数理解析研究所講究録
1482 巻 2006 年 118-152

Asymptotic theory of tsunami waves: geometrical aspects and the generalized Maslov representation.

Sergey Dobrokhotov* Sergey Sekerzh-Zenkovich† Brunello Tirozzi‡
Timur Tudorovskiy §

April 7, 2006

Abstract

We suggest a new asymptotic representation for the solutions to the 2-D wave equation with variable velocity and localized initial data. This representation is a generalization of the Maslov canonical operator and gives the formulas for the relationship between initial localized perturbations and wave profiles near the wave fronts including the neighborhood of backtracking (focal or turning) and self intersection points. We apply these formulas to the problem of a propagation of tsunami waves in the frame of so-called "piston model". Finally we suggest a fast asymptotically-numerical algorithm for simulation of tsunami wave over nonuniform bottom. Different scenarios of the distribution of the waves are considered, the wave profiles of the front are obtained in connection with the different shapes of the source and with the diverse rays generating the fronts. It is possible to use the suggested algorithm to predict in real time the zones of the beaches where the amplitude of the tsunami wave has dangerous high values. The paper concentrates mainly on the final formulas and geometrical aspects of the proposed asymptotic theory.

1 Introduction

The traditional calculations of the diffusion of the tsunami waves are done by solving the linear shallow water equations in the framework of the so called "piston model", which assumes that the source of the perturbation of the wave is given by an instantaneous vertical velocity of a certain region of the bottom of the ocean. The corresponding mathematical problem is the search of the solution of the two-dimensional wave equation with variable velocity and localized initial conditions:

$$\frac{\partial^2\eta}{\partial t^2} = <\nabla, C(x)\nabla>\eta, \tag{1.1}$$

$$\eta|_{t=0} = V(\frac{x}{\mu}), \quad \eta_t|_{t=0} = 0. \tag{1.2}$$

*Institute for Problems in Mechanics, RAS, Moscow; E-mail: dobr@ipmnet.ru
†Institute for Problems in Mechanics, RAS, Moscow; E-mail: seker@ipmnet.ru
‡Department of Physics, University "La Sapienza", Rome; E-mail: b.tirozzi@libero.it
§Institute for Problems in Mechanics, RAS, Moscow; E-mail: timtudor@ipmnet.ru

Here $\mu \ll 1$ and the function $V(y)$ decays fast as $|y| \to \infty$. It is usual to solve directly with numerical methods this equation for computing the tsunami in basins with non uniform bottom. In this way the position of the front is rather well defined but there are errors in the estimate of the amplitudes ([2]), this could be the cause of the not very high effectiveness of the tsunami alarm system ([1]). In particular, in order to obtain a good accuracy in a neighborhood of a caustic, it is necessary to spend a large amount of computer time and this makes almost impossible to use the direct numerical solution of the wave equation for real time simulation of the propagation of tsunami. From our point of view the existing methods of computing the wave field for the case of the ocean with non uniform bottom are good only for a qualitative description of the distribution of the wave but satisfactory quantitative calculations are still missing. The mathematical complications encountered in solving the problem are connected with the metamorphosis of the solution: at the initial time the wave is concentrated in a point and after sometime in a neighborhood of a curve (i.e. the front of the wave). The problem is essentially two dimensional with the effect, typical of the multi dimensional wave equation with variable velocity, of the intersection of the characteristics. These arguments for the problem of localized initial conditions have been treated with accuracy in the paper [15] but the final formulas, based on the representation of the asymptotic [29] for the equations with constant coefficients, are not very effective from the point of view of the real applications. The main result of this paper consists in the derivation from the mentioned formulas of essentially simple asymptotic equations for the wave amplitude ((4.4), (4.7), (4.14)) generated by some localized source. It is necessary to emphasize that these formulas refer only to the well known wave theory and geometrical optic and that can be implemented in a computer in a relatively easy way by means of programs of the type of Mathematica and Maple. In this paper we concentrate mainly on the construction of the geometrical and topological concepts (like the wave front, the Morse and Maslov index etc) playing a fundamental role in the asymptotic behavior. As we mentioned above our final results are based on the relatively simple piston model. We observe that until now, despite its simple formulation and the numerous publications about it, no complete and accurate asymptotic solutions of this model have been published. On the contrary, we show that many features, not only qualitative but also quantitative, of the tsunami waves can be explained by means of the piston model without any useless complications. We briefly describe the plane of the work. In Sect. 2 we give a detailed description of the linear case, in Sect. 3, starting from the example of the problem with constant coefficients, we justify the utilization of the wave equation for analyzing the tsunami waves. In Sec. 4 we give the asymptotic formulas for the case when the front passes through a focal point and the self-intersections of the wave front appear. In Sec. 5 the topological and geometrical concepts, on which the formulas ((4.4), (4.7), (4.14)) are based, are shown. The global uniform asymptotic solution (6.9)-(6.10) to problem (1.1)-(1.2) based on the generalization of the Maslov canonical operator (6.1) (and realized in different situations in the various forms of the equations (4.4), (4.7), (4.14)) is presented in Sec. 6. The proofs of the main theorems given in this paper are omitted, they will be presented in a forthcoming paper.

2 The main equations and a simple example: the wave field in the case of constant bottom

2.1 Some notations

Let us introduce the notations used in this paper. A two dimensional vector can be written with capital or small letters $X = (X_1, X_2)$ or $x = (x_1, x_2)$. The vector can be written also as a column vector $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Two dimensional vectors X and Y can form a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ with four rows. The real scalar product between two vectors X and Y, with real components, is indicated by $< X, Y >$, the complex scalar product among bi-dimensional vectors Z, W, with complex components, is written as $< Z, W >_c$, the two by two matrix generated by two bi-dimensional vectors X, Y is written as (X, Y) where in the first column there are the components of the vector X and in the second column those of the vector Y; the transposed matrix of C is denoted by ${}^tC$.

2.2 The main equations

Let us remind the statements of problems used in tsunami wave problems as well as in general linear water wave theory; see e.g. ([2]-[14]) where it is possible to find a more complete bibliography.

Let us assume that the bottom of the basin is moving $H = H_0(x) - H_1(x,t)$. We assume also that the perturbation $H_1(x,t)$ is small with respect to H_0 $|H_1| << H_0(x)$, and that H_1 is localized in a neighborhood of some given point x_0. If L is the dimension of the region where the wave phenomena is studied, and l is the dimension of the perturbed region, then our hypothesis implies that $l << L$. Another assumption is that the bottom "changes slowly", i.e. that $\nabla H_0 \sim \mu$, where μ is some small ("adiabatic parameter"). We discuss below its meaning. Introducing the scaled variables $x' = \frac{x}{L}$, then $H = H_0(x') - H_1(\frac{x'}{\mu}, t)$, where $\mu = \frac{l}{L} << 1$.

The equation for the velocity potential Φ in the water $-H \le z \le \eta$, where $\eta(x,t)$ is the sea elevation in the linear approximation, has the form, in dimensional variables,:

$$\Delta\Phi = 0, \tag{2.1}$$

$$\eta_t - \frac{\partial \Phi}{\partial z}|_{z=0} = 0, \quad \Phi_t + g\eta|_{z=0} = 0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial n} \equiv \frac{\partial \Phi}{\partial z} + < \nabla H, \nabla \Phi >= v(x,t)|_{z=-H}. \tag{2.3}$$

where $v(x,t)$ is the normal component of the velocity of the motion of the bottom in the point x. The velocity v can be expressed by means of the derivative $\frac{\partial H_1}{\partial t}$ by : $\frac{\partial H_1}{\partial t}/\sqrt{(\nabla H)^2 + 1}$, since v is the projection of the velocity on the vector $\frac{1}{\sqrt{(\nabla H)^2+1}}{}^t(\nabla H, 1)$ normal to the surface $z = -H$. If we consider ∇H_0 to be small (because of the slow variation of the bottom relief), and that also ∇H_1 is small (because of the small amplitude H_1), then we have $v = \frac{\partial H_1}{\partial t}$.

3 A simple example: the wave field in the case of constant bottom

3.1 The solution in the form of the Fourier transform

Let us begin considering the system (2.1)-(2.3) in the case of constant bottom. In this case the velocity potential and its derivatives are zero for $t = 0$. We make the Fourier transform of the system (2.1)-(2.3) with respect to the variables x_1, x_2. The dual variables will be denoted with p_1, p_2 and the Fourier transform of the corresponding function will be considered as a "wave". Then (2.1)-(2.3) get the form

$$\widetilde{\Phi}_{zz} - p^2\widetilde{\Phi} = 0, \tag{3.1}$$

$$\widetilde{\eta}_t - \frac{\partial\widetilde{\Phi}}{\partial z}|_{z=0} = 0, \tag{3.2}$$

$$(\widetilde{\Phi}_t + g\widetilde{\eta})|_{z=0} = 0, \tag{3.3}$$

$$\frac{\partial\widetilde{\Phi}}{\partial z}|_{z=-H} = \tilde{v} \equiv \frac{\partial\tilde{H}_1}{\partial t}. \tag{3.4}$$

Solving ((3.1))-((3.4)), we find

$$\widetilde{\Phi} = \frac{\operatorname{ch}((z+H)|p|)}{\operatorname{ch} H|p|}\tilde{\varphi} + \frac{\operatorname{sh}(z|p|)}{|p|\operatorname{ch}(H|p|)}\frac{\partial\widetilde{H_1}}{\partial t} \tag{3.5}$$

and

$$\widetilde{\Phi}_z|_{z=0} = |p|\tanh(H|p|)\tilde{\varphi} + \frac{1}{\operatorname{ch} H|p|}\frac{\partial\widetilde{H_1}}{\partial t} \tag{3.6}$$

