the results reads as follows:

Proposition 4.5. *Let $u \in C^2(B_1)$ satisfy $F(D^2u) = 0$ in B_1, where F is of the form* (4.1). *Then, there exists a universal constant $c_f > 0$ such that*

$$if\ F^{\cup}(0) > c_f\|u\|_{L^\infty(B_1)}\ then\ F^{\cap}(D^2u) = 0 \quad in\ B_{1/2} .$$

Recall that, by assumption, $F(0) = \min(F^{\cap}(0), F^{\cup}(0)) = 0$. The previous proposition gives that if $F^{\cup}(0)$ is positive and too large compared to $\|u\|_{L^\infty(B_1)}$, then we have $F^{\cap}(D^2u) = 0$ in $B_{1/2}$ —that is, only $F^{\cap}$ acts on D^2u in the smaller ball $B_{1/2}$, in which case regularity is automatic since $F^{\cap}$ is concave.

After translations in $\mathcal{S}$, this result allows to control $F^{\cup}(D^2P)$ (and not only $F^{\cup}(0)$) for every quadratic polynomial P with $F(D^2P) = 0$ —unless $F^{\cap}(D^2u) = 0$ in $B_{1/2}$. This will be crucial when deriving $C^{2,\alpha}$ estimates through approximations of u by quadratic polynomials P, that we describe next.

The proof of Theorem 4.1 uses the two previous propositions and the $C^{2,\alpha}$ iteration scheme developed in [Cf]. The goal is to approximate u by polynomials of degree two in $L^\infty(B_{\mu^k}(0))$–norm, where $0 < \mu < 1$, and to do it better and better as k increases. For this, we set $S_0 := \sup_{B_{1/2}} F^{\cup}(D^2u)$ and we distinguish two cases. The first case is when most points x, in measure, have $F^{\cup}(D^2u(x))$ close to S_0. Then we can approximate u by a solution of $F^{\cup}(D^2v) = S_0$, which is $C^{2,\alpha}$ at the origin since $F^{\cup}$ is convex. In the other case, the weak Harnack inequality of Krylov–Safonov, applied to the supersolution $S_0 - F^{\cup}(D^2u) \ge 0$, forces the supremum of $F^{\cup}(D^2u)$ in a smaller ball to decrease by a factor (with respect to S_0). Heuristically, if this second case happens "often" as $k \to \infty$, then $F^{\cup}(D^2u)$ is concentrating near $\{F^{\cup} = 0\}$, and hence u can be approximated by the quadratic part of a solution of $F^{\cup}(D^2v) = 0$.

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