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A note on asymptotic solutions of Hamilton-Jacobi equations

Yasuhiro FUJITA (藤田 安啓)
University of Toyama, 930-8555 Toyama, Japan
(富山大学 理学部)
yfujita@sci.toyama-u.ac.jp

This is a survey of my result [10]. In this talk, we consider the viscosity solutions of the Cauchy problem

$$u_t + \alpha x \cdot Du + H(Du) = f \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1}$$

$$u|_{t=0} = \phi \quad \text{for } x \in \mathbb{R}^N, \tag{2}$$

where α is a positive constant. Our goal is to investigate convergence rates of $u(t,x)$ to the stationary state as $t \to \infty$. We assume the following:

(A1) $H, f, \phi \in C(\mathbb{R}^N)$.

(A2) H is convex on $\mathbb{R}^N$.

(A3) $\displaystyle\lim_{|x|\to\infty} \frac{H(x)}{|x|} = \infty.$

We denote by L the convex conjugate of H defined by

$$L(x) = \sup\left\{ z \cdot x - H(z) \,|\, z \in \mathbb{R}^N \right\}.$$

Then, L satisfies (A2) and (A3) in place of H. Furthermore, we assume that there is a convex function ℓ on $\mathbb{R}^N$ satisfying

(A4) $\displaystyle\lim_{|x|\to\infty} (L(x) - \ell(x)) = \infty.$

(A5) $\inf\left\{ f(x) + \ell(-\alpha x) \,|\, x \in \mathbb{R}^N \right\} > -\infty.$

(A6) $\inf\left\{\phi(x)+\dfrac{1}{\alpha}\ell(-\alpha x)\,|\,x\in\mathbb{R}^N\right\}>-\infty.$

Now, we introduce several notations.

$$c:=\min\left\{f(x)+L(-\alpha x)\,|\,x\in\mathbb{R}^N\right\},\qquad f_c(x):=f(x)-c,$$

$$Z:=\{x\in\mathbb{R}^N\,|\,f_c(x)+L(-\alpha x)=0\},$$
$$\mathcal{C}(x,T)=\{X\in \mathrm{AC}([0,T])\,|\,X(0)=x\},$$
$$\mathcal{C}(x,y,T)=\{X\in\mathcal{C}(x,T)\,|\,X(T)=y\},$$

$$d(x,y)=\inf\left\{\int_0^T[f_c(X(t))+L(-\alpha X(t)-\dot X(t))]\,dt\,\middle|\,T>0,\ X\in\mathcal{C}(x,y,T)\right\},$$

$$\psi(x)=\inf\left\{\int_0^T[f_c(X(t))+L(-\alpha X(t)-\dot X(t))]\,dt+\phi(X(T))\,\middle|\,T>0,\ X\in\mathcal{C}(x,T)\right\},$$

$$v(x)=\min_{z\in Z}\Big(d(x,z)+\psi(z)\Big).$$

The following propositions were proved in [11] (see also the paper of Professor Hitoshi Ishii in this volume).

Proposition 1. There is the unique viscosity solution $u\in C(\mathbb{R}^N\times[0,\infty))$ of (1)-(2) satisfying for any $T>0$

$$\lim_{r\to\infty}\inf\left\{u(x,t)+\frac{1}{\alpha}L(-\alpha x)\,|\,(x,t)\in(\mathbb{R}^N\setminus B(0,r))\times[0,T)\right\}=\infty, \tag{3}$$

where $B(a,r)=\{x\in\mathbb{R}^N\,|\,|x-a|\le r\}$ for $a\in\mathbb{R}^N$ and $r>0$. □

Proposition 2. For the unique viscosity solution $u\in C(\mathbb{R}^N\times[0,\infty))$ of (1)-(2) satisfying (3), we have

$$\lim_{t\to\infty}\max_{x\in B(0,R)}|u(x,t)-(ct+v(x))|=0\quad\text{for } R>0. \tag{4}$$ □