Thus the equations ((3.2))-((3.3)) take the form

$$\begin{gathered} \frac{\partial\tilde{\eta}}{\partial t} - |p|\tanh(H|p|)\tilde{\varphi} - \frac{1}{\operatorname{ch}(H|p|)}\frac{\partial\widetilde{H_1}}{\partial t} = 0 \\ \frac{\partial\tilde{\varphi}}{\partial t} + g\tilde{\eta} = 0 \end{gathered} \tag{3.7}$$

Where $\tilde{\varphi} = \widetilde{\Phi}_t|_{z=0}$, and we have the initial conditions $t = 0$

$$\tilde{\varphi}|_{t=0} = 0, \qquad \tilde{\varphi}_t|_{t=0} = 0 \Longleftrightarrow \tilde{\eta}|_{t=0} = 0. \tag{3.8}$$

These conditions define the so called Cauchy-Poisson problem for the system (3.7). They are compatible with the perturbation of the bottom only if we suppose that the earthquake starts at a time different from zero. So we assume that the bottom has an "instantaneous" movement at a small time $t = \varepsilon$:

$$H_1(x,t) = \theta(t-\varepsilon)V(x), \tag{3.9}$$

then we send ε to zero at the end of the calculation; the smooth function $V(x)$ decays rapidly at infinity.

Differentiating the first equation in (3.7) with respect to t and substituting $\frac{\partial\tilde{\varphi}}{\partial t}$ with $-g\tilde{\eta}$ we get the equation for $\tilde{\eta}$:

$$\tilde{\eta}_{tt} + \mathcal{L}\tilde{\eta} - \frac{1}{\operatorname{ch}(H|p|)}\frac{\partial^2\widetilde{H_1}}{\partial t^2} = 0, \qquad \mathcal{L} = g|p|\tanh(H|p|). \tag{3.10}$$

Differentiating the second equation of the system (3.7) with respect to t and substituting the derivative η_t with the expression of the first equation and considering the condition that the source is active at the moment $t = \varepsilon > 0$, we get $\varphi_{tt}|_{t=0} = -g|p|\tanh(H|p|)\tilde{\varphi}|_{t=0} = 0$ and the initial condition for (3.10)

$$\eta_{t=0} = 0 \quad \eta_t|_{t=0} = 0. \tag{3.11}$$

It is easy to find the solution $\widetilde{G}$ of the homogeneous equation associated with (3.10):

$$\widetilde{G}_{tt} + \mathcal{L}(p,H)\widetilde{G} = 0, \quad \widetilde{G}|_{t=\tau} = 0, \quad \widetilde{G}_t|_{t=\tau} = 1,$$
$$\widetilde{G}(t,\tau,p) = \frac{e^{i\sqrt{\mathcal{L}}(t-\tau)} - e^{-i\sqrt{\mathcal{L}}(t-\tau)}}{2i\sqrt{\mathcal{L}}} = \frac{\sin\sqrt{\mathcal{L}}(t-\tau)}{\sqrt{\mathcal{L}}}.$$

In this way the solution of the non homogeneous equation (3.10) is

$$\tilde{\eta} = \int_0^t \widetilde{G}(t,\tau,p)\frac{1}{\cosh(H|p|)}\frac{\partial^2 \widetilde{H}_1(\tau,p)}{\partial t^2}d\tau.$$

The inverse Fourier transform of the function $\tilde{\eta}$ gives the elevation of the free surface. Under our assumption of instantaneous motion at time ε we have $\frac{\partial^2 \widetilde{H}_1(\tau,p)}{\partial t^2} = \delta'(t-\varepsilon)\widetilde{V}$ and so:

$$\tilde{\eta} = \int_0^t \widetilde{G}(t,\tau,p)\frac{1}{\cosh(H|p|)}\frac{\partial^2 \widetilde{H}_1(\tau,p)}{\partial t^2}d\tau = \frac{\widetilde{V}}{\cosh H|p|}\int_0^t \frac{\sin\sqrt{\mathcal{L}}(t-\tau)}{\sqrt{\mathcal{L}}}\delta'(\tau-\varepsilon)d\tau =$$
$$-\frac{\widetilde{V}}{\cosh H|p|}\frac{\partial}{\partial\tau}\left(\frac{\sin\sqrt{\mathcal{L}}(t-\tau)}{\sqrt{\mathcal{L}}}\right)\Big|_{\tau=\varepsilon} = \frac{\widetilde{V}}{\cosh H|p|}\cos\sqrt{\mathcal{L}}(t-\varepsilon).$$

We send now ε to zero so we get the function $\tilde{\eta} = \frac{\widetilde{V}}{\cosh(H|p|)}\cos\sqrt{\mathcal{L}}t$. It is evident that $\tilde{\eta}$ is the solution of the equation (3.10) with the following Cauchy conditions

$$\tilde{\eta}|_{t=0} \equiv \frac{\widetilde{V}}{\cosh(H|p|)}, \quad \tilde{\eta}'|_{t=0} = 0. \tag{3.12}$$

We shall discuss the relevance of such initial conditions for the function η in the next section.

3.2 The solution of the Cauchy problem for constant bottom and instantaneous source

Let us study the solution η corresponding to (3.12). It is not restrictive to assume that the center of the source is located in the origin of the coordinates $x_0 = 0$ and that the perturbation decays rapidly with the distance from the origin and that it has a maximum in a small neighborhood of the origin. We use also dimensionless variables:

$$V = V(\frac{x}{l}),$$

where l is the size of the shifted region and

$$\tilde{V} = \frac{1}{2\pi}\int V(\tfrac{\xi}{l})e^{-ip\cdot\xi}d\xi = \frac{l}{2\pi}\int V(y)e^{-il<p,y>}dy = l\tilde{V}(pl),$$
$$\tilde{\eta}_0(p) = \frac{l}{\cosh(|p|H)}\tilde{V}(pl),$$

where we made the substitution $\xi = yl$ and $\tilde{V}(p)$ is the usual Fourier transform of the function $V(y)$. We assume that $V(y)$ is a smooth function rapidly decaying as $|y| \to \infty$.

Then we can make the inverse Fourier transform:

$$\eta = \frac{l}{4\pi}\sum_{\pm}\int e^{\pm it\sqrt{\mathcal{L}(p,H)}+i<p,x>}\tilde{\eta}_0(p)dp = \frac{l}{4\pi}\Sigma_{\pm}\int e^{\pm it\sqrt{\mathcal{L}(p,H)}+i<p,x>}\frac{1}{\cosh(|p|H)}\tilde{V}(pl)dp.$$

Changing the variables $p = p'/l$, we get

$$\eta = \frac{1}{4\pi}\Sigma_{\pm}\int e^{\pm i\sqrt{\frac{g|p|}{l}\tanh(|p|\frac{H}{l})}+i\frac{<p,x>}{l}}\frac{1}{\cosh(|p|\frac{H}{l})}\tilde{V}(p)dp.$$

In this way the problem is reduced to the computation of the asymptotic behavior of the integral.

We will study the asymptotic values for $|x| >> l$. We change variables inside the integral and pass to polar coordinates (ρ,φ), where φ is defined as the angle among p and $x - x_0$. Thus $p = \rho\Theta(\varphi)\frac{x}{|x|}$, where $\Theta(\varphi)$ is the two dimensional matrix defining the rotation of an angle φ.

$$\Theta(\varphi) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

Then the last integral has the form

$$\eta = \frac{1}{4\pi}\Sigma_{\pm}\int_0^{\infty}\rho d\rho\int_0^{2\pi}d\varphi\exp\big(\pm it\sqrt{\frac{g\rho}{l}\tanh(\rho\frac{H}{l})}\big)\exp\big(i\frac{\rho|x|}{l}\cos\varphi\big)\frac{1}{\cosh(\rho\frac{H}{l})}\tilde{V}(\rho\Theta\frac{x}{|x|}).$$

The internal integral can be computed using the method of stationary phase. The phase has the form: $\Phi = \frac{\rho|x|}{l}\cos\varphi$, the equation $\frac{\partial\Phi}{\partial\varphi} = 0$ gives $\varphi = 0, \varphi = \pi$; however it is not possible to apply the method of the stationary phase in the point $\rho = 0$. One can reduce the interval of integration to a sufficiently small neighborhood of the saddle points of the variable φ and show that, [12, 13, 14], the error is smaller than the contribution of the terms that have been neglected. The result is:

$$\eta \approx \frac{1}{2\sqrt{2\pi}}\sqrt{\frac{l}{|x|}}\Sigma_{\pm}\int_0^{\infty}d\rho\frac{\sqrt{\rho}}{\cosh(\rho\frac{H}{l})}$$
$$\exp\big(\pm it\sqrt{\frac{g\rho}{l}\tanh(\rho\frac{H}{l})}\big)\Sigma_{\pm}\big(e^{\mp i\pi/4}e^{\frac{\pm i\rho|x|}{l}}\tilde{V}(\pm\rho\frac{x}{|x|})\big).$$

Let us consider the last integral. Its global phases are:

$$\Phi_{\pm,\pm}/l = \pm(t\sqrt{gl\rho\tanh(\rho\frac{H}{l})}\pm\rho|x|)/l.$$

For $t > 0, \rho > 0$ the derivative $\frac{\partial\Phi_{\pm,+}}{\partial\rho}$ is strictly positive, this implies the absence of critical points for the functions $\Phi_{\pm,+}$. It follows that these terms give, for $t > 0$, a contribution to the wave field which is asymptotically small with respect to the other contributions and so it can be dropped. Furthermore since V is a real function then

$\tilde{V}(\rho\frac{x}{|x|})$ and $\tilde{V}(-\rho\frac{x}{|x|})$ are complex conjugates so the last integral may be written in the form

$$\eta \approx \frac{1}{\sqrt{2\pi}}\sqrt{\frac{l}{|x|}}\times$$

$$\mathrm{Re}\int_0^\infty \mathrm{d}\rho\frac{\sqrt{\varrho}}{\cosh(\rho\frac{\mathrm{H}}{\mathrm{l}})}\tilde{\mathrm{V}}(\rho\frac{\mathrm{x}}{|\mathrm{x}|})\mathrm{e}^{-\mathrm{i}\pi/4}\exp\big(\frac{\mathrm{i}}{\mathrm{l}}(\varrho|\mathrm{x}| - \mathrm{t}\rho\sqrt{\mathrm{gH}}\sqrt{\frac{1}{\rho\mathrm{H}}\tanh(\frac{\rho\mathrm{H}}{\mathrm{l}})}\big).$$

Since the ratio $\frac{H}{l}$ is rather small, the source is localized, and the function $\tilde{V}(\rho\frac{x}{|x|})$ decays rapidly as a function of ρ, then the main contribution to the last integral is coming from the small values of ρ. Then we get that the functions $\frac{1}{\cosh(\rho\frac{H}{l})}$ and $t\rho\sqrt{gH}\sqrt{\frac{l}{\rho H}\tanh(\frac{\rho H}{l})}$ can be expanded in Taylor series. If we substitute the first function with 1 we neglect a term of the order of $O(\frac{H}{l})^2$. The second function can be approximated by the first two non zero terms of its expansion $t\rho\sqrt{gH}(\frac{1}{l} - \frac{1}{6}(\frac{\rho H}{l})^2)$ making an error of the order of $t\sqrt{gH}(\frac{H}{l})^4$. It is clear from the previous estimates that these terms are small and so we obtain

$$\eta \approx \frac{1}{\sqrt{2\pi}}\sqrt{\frac{l}{|x|}}\mathrm{Re}\int_0^\infty \mathrm{d}\rho\sqrt{\varrho}\tilde{\mathrm{V}}(\rho\frac{\mathrm{x}}{|\mathrm{x}|})\mathrm{e}^{-\mathrm{i}\pi/4}\exp\big(\frac{\mathrm{i}}{\mathrm{l}}(\varrho|\mathrm{x}| - \mathrm{t}\rho\sqrt{\mathrm{gH}}(1 - \frac{\rho^2}{6}(\frac{\mathrm{H}}{\mathrm{l}})^2))\big).$$

It will be explained below that the integral gets its larger values in the neighborhood of the front, i.e. near the curve (circle) $|x| = \sqrt{gH}t$. In this way the dispersion effects can influence the asymptotic values in the far wave field under the condition that the coefficient of ρ^3 in the exponent is larger or equal to one. Thus we obtain different behaviors, putting $\sqrt{gH}t$ equal to $|x|$ in this coefficient, according to the possible relations among $|x|, H, l$ (compare [3]-[8],[14]):

a) For $|x| >> \frac{l^3}{H^2}$ the dispersion has an important influence in the neighborhood of the front, and the asymptotic can be expressed by means of a function similar to the Airy function. In this case the behavior of the function V is not important for the definition of the profile of the front.

b) For $|x| \sim \frac{l^3}{H^2}$ the weak dispersion and the function $\tilde{V}$ influences the formation of the wave profile;

c) For $|x| << \frac{l^3}{H^2}$ the dispersion is not important and the function $\tilde{V}$ is important for determining the profile. If the term with ρ^3, is dropped from the phase of the integral an error of the order of $|x|H^2/l^3$ is done.

Let us consider the example where $H = 4km$, $l = 40km$, thus $l^3/H^2 = 4000km$. Thus a (weak) effect of the dispersion starts at $4000km$. If the size of the source increases twice this distance increases 8 times and becomes 32000 km, a distance larger than any ocean. Thus we will start analyzing the point **c** (it possible to neglect the effect of the dispersion).

3.3 Asymptotic behavior of the wave field with very small dispersion in the case of constant depth

Thus, assuming that the inequality $|x| << l^3/H^2$ is satisfied, we have

$$\eta \approx \frac{1}{\sqrt{2\pi}}\sqrt{\frac{l}{|x|}}\mathrm{Re}\int_0^\infty \mathrm{d}\rho\sqrt{\varrho}\tilde{\mathrm{V}}(\rho\frac{\mathrm{x}}{|\mathrm{x}|})\mathrm{e}^{-\mathrm{i}\pi/4}\exp\big(\frac{\mathrm{i}}{\mathrm{l}}(\varrho|\mathrm{x}| - \mathrm{t}\rho\sqrt{\mathrm{gH}}\big) =$$

$$= \frac{l^{1/2}}{\sqrt{|x|}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{F}(\frac{\Phi(\mathrm{x},\mathrm{t})}{1}, \frac{\mathrm{x}}{|\mathrm{x}|})), \qquad \Phi(\mathrm{x},\mathrm{t}) = |\mathrm{x}| - \mathrm{t}\sqrt{\mathrm{gH}}, \tag{3.13}$$

where

$$F(z,\mathbf{n}) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{iz\rho}\sqrt{\rho}\widetilde{V}(\rho\mathbf{n})d\rho. \tag{3.14}$$

Here $\mathbf{n}$ is the unit vector parallel to the vector x

$$\mathbf{n} = \mathbf{n}(\psi) = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}. \tag{3.15}$$

The angle ψ is chosen in such a way that $\psi = 0$ correspondsand to change in final asymptotic formulas μ by l.

to the axis x_1. Hence the function $\widetilde{V}(\rho,\mathbf{n}(\psi))$ depends on (ρ,ψ) and the function $F(z,\mathbf{n}(\psi))$ depends on (z,ψ). For avoiding complicate notations we use the same symbols $\widetilde{V}$ and F for them and sometimes write $\widetilde{V}(\rho,\psi)$ and $F(z,\psi)$ instead $\widetilde{V}(\rho,\mathbf{n}(\psi))$ and $F(z,\mathbf{n}(\psi))$ respectively.

We note, that the function $F(z,\mathbf{n})$ decreases for $|z| \to \infty$ as an inverse power. Indeed, let us change variable in the last integral $\rho = \frac{y^2}{2}$; then

$$F(z,\mathbf{n}) = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\{\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\int_0^\infty \mathrm{y}^2\,\mathrm{e}^{\frac{\mathrm{i}z\mathrm{y}^2}{2}}\,\widetilde{\mathrm{V}}(\frac{\mathrm{y}^2}{2}\mathbf{n})\mathrm{dy}\}.$$

Using the method of the stationary phase we get, because of the presence of the factor y^2 under the integral, $F(z,\omega) \sim \frac{1}{z^{3/2}}$, if $\widetilde{V}(0) \neq 0$. Thus for $\big||x| - \sqrt{gH}t\big| >> l$ and $|x| >> l$, we have that $\eta \sim \frac{l^3}{|x|^2}\widetilde{V}(0)$.

Example 1. *Let us give some example of the function $F(z,\omega)$. We choose for the function V, defining the source, the function*

$$V(y) = \bar{V}\cos(a_1Y_1 + a_2Y_2 + \delta)e^{-b_1Y_1^2 - b_2Y_2^2},\ Y = \Theta(\theta)y, \tag{3.16}$$
$$\Theta(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},$$

where $\bar{V}, a_1, a_2, b_1,\ b_2 > 0,\ \theta, \chi$ are parameters. In this case the function $F(z,\psi)$ can be expressed in terms of parabolic cylinder functions $D_{-3/2}$ or confluent hypergeometric functions ${}_1F_1$

$$\begin{aligned}
\widetilde{V}(\rho,\psi) &= \frac{\bar{V}\sqrt{\rho}}{2\sqrt{b_1b_2}}e^{-\alpha-\beta\rho^2}\cosh(i\delta+\gamma\rho), \qquad (3.17)\\
F(z,\psi) &= \frac{\bar{V}\sqrt{b_1b_2}e^{-\delta}}{2\sqrt{2\pi}}\mathrm{Re}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\int_0^\infty \sqrt{\rho}(\mathrm{e}^{-\frac{\rho^2\beta}{2}+\gamma\rho+\mathrm{i}\rho z}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\frac{\rho^2\beta}{2}-\gamma\rho+\mathrm{i}\rho z}\mathrm{e}^{-\mathrm{i}\theta})\mathrm{d}\rho)\\
&\equiv \frac{\bar{V}\sqrt{b_1b_2}}{4e^{\delta}\beta^{3/4}}\mathrm{Re}\big(\exp(\frac{(\gamma+\mathrm{i}z)^2}{4\beta})\mathrm{D}_{-3/2}(-\frac{\gamma+\mathrm{i}z}{\sqrt{\beta}})\mathrm{e}^{\mathrm{i}\theta}\\
&\qquad + \exp(\frac{(-\gamma+iz)^2}{4\beta})D_{-3/2}(-\frac{-\gamma+iz}{\sqrt{\beta}})e^{-i\theta}\big)\\
&\equiv \bar{V}\sqrt{\frac{1}{32\pi b_1b_2}}Re[(Q_+ + Q_-)],
\end{aligned}$$

$$Q_\pm(\Phi,\psi) = \frac{e^{-i\frac{\pi}{4}-\sigma\pm i\chi}}{8\beta^{5/4}}\Big(-\sqrt{\beta}\,\Gamma(-\frac{1}{4})\ {}_1F_1(\frac{3}{4},\frac{1}{2},\frac{w_\pm^2}{4\beta}) + w_\pm\,\Gamma(\frac{1}{4})\ {}_1F_1(\frac{5}{4},\frac{3}{2},\frac{w_\pm^2}{4\beta})\Big),$$