Note that by the stability theorem of viscosity solutions, v is a viscosity solution of the equation

$$c+\alpha x\cdot Dv+H(Dv)=f\qquad\text{for } x\in\mathbb{R}^N. \tag{5}$$

Next, we consider the convergence rate of (4). First, we consider the case such that the convergence rate of (4) is faster than $e^{-\theta t}$ for some constant $\theta > 0$. Besides (A1) $\sim$ (A6), we assume the following:

(A7) $H \geq 0$ in $\mathbb{R}^N$ with $H(0) = 0$.

(A8) $f \geq 0$ in $\mathbb{R}^N$ with $f(0) = 0$, and there exists a constant $\theta > 0$ such that

$$\theta \int_0^\infty f(xe^{-\alpha t})\, dt \leq f(x) \quad \text{for } x \in \mathbb{R}^N.$$

(A9) There exists a constant $m > 0$ such that

$$0 \leq \phi(x) \leq m\, w(x) \quad \text{for } x \in \mathbb{R}^N,$$

where $w \in C(\mathbb{R}^N)$ is a subsolution of

$$\alpha x \cdot Dv(x) + H(Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N,$$

and satisfies the following inequality for a constant $\lambda > 0$:

$$0 \leq \lambda w(x) \leq f(x) \quad \text{for } x \in \mathbb{R}^N.$$

Example 1. Let f be a nonnegative and convex function on $\mathbb{R}^N$ with $f(0) = 0$. Then, f satisfies (A8) for $\theta = \alpha$.

Example 2. Let G be a nonnegative and convex function on $\mathbb{R}^N$ with $G(0) = 0$. Assume that there exist constants δ_1, δ_2 $(0 < \delta_1 < \delta_2)$ and $f \in C(\mathbb{R}^N)$ such that

$$\delta_1 G(x) \leq f(x) \leq \delta_2 G(x) \quad \text{for } x \in \mathbb{R}^N.$$

Then, f satisfies (A8) for $\theta = \alpha\delta_1/\delta_2$.

Example 3. Assume that there are constants $p \in (1, \infty)$ and $a \in (0, 1)$ such that

$$a\Big(\alpha|x|^p + H(|x|^{p-2}x)\Big) \leq f(x) \quad \text{for } x \in \mathbb{R}^N.$$

Then, as a function $\phi \in C(\mathbb{R}^N)$ of (A9), we can take any one satisfying

$$0 \leq \phi(x) \leq k|x|^p \quad \text{for } x \in \mathbb{R}^N,$$

where $k > 0$ is a constant.

Theorem 3. Assume (A1)-(A9). Let $u \in C(\mathbb{R}^N \times [0,\infty))$ be the unique viscosity solution of (1)-(2) satisfying (3). Then, we have $c = 0$, $Z \ni 0$, and,

$$-v(x)e^{-\theta t} \leq u(x,t) - v(x) \leq mv(x)e^{-\theta t} \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,\infty). \tag{6}$$

Finally, we give an example, which shows that there is the case such that the convergence rate of (4) is not faster than t^{-1}.

Example 4. Let $H(x) = |x|^p/p$ for some constant $p > 1$. Then, $L(x) = |x|^q/q$, where $(1/p) + (1/q) = 1$. For $r > 0$, let

$$f(x) = -\frac{\alpha^q}{q} \min\{|x|^q, r^q\} \quad \text{for } x \in \mathbb{R}^N.$$

Let $\phi \in C(\mathbb{R}^N)$ be a function satisfying $\phi(x) \geq 0$ for $x \in \mathbb{R}^N$. Then, we have $c = 0$, $Z = B(0,r)$, and,

$$\frac{1}{\alpha}L(-\alpha x)(t+1)^{-1} \leq u(x,t) - v(x) \quad \text{for } (x,t) \in Z \times [0,\infty). \tag{7}$$

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