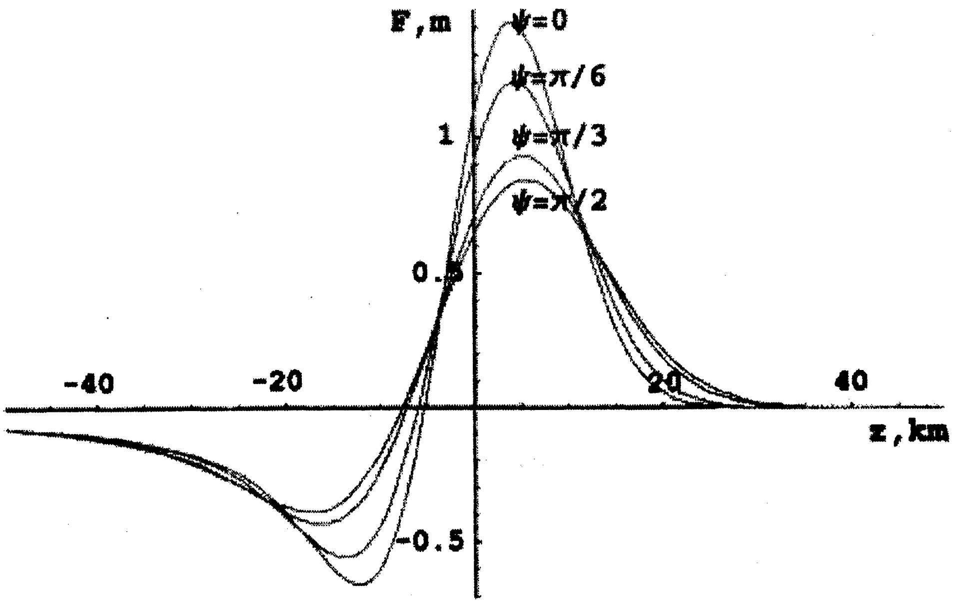

Figure 1: Source function F for Gaussian perturbation.
The form of the source is determined by (3.16) where $\bar{V} = 10\,\mathrm{m}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0.01\,\mathrm{km}^{-2}$, $b_2 = 0.005\,\mathrm{km}^{-2}$, $\psi = 0$, $\delta = 0$

where $\sigma = (b_1\alpha_2^2 + b_2\alpha_1^2)/(4b_1b_2)$, $\beta = (b_1\sin^2(\psi-\theta) + b_2\cos^2(\psi-\theta))/(4b_1b_2)$, $\gamma = (b_1\alpha_2\sin(\psi-\theta) + b_2\alpha_1\cos(\psi-\theta))/(2b_1b_2)$, $w_\pm = \pm\gamma + i\Phi$, ${}_1F_1(.)$ *is hypergeometric Kummer function,* Γ *is a gamma function (see Fig.1, Fig.2).*

Main conclusion: the phase in the neighborhood of the front defines completely a one parameter family of trajectories which generate the front. Further we remark that, since the function F decreases, we can expand in the formula (3.13) $|x|$ in a neighborhood of the front, keeping in the expansion only the zero order term, and that we can substitute the factor $\frac{1}{\sqrt{|x|}}$ (the amplitude of the wave) with the term $\frac{1}{\sqrt{\sqrt{gH}t}}$. We want to find analogous formulas for the wave field in the case of negligible small dispersion and for variable bottom.

4 Localized solutions to the wave equation and asymptotic behavior of the wave field over nonuniform bottom for very small dispersion

4.1 The wave equation, rays and wave fronts

In this section we start the analysis of the behavior of the amplitude of the wave when the bottom is not constant. We use here well known objects and their characteristics which one can find in books connected with the semiclassical asymptotic and ray method, geometrical optics and wave fronts, Hamiltonian mechanics, catastrophe theory etc. We try to collect here all necessary concepts and give their description in elementary form. More complete presentations and details one can find in [16]-[25]. It is clear that in practice we have studied the solution of the wave equation in the previous section. In order to construct a meaningful asymptotic theory we introduced

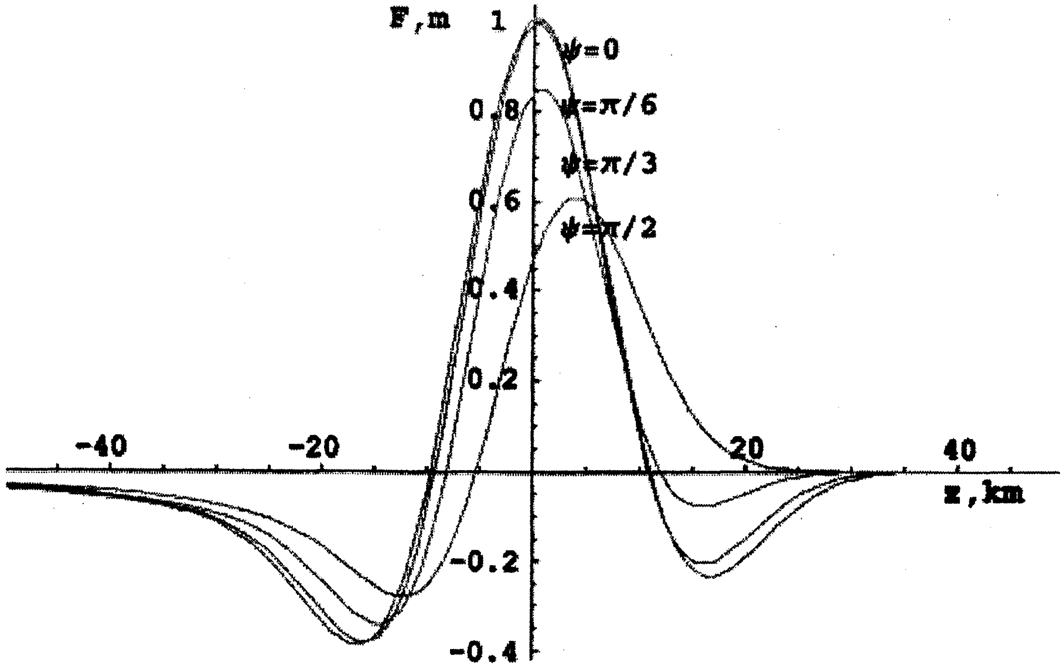

Figure 2: Source function F for "modulated" Gaussian perturbation.
The form of the source is determined by (3.16) where $\bar{V} = 10\,\mathrm{m}$, $a_1 = 0$, $a_2 = 0.1\,\mathrm{km}^{-1}$, $b_1 = 0.01\,\mathrm{km}^{-2}$, $b_2 = 0.005\,\mathrm{km}^{-2}$, $\psi = 0$, $\delta = \pi/4$

there the small parameter

$$\mu = \frac{l}{L} \tag{4.1}$$

expressing the relationship among the characteristic size of the source and the characteristic size of the basin. We begin introducing non dimensional variables in the equations and scale using the characteristic depth of the basin H_0 . After we make the change of variables $x' = x/L$, $t' = t\sqrt{gH_0}/L$, $H = H_0H'(x')$ our equations and initial data will take the form (1.2). Our asymptotic expansions will be done in term of this parameter under the assumption $\mu \ll 1$. *To come back to original variables it is enough to use the original variables x, t to change in final asymptotic formulas μ by l and in (4.2) $C(x) = \sqrt{H(x)}$ by $C(x) = \sqrt{gH(x)}$.*

We assume that the source of the perturbation is localized in $x = 0$. It is easy to see that finding the field far from the source,$|x| \gg l$, is similar to find the asymptotic values for $\mu \to 0$ in the problem (1.1). The problem now is to study the wave equation with variable coefficient. The asymptotic values of the wave amplitude η can be expressed by means of the wave front formed by rays. It is a known fact that instead of the straight rays one has to introduce curved rays and characteristics given by the one dimensional family of trajectories $P(\psi, t), X(\psi, t)$ of an appropriate Hamiltonian system . The ends of the rays form the wavefront, a complicated closed curve probably with cusps and self intersection points. In the considered situation these rays and characteristics are determined in the following way.

We introduce the function $C(x) = \sqrt{H(x)}$ and, as before, let $\mathbf{n}$ be the unit vector (3.15) directed as the external normal to the unit circle. Then the Hamilton system is:

$$\dot{x} = \frac{p}{|p|}C(x), \quad \dot{p} = -|p|\frac{\partial C}{\partial x}, \qquad x|_{t=0} = 0, \quad p|_{t=0} = \mathbf{n}(\psi), \tag{4.2}$$

i.e. the family of trajectories $P(\psi,t), X(\psi,t)$ going out from the point $x=0$ with unit impulse $p=\mathbf{n}(\psi)$. Let us indicate $C(0)=C_0$. The Hamiltonian corresponding to (4.2) is $\mathcal{H}=C(X)|p|$. From the conservation of the Hamiltonian on the trajectories we have the important equation

$$|P|C(X)=C_0. \tag{4.3}$$

The projections $x=X(\psi,t)$ of the trajectories on the plane $\mathbb{R}^2_x$ are called the rays. Recall that the *front* in the plane $\mathbb{R}^2_x$ at the time $t>0$ is the curve $\gamma_t=\{x\in\mathbb{R}^2|x=X(\psi,t)\}$, [25, 16]. The points on this curve are parameterized by the angle $\psi\in(0,2\pi]$. If in each point x of the front γ_t $\frac{\partial X}{\partial\psi}\neq 0$, then the front is a smooth curve. The points where $\frac{\partial X}{\partial\psi}=0$ are named *focals*, in these points the front looses its smoothness. In the situation in which the focal points appear, (they are very interesting from the point of view of tsunami), it is reasonable to introduce the concept of the front in the phase space $\mathbb{R}^4_{p,x}$ at the moment $t>0$, i.e. the curve $\Gamma_t=\{p=P(\psi,t), x=X(\psi,t), \psi\in[0,2\pi]\}$. We note that at least one of the component of the vector P_ψ, X_Ψ is different from zero and also the rays $x=X(t,\psi)$ are orthogonal to the front γ_t: $\langle\dot{X},X_\psi\rangle=0$ see Lemma 3.

4.2 The wave field before critical times.

It is not difficult to check that a (possibly sufficiently small) t_1 exists such that, for any t, $t_1\geq t>\delta>0$, there are no focal points in γ_t. The first instant of time t_{cr}, in which focal points are formed is called *critical.* Let us first write the solution before critical times, larger than δ, when the front is already defined. In this case the asymptotic solution is defined in the following way. We define a neighborhood of the front for sufficiently small (but independent of μ) coordinates ψ, y, where $|y|$ is the distance among the point x belonging to a neighborhood of the front and the front. For this aim we will take $y\geq 0$ for the external subset of the front and $y\leq 0$ and for the internal subset of the front. Then a point x of the neighborhood of the front is characterized by two coordinates: $\psi(t,x)$ and $y(t,x)$, where $\psi(t,x)$ is defined by the condition that the vector $y=x-X(\psi,t)$ is orthogonal to the vector tangent to the front in the point $X(\psi,t)$. Thus we have the condition $\langle y, X_\psi(\psi,t)\rangle=0$. Let us find the phase

$$S(t,x)=\langle P(\psi(t,x),t), x-X(\psi(t,x),t)\rangle=\frac{C(0)}{C(X(\psi(t,x),t))}y=\sqrt{\frac{H(0)}{H(X(\psi(t,x),t))}}y$$

The second equality is a consequence of the equation (4.3).

Now we state the first important theorem of this paper connecting the wave amplitude with the initial perturbation $V(x)$ and the profile of the bottom and the integration over the characteristics.

Theorem 1. *For $t_{cr}>t>\delta>0$, in some neighborhood of the front γ_t, not depending on μ, η, the asymptotic elevation of the free surface, has the form:*

$$\eta=\frac{\sqrt{\mu}}{\sqrt{|X_\psi(\psi,t)|}}\sqrt[4]{\frac{H(0)}{H(X(\psi,t))}}\operatorname{Re}\left[e^{-\frac{i\pi}{4}}F\left(\frac{S(t,x)}{\mu},\mathbf{n}(\psi)\right)\right]|_{\psi=\psi(t,x)}+O(\mu^{3/2}) \tag{4.4}$$

Outside this region $\eta=O(\mu^{3/2})$. The function $F(z,\mathbf{n})$ is defined in (3.14).

In this way till the critical time the asymptotic elevation of the free surface is completely defined by means of the trajectory, which forms the front of the wave, and of the function V, corresponding to the source of the perturbation. Despite of the simple and natural form of the asymptotic of η, the proof of the formula (4.4) is not trivial at all; the main step is the computation of the function V, more exactly the proof of the fact that the formula is the same as in the case of constant bottom, if the right choice of the rays is made. We will give below the necessary tools for a constructive approach of the proof of this formula, in the meantime we now show some elementary consequence of the equation (4.4). Since the phase $S(x,t)$ is equal to zero on the front and $S(x,t)/\mu$ gets large going out from the front, then η, as one could expect, decreases enough quickly and the maximum of $|\eta|$ is attained in a neighborhood of the front. As a consequence, η can have some oscillations depending on the form of the source. The second factor in (4.4) is the two dimensional analogue of the Green rule, well known in the theory of water waves in the channels: the amplitude η increases when the depth decreases as the inverse of the fourth root of the depth $1/\sqrt{C(x)} = 1/\sqrt[4]{H(x)}$; the factor $1/\sqrt{|X_\psi|}$ is connected to the divergence of the rays, in other words if a smaller number of rays goes through a neighborhood of the point $X(\psi,t)$, the smaller will be the amplitude of the wave field. The factor $\frac{C_0}{C(X(\psi(t,x),t))}$ appearing in the formula of the phase expresses the phenomena, also well known, of the "contraction" of the wave profile and the increase of its amplitude as the depth decreases. In fact the amplitude increases because of the factor in front of the function V but also the phase $S(x,t)$ increases and this makes the wave profile narrower. This result explains the well know fact that the wave length of the tsunami decreases when the wave approaches the coast and that its amplitude increases. The same profile (i.e. a section of $\eta(x,t)$ for fixed t and ψ) can depend on the way the trajectory (ray) intersects the initial perturbation of the bottom at $t=0$. It is just this fact to give the dependence of the diagram of the directions on two factors: the shape of the source and the angle of its intersection with the ray passing through a given point of the front. For this reason, depending on the form of the bottom, two rays going out with two very different angles, can arrive near the same point of the front and contribute to the profile with very different amplitudes. These effects can be well seen in Fig.3, Fig.4.

4.3 The structure and metamorphosis of wave profiles after critical time.

4.3.1 The Maslov index and metamorphosis of the wave profile.

For $t > t_{\rm cr}$ when the focal points appear, as it is well known in the wave theory, the front can have "angles" and sometimes the front lines can have self intersection points. The ends of the arcs corresponding to these angles are the *focal points* (or backtracking or turning points). For $t > t_{\rm cr}$ the front divides in some arcs γ_t^j, indexed by the number j, separated by focal points. The internal points of these arcs are the ends of the trajectories $P(\psi,t), X(\psi,t)$ with the same topological structure. Namely these equivalent trajectories cross the same numbers of focal points at times t^F before t, $t^F < t$. They are characterized, from the topological point of view, by the *Maslov index, an integer number* $m(\psi,t)$ *depending on* ψ, t. The Maslov index m can be defined on the *regular* points of the front in different ways, we give below a more practical definition of this important concept by means of a simple definition of its increments in the problem under examination. The index m is related to the sign of the Jacobian $J = \frac{\partial X}{\partial(t,\psi)} \equiv (\dot{X}, X_\psi)$. The function J is equal to zero in the focal points and only in these points. Thus moving along the front γ_t or along the trajectory (P,X) after crossing the focal point, the Jacobian can change its sign. Actually the Maslov index

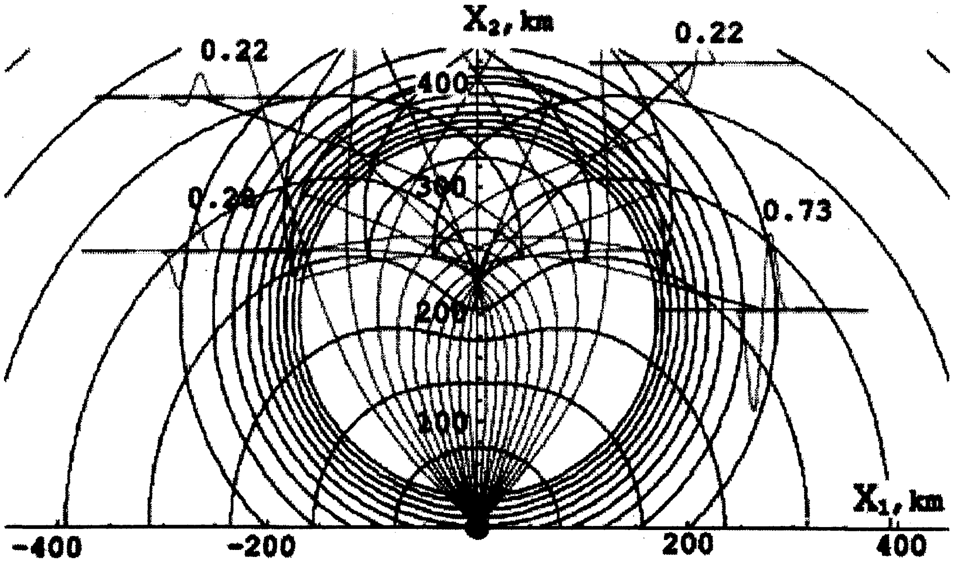

Figure 3: Tsunami spreading over bottom with bank-like mountain.
Amplitudes maxima are given in meters.
The depth of the bottom in [km] is $H(x_1, x_2) = 4.5 - 4\exp[-(x_1/100)^2 - (x_2/100 - 2)^2]$.
The form of the source is determined by (3.16) where $\bar{V} = 10\,\mathrm{m}$, $a_1 = 0$, $a_2 = 0.1\,\mathrm{km}^{-1}$, $b_1 = 0.01\,\mathrm{km}^{-2}$, $b_2 = 0.005\,\mathrm{km}^{-2}$, $\psi = 0$, $\delta = \pi/4$

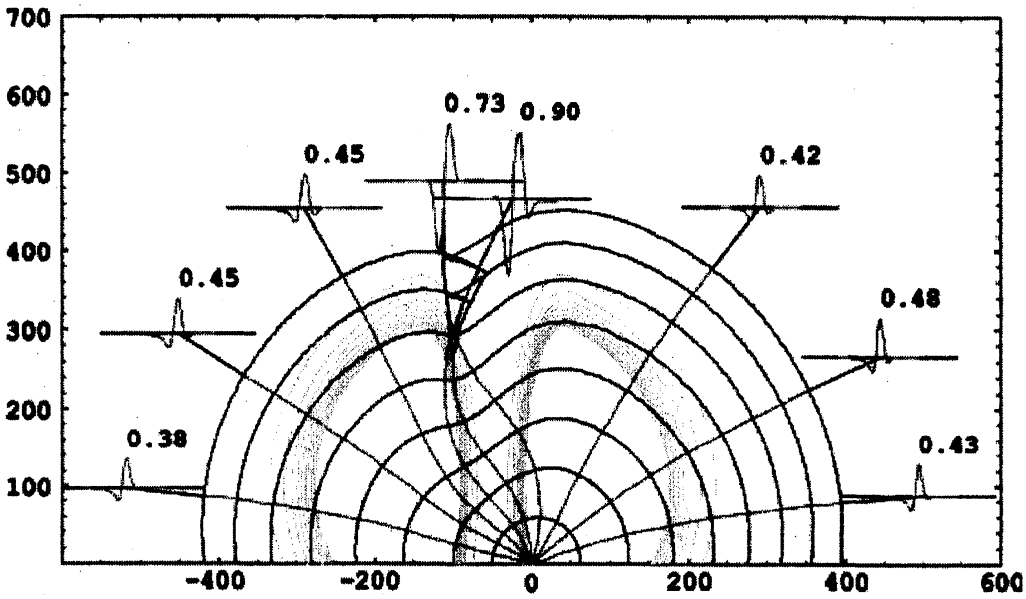

Figure 4: Tsunami spreading over the well with ridge.
Amplitudes maxima are given in meters.
The depth of the bottom in [km] is $H(x_1, x_2) = 1 + d_1(x_1 + 50, x_2 - 100)d_2(x_1 + 50, x_2 - 100)$,
$d_1(x_1, x_2) = 1.3$ if $|x_1| < 100$ and $d_1(x_1, x_2) = 1.3 - \cos^2(\pi x_1/200)$ if $|x_1| \geq 100$.
$d_2(x_1, x_2) = 2\cos^2(\pi(x_1^2 + x_2^2)^{1/2}/900)$ if $(x_1^2 + x_2^2)^{1/2} < 450$ and $d_2(x_1, x_2) = 0$ otherwise. The form of the source is determined by (3.16) where $\bar{V} = 10\,\mathrm{m}$, $a_1 = 0$, $a_2 = 0.1\,\mathrm{km}^{-1}$, $b_1 = 0.01\,\mathrm{km}^{-2}$, $b_2 = 0.005\,\mathrm{km}^{-2}$, $\psi = 0$, $\delta = \pi/4$

prescribes a receipt for assigning the correct sign to the square root of J and it can be defined in a way independent from the trajectories. But if we move along a trajectory there is, in this problem, the nice and useful fact that the Maslov *index coincides with the* **simpler** *Morse index.* So, considering the trajectories arriving to γ_t^j, we have that *the Morse index $m(\psi,t)$ of the point $x = X(\psi,t) \in \mathbb{R}_x^2$ is equal to the number of focal points on the trajectory $p = P(\psi,\tau), x = X(\psi,\tau), \tau \in (0,t)$ arriving to $x = X(\psi,t)$.* Note also that, as the time t changes, the ends of the arcs γ_t^j produce the entire set of focal points. It is also a well known fact that these sets constitute the (space-time) *caustics* which are the singularities of the projections of some Lagrangian manifold (we denote it M^2) from the phase space $\mathbb{R}_{p,x}^4$ to the plane (configuration space) $\mathbb{R}_x^2$.

Example 2.

Let us illustrate the concepts explained above by the example (considered in [9] for the scattering problems)about the waves on an axially symmetrical bank described by the depth function (see Fig.3)

$$H = H(\rho),\ \rho = \sqrt{x_1^2 + x_2^2}. \tag{4.5}$$

In this case an additional integral exists

$$p_\varphi = x_1 p_2 - x_2 p_1 \tag{4.6}$$

and the Hamiltonian system (4.5) is completely integrable.

We assume that the source is located in a neighborhood of the point $x_1 = 0, x_2 = -\rho_0$.

For each fixed time t the front γ_t is separated into two arcs: the first, a long one, is γ_t^1 with self-intersection points, and the second, a short one, is γ_t^2, located between the angles on the fronts. The union of the ends of the arc γ_t^2 for different times t gives a caustic. The arc γ_t^1 consists of the ends of trajectories (rays) without focal points on them (except $t = 0$). Thus the Jacobian $J(\psi,t) = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)(\psi,\tau) > 0$ for fixed ψ and for each $\tau \in (0,t]$; hence the Morse index $m(x \in \gamma_t^1) = 0$. On the contrary the arc γ_t^2 consists of the final points of the trajectories (rays) which cross one focal point at the time $t = t_F(\psi), 0 < t_F(\psi) < t$ when they touch the caustic. In this case before $t_F(\psi)$ $J > 0$, $J(\psi, t_F(\psi)) = 0$, and $J < 0$ for $t > t_F(\psi)$. Hence $m(x \in \gamma_t^2) = 1$.

Now let us fix the time t and move along the front γ_t. Then after the passage through the focal points the phase $-\pi/4$ in formula (4.4) increases by a quantity $-\pi/4 \pm \pi/2$, where ± 1 is the jump of the Maslov index. Finally after passing through several focal points instead of the factor $e^{-\frac{i\pi}{4}}$ one has the factor $e^{-\frac{i\pi}{4} - \frac{i\pi m(\psi,t)}{2}}$. The number m is defined mod 4. The appearance of this new factor *produces crucial changes of the form of the wave profile* in the formula (4.7) i.e. in the function $\mathrm{Re}(e^{-\frac{i\pi}{4} - \frac{i\pi m}{2}}\mathrm{F})$. This fact is analogous to the well known metamorphosis of the discontinuity in the theory of hyperbolic systems (see e.g. [10, 17, 21]), and the formula (4.7) describes explicitly the appearance of the same fact in the case of localized initial perturbations.

Let us present the formula for the wave amplitude in a neighborhood of the front but outside of some neighborhood of the focal points. As we have just seen in the previous example, points of self-intersection can appear for $t > t_{\mathrm{cr}}$. The amplitude of the wave in a point x belonging to a neighborhood of these points now is the sum of the contributions coming from different $\psi_j(x,t)$, $y_j(x,t)$, and $S_j(x,t)$ with index j, and with the Maslov index $m(\psi_j(x,t),t)$ (see Fig.3, Fig.4).

Theorem 2. *In a neighborhood of the front but outside of some neighborhood of the focal points the wave field is the sum of the fields*

$$\eta = \sum_j \{ \frac{\sqrt{\mu}}{\sqrt{|X_\psi(\psi,t)|}} \sqrt[4]{\frac{H(0)}{H(X(\psi,t))}} \operatorname{Re}\left[e^{-\frac{i\pi}{4}-\frac{i\pi m}{2}} F(\frac{S_j(x,t)}{\mu}, \mathbf{n}(\psi))\right] \}|_{\psi=\psi_j(x,t)} + O(\mu^{3/2}). \tag{4.7}$$

Outside this neighborhood of the front γ_t $\eta(x,t) = O(\mu^{3/2})$. *Again the function* $F(z, \mathbf{n}(\psi))$ *is determined in* (3.14).

Let us emphasize that the number m has a pure topological and geometrical character and can be calculated without any relation with the asymptotic formulas for the wave field. From the theorem 2 it follows that, in order to construct the wave field at some time t in a point x, one has to know only the initial values $\eta|_{t=0}$ and $\eta_t|_{t=0}$ and has not to know the wave field η for all previous time between 0 and t. The trajectories and the Maslov (Morse) index take into account all metamorphosis of the wave field during the evolution from time zero until time t.

4.4 Wave field asymptotic in a neighborhood of focal point

4.4.1 Completely non degenerate focal points and coordinate system

Now we consider the situation when for some t the point $(P^F, X^F) = (P(\psi^F(t),t), X(\psi^F(t),t))$ corresponding to the angle $\psi^F(t)$ is a *focal* one. In this point X_ψ=0 and one has to use another asymptotic representation for the solution. Roughly speaking the neighborhood of the point $X(\psi^F(t),t)$ on the plane $\mathbb{R}^2_x$ can include several arcs of γ_t with the angles ψ different from $\psi^F(t)$. This means that one has to take into account the contribution of all of these arcs in the final formulas for η in the neighborhood of the point $x = X(\psi^F(t),t)$. The influence of nonsingular points are defined by formula (4.7) and the influence of the points from the neighborhood of the focal points are described by formulas (4.14) given below. Thus it is necessary to enumerate the focal points with nearby projections and write $P(\psi_j^F(t),t), X(\psi_j^F(t),t)$. These points have the same position $X^F = X(\psi_j^F(t),t)$, but different momentum $P^F = P(\psi_j^F(t),t)$. To simplify the notations we discuss here the influence on η of only one focal point omitting the subindex j but keeping P^F.

We present the corresponding formula under the assumption that some derivative

$$X_\psi^{(n)F} = \frac{\partial^n X}{\partial \psi^n}(\psi^F(t),t) \neq 0, \tag{4.8}$$

and the derivatives $X_\psi^{(k)F} = 0$ for $1 \leq k < n$. It means that this focal point is not completely degenerate. For future convenience we introduce the "mixed" Jacobian

$$\tilde{J} = \det(\dot{X}, P_\psi)(\psi,t) = \frac{C^2(X)\det(P, P_\psi)}{C_0}(\psi,t) \tag{4.9}$$

and some characteristics of the focal point (P^F, X^F):

$$C_F = C(X^F),\ \dot{X}^F = \dot{X}(\psi^F(t),t) = \frac{P^F C_F^2}{C_0},\ P_\psi^F = P_\psi(\psi^F(t),t),$$

$$\tilde{J}_F = \det(\dot{X}^F, P_\psi^F) = \frac{C_F^2\det(P,P_\psi)}{C_0},\ J_F^{(n)} = \det(\dot{X}^F, X_\psi^{(n)F}). \tag{4.10}$$

Again the topological characteristic appears, i.e. the Maslov index of this focal point or its neighborhood (it is the same), but now it depends on the choice of the coordinates in the neighborhood of (P^F, X^F). It is natural to choose the new coordinates (x_1', x_2') associated with the nonzero vector $\dot{X}^F = \dot{X}(\psi^F(t), t)$; namely we assume that the direction of the new vertical axis x_2' coincides with the vector $\dot{X}^F$. We put $\mathbf{k}_2 = {}^t(k_{21}, k_{22}) = \dot{X}^F/|\dot{X}^F| = \dot{X}^F/C_F = P^F C_F/C_0$, $\mathbf{k}_1 = {}^t(k_{11}, k_{12}) = (k_{22}, -k_{21})$ and introduce the new coordinates p', x' in the neighborhood of (P^F, X^F) in the phase space $\mathbb{R}^4_{p,x}$ by the formulas:

$$x_1' = \langle \mathbf{k}_1, x - X^F\rangle = -\frac{\det(\dot{X}^F, x - X^F)}{C_F} = -\frac{C_F}{C_0}\det(P^F, x - X^F),$$
$$x_2' = \langle \mathbf{k}_2, x - X^F\rangle = \frac{\langle \dot{X}^F, x - X^F\rangle}{C_F} = \frac{C_F}{C_0}\langle P^F, x - X^F\rangle,$$
$$p_1' = \langle \mathbf{k}_1, p\rangle,\ p_2' = \langle \mathbf{k}_2, p\rangle. \tag{4.11}$$

It is easy to see that

$$\det\begin{pmatrix} \dot{P}_1' & P_{1\psi}' \\ \dot{X}_2' & X_{2\psi}' \end{pmatrix} = \tilde{J}_F. \tag{4.12}$$

4.4.2 The Maslov index of a focal point.

Since the determinant $\tilde{J} \neq 0$ in the focal point (P^F, X^F), the same inequality takes place in some of its neighborhood, thus $\tilde{J}$ has a constant sign. On the contrary the Jacobian J changes sign in this neighborhood. *We define the Maslov index* $\mathbf{m}(P^F, X^F)$ *of the non (completely) degenerate focal point* $(P^F, X^F) = (P, X)(\psi^F(t), t)$ *as the index* $m(\tilde{P}, \tilde{X})(\psi, t)$ *of a regular point* $(\tilde{P}, \tilde{X}) = (P, X)(\tilde{\psi}, \tilde{t})$ *in the neighborhood of* (P^F, X^F) *such that the signs of the determinants* $J(\tilde{\psi}, \tilde{t})$ *and* $\tilde{J}(\tilde{\psi}, \tilde{t})$ *coincide.* For instance one can choose $\tilde{\psi} = \psi^F(t), \tilde{t} = t \pm \delta$, where delta is small enough. This means that we compare the sign of J with the sign of $\tilde{J}$ on the trajectory (P, X) crossing the curve Γ_t in the focal point (P^F, X^F) before and after this crossing.

4.4.3 The model functions and the wave profile in a neighborhood of the focal point.

Now we present the formulas for the wave field in the neighborhood of a focal point $x = X^F$. Let us put $\sigma = \operatorname{sign}(\tilde{J}_F J_F^{(n)})$ and introduce the function (or more precisely the linear operator acting to the source function $V(y_1, y_2)$)

$$g_n^\sigma(z_1, z_2, \psi) = \int_{-\infty}^{\infty} d\xi \int_0^\infty \rho d\rho \tilde{V}(\rho\mathbf{n}(\psi)) \exp\{i\rho(z_2 - \xi z_1 - \sigma\frac{\xi^{n+1}}{(n+1)!})\}$$
$$= \int_{-\infty}^{\infty} d\xi \int_0^\infty \sqrt{\rho} d\rho \tilde{f}(\rho\mathbf{n}(\psi)) \exp\{i\rho(z_2 - \xi z_1 - \sigma\frac{\xi^{n+1}}{(n+1)!})\}. \tag{4.13}$$

We put

$$z_1^F = \frac{x_1'}{\mu^{\frac{n}{n+1}}} \frac{\tilde{J}_F}{|\tilde{J}_F J_F^{(n)}|^{\frac{1}{n+1}} C_F^{\frac{n}{n-1}}} \equiv -\frac{\det(P^F, x - X^F)}{C_0 C_F^{\frac{1}{n-1}} \mu^{\frac{n}{n+1}}} \frac{\tilde{J}_F}{|\tilde{J}_F J_F^{(n)}|^{\frac{1}{n+1}}},$$
$$z_2^F = \frac{x_2'}{\mu} \frac{C_0}{C_F} \equiv \frac{\langle P^F, x - X^F\rangle}{\mu}$$

Theorem 3. *In a neighborhood of the front* γ_t *each focal point* (P^F, X^F) *gives the following contribution to the asymptotic values of the solution* η

$$\eta^F = \mu^{\frac{1}{n+1}}\{\frac{\sqrt{C_0|\tilde{J}_F|^{\frac{n-1}{n+1}}}}{|J_F^{(n)}|^{\frac{1}{n+1}}C_F}\mathrm{Re}[\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\mathrm{m}(\mathrm{P^F},\mathrm{X^F})}\mathrm{g}_\mathrm{n}^\sigma(\mathrm{z}_1^\mathrm{F},\mathrm{z}_2^\mathrm{F},\psi^\mathrm{F})] + \mathrm{O}(\mu)\}. \tag{4.14}$$

If several arcs of γ_t *belong to the neighborhood of the point* x*, then one needs to sum over all the corresponding functions* (4.14) *and* (4.7).

5 The geometric base of the asymptotic theory: Lagrangian manifolds, the Maslov and Morse indices.

The aim of the next section is the discussion of the geometrical objects appearing in Theorems 1-3. Let us first recall the geometrical concepts and the important properties of the Hamiltonian system (4.2), in order to give an asymptotic solution to problem (4.2) including the behavior in a neighborhood of the focal points, initial moment of time, calculation of the Maslov and Morse indices etc. The majority of these constructions and properties are well known, we present them in the most simple form and collect them in our paper for giving a self-contained treatment. An exhaustive description of the wave fronts and the focal points, their connection with the ray method and the semiclassical asymptotic, can be found for instance in [25, 20, 16, 18]. There exist different equivalent definitions of the Maslov index; one of the aims of the next subsection is to recall the definition from [26, 17, 28] which, in our opinion, is more suitable for concrete calculations.

5.1 Lagrangian manifolds ("bands") and their properties.

As we have already said, taking into account the fact that after the appearance of the focal points the front line can intersect itself, it is convenient to add to the point $x = X(\psi, t)$ the corresponding momentum component $p = P(\psi, t)$, and consider the point $\mathbf{r} = \mathbf{r}(\psi, t) = (P(\psi, t), X(\psi, t))$ in the 4 dimensional phase space $\mathbb{R}^4_{p,x}$. Each point $\mathbf{r}(\psi, t)$ is completely defined by its coordinates, which are the angle ψ (defined mod 2π) and the "proper time" t.

Fixing the angle ψ we obtain the trajectories (bi-characteristics) of the Hamiltonian system (4.2) in the phase space $\mathbb{R}^4_{p,x}$, and, fixing the time t, we obtain the front Γ_t in the phase space $\mathbb{R}^4_{p,x}$. The projections of the trajectories from $\mathbb{R}^4_{p,x}$ to the configuration space (plane) $\mathbb{R}^2_x$ are the *rays*. The projection of the curve Γ_t from $\mathbb{R}^4_{p,x}$ to the configuration space (plane) $\mathbb{R}^2_x$ are the fronts γ_t. Different points $\mathbf{r}(\psi_j, t)$ on Γ_t can have the same projection $x = X(\psi_j, t)$ on γ_t, but now we distinguish them by different angles ψ_j.

Let us fix some small number δ, independent of μ. According to [18] changing both the angle ψ and the time $\tau \in (t-\delta, t+\delta)$ on the cylinder $\mathbb{S} \times (t-\delta, t+\delta)$ we obtain the 2-D Lagrangian manifold (with the boundary) $M_t^2 = \{p = P(\psi, \tau), x = X(\psi, \tau) | \psi \in \mathbb{S}, \tau \in (t-\delta, t+\delta)\}$; the angle ψ from the unit circle $\mathbb{S}$ and the time t from $(t-\delta, t+\delta) \in \mathbb{R}$ are the coordinates on M_t^2, sometimes we shall use the notation $\alpha = \tau - t$ instead of the time t. Actually the manifold M_t^2 has the structure of a cylindrical "band" (or closed strip) with the width 2δ, thus we call it a *Lagrangian* *band*; of course it depends on δ, we omit this dependence to simplify the notation. The family of Lagrangian bands M_t^2 is *invariant* with respect to the phase flow $g^t_{\mathcal{H}}$ generated by the system (4.2). This means that the point $\mathbf{r}(\psi_j, \tau)$ from $M^2_{t_0}$ shifted by the action of the flow $g^t_{\mathcal{H}}$ gives again the point $\mathbf{r}(\psi_j, \tau+t)$ on $M^2_{t_0+t}$ but with the shifted time $\tau+t$. Due to definition the coordinate α does not change. That is why the coordinate τ (corresponding to t) is called the *proper time*. Sometimes it is possible to choose δ arbitrary large, even infinity (e.g. in the case $C = const$). But in many situation the set $\{p = P(\psi,\tau), x = X(\psi,\tau)|\psi \in \mathbb{S}, \tau \in -\infty\}$ has the intersection points (e.g. if the trajectories $P(\psi,\tau), x = X(\psi,\tau)$ belong to the Liouville tori), and this set is not even a manifold. But for our purpose it is enough to work with the "Lagrangian band" M_t^2 only. Along with the general properties of Lagrangian manifolds, the band M_t^2 has very useful additional ones. Let us present all of them for completeness.

Let us introduce the matrices

$$\mathcal{B} = \frac{\partial P}{\partial(t,\psi)} \equiv (\dot{P}, P_\psi), \quad \mathcal{C} = \frac{\partial X}{\partial(t,\psi)} \equiv (\dot{X}, X_\psi).$$

Each column-vector $\begin{pmatrix}\dot{P}\\ \dot{X}\end{pmatrix}$, $\begin{pmatrix}P_\psi\\ X_\psi\end{pmatrix}$ and $\begin{pmatrix}P\\ 0\end{pmatrix}$ satisfies the variational system

$$\dot{\delta x} = \mathcal{H}_{pp}\delta p + \mathcal{H}_{px}\delta x, \quad \dot{\delta p} = -(\mathcal{H}_{xp}\delta p + \mathcal{H}_{xx}\delta x) \tag{5.1}$$

It is easy to check that these vectors are linearly independent and obviously that the first two vectors are tangent to M_t^2.

Lemma 1. *(see e.g.[17, 18]) The following properties are true:*

1)the rank of the matrix $\begin{pmatrix}\mathcal{B}\\ \mathcal{C}\end{pmatrix}$ *is equal to 2 which actually means that dimension of* M_t^2 *is 2.*

2)${}^t\mathcal{B}\mathcal{C} = {}^t\mathcal{C}\mathcal{B}$ *which means that* M^2 *is Lagrangian,*

3)for any positive ε *the matrix* $\mathcal{C} - i\varepsilon\mathcal{B}$ *is not degenerate.*

Proof. The first two propositions follow from the properties of the variational system. They hold for $t = 0$ because $\mathcal{B} = (-\nabla C(0), \mathbf{n}_\perp), \mathcal{C} = (C(0)\mathbf{n}, 0)$ where $\mathbf{n}_\perp = {}^t(-\sin\psi, \cos\psi)$. In this argument we use the definition of the trajectories (P, X), namely the property $P|_{t=0} = \mathbf{n}(\psi), X|_{t=0} = 0$, $\mathbf{n} = {}^t(\cos\psi, \sin\psi)$. Thus according to the variational system (5.1) the vector columns $\begin{pmatrix}\dot{P}\\ \dot{X}\end{pmatrix}$ and $\begin{pmatrix}P_\psi\\ X_\psi\end{pmatrix}$ are linearly independent for each t which gives 1). 2) Follows from a direct calculation: $({}^t\mathcal{B}\mathcal{C} - {}^t\mathcal{C}\mathcal{B})_{ii} = 0$ for $i = 1, 2$ and

$$({}^t\mathcal{B}\mathcal{C} - {}^t\mathcal{C}\mathcal{B})_{12} = ({}^t\mathcal{B}\mathcal{C} - {}^t\mathcal{C}\mathcal{B})_{21} = -|p|\frac{\partial C}{\partial\psi} - C\frac{\partial|p|}{\partial\psi} = -\frac{\partial|p|C}{\partial\psi} = 0$$

since $|p|C(x)$ is the Hamiltonian. To prove 3) assume that $\mathcal{C} - i\varepsilon\mathcal{B}$ is degenerate, then there exists a 2-D vector $\xi \neq 0$ such that $\mathcal{C}\xi = i\varepsilon\mathcal{B}\xi$. Consider the (complex) scalar product

$$\begin{aligned} 0 &= <\xi, ({}^t\mathcal{B}\mathcal{C} - {}^t\mathcal{C}\mathcal{B})\xi>_c = <\mathcal{B}\xi, \mathcal{C}\xi>_c - <\mathcal{C}\xi, \mathcal{B}\xi>_c \\ &= i(\varepsilon <\mathcal{C}\xi, \mathcal{C}\xi>_c + \frac{1}{\varepsilon} <\mathcal{B}\xi, \mathcal{B}\xi>_c) = 0. \end{aligned}$$

From this equation it follows that both $\mathcal{B}\xi = 0, \mathcal{C}\xi = 0$ which contradicts 1). □

The same considerations allow one to obtain a similar result.

Lemma 2. *The propositions of the previous Lemma concerning the matrices $\mathcal{B},\mathcal{C}$ are true if one changes the matrix $\mathcal{B}$ by the matrix*

$$\widetilde{\mathcal{B}} = (\dot{P} - \lambda P, P_\psi),$$

where $\lambda = \langle C_x(0), \mathbf{n}(\psi)\rangle$.

Let us recall that the points $x = X(\psi^F, t) = X^F$ on M_t^2 where the Jacobian

$$J \equiv \det\mathcal{C} \equiv \det(\dot{\mathrm{X}}, \mathrm{X}_\psi) = 0$$

are the *focal* points [1]. Since the manifold M_t^2 is generated by the curves Γ_t as well as by the trajectories (P, X) each focal point of one of these objects simultaneously is a focal point for the other ones. Later we shall show that this definition of the focal points coincides with the definition, based on the equality $X_\psi = 0$, used in the previous sections.

Let us fix a time t and consider the smooth curve $\Gamma_t = \{p = P(\psi, t),\ x = X(\psi, t)\}$ on $M_t^2 \in \mathbb{R}^4_{p,x}$ (the "time cut" of M_t^2). Then obviously the front $\gamma_t = \{x = X(\psi, t)\}$ is nothing but the projection of Γ_t to $\mathbb{R}^2_x$. Hence the focal points on the front are also the focal points of the manifold M_t^2, and from this point of view the caustics of M_t^2 are called space-time ones.

Lemma 3. *The vector-functions $\dot{X}$ and X_ψ as well as vector-functions P and X_ψ are orthogonal:* $\langle \dot{X}, X_\psi\rangle = \langle P, X_\psi\rangle = 0$.

PROOF. According to system (4.2) the vectors P and $\dot{X}$ are parallel and it is enough to prove the second equality. Let us differentiate $\langle P, X_\psi\rangle$ along the trajectories of the system (4.2). We have

$$\begin{aligned}
\frac{d}{dt}\langle P, X_\psi\rangle &= \langle \dot{P}, X_\psi\rangle + \langle P, \dot{X}_\psi\rangle \\
&= -|P|\langle C_x, X_\psi\rangle + \frac{C^2}{C_0}\langle P, P_\psi\rangle + \frac{\partial C^2}{\partial \psi}\frac{1}{C_0}\langle P, P\rangle \quad (\text{using } 4.3) \\
&= -|P|\frac{\partial C}{\partial \psi} + \frac{1}{2C_0}\frac{\partial (C^2P^2)}{\partial \psi} + \frac{C|P|}{C_0}|P|\frac{\partial C}{\partial \psi} \\
&= -|P|\frac{\partial C}{\partial \psi} + \frac{1}{2C_0}\frac{\partial (C_0^2)}{\partial \psi} + |P|\frac{\partial C}{\partial \psi} = 0.
\end{aligned}$$

But $X|_{t=0} = 0$, thus $\langle P, X_\psi\rangle|_{t=0} = 0$ and Lemma is proved. □

Corollary. *1) The following equality is true* $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi) = \pm|\dot{\mathrm{X}}|\,|\mathrm{X}_\psi|$. *2) The point $x = X(\psi, t)$ on the front γ_t, or the point $\mathbf{r} = (p = P(\psi, t), x = X(\psi, t))$ on $\Gamma_t \in M_t^2$ is a focal one if and only if* $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi) = 0$.

According to the equality $|\dot{X}| = C(X)$ $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)$ as well as the Jacobian in some neighborhood of γ_t can be equal to zero if and only if $X_\psi = 0$. Thus the last equation really determines the focal points from the point of view of the Lagrangian manifold also.

Lemma 4. *In the focal point $x = x^F = X(\psi^F, t)$ 1) $\langle P^F, P^F_\psi\rangle = 0$, but 2) $P^F_\psi \neq 0$, 3) $\frac{dJ}{dt} = \frac{C_F^2}{C_0}\det(\dot{\mathrm{X}}^F, \mathrm{P}^F_\psi)$, where as it was before $C_0 = C(0)$ and $C_F = C(X^F)$.*

[1] Note that using the Hamiltonian system we can change $\dot{X}$ by P in last formula as well as in many formulas containing $\dot{X}$.

PROOF. According to the conservation law (4.3) $\langle P, P_\psi\rangle(\psi^F, t) = \langle\nabla(\frac{C_0^2}{C^2(x)}), X_\psi\rangle(\psi^F, t) = 0$. To prove the second inequality one can mention that the vector-function $(P_\psi, X_\psi)^T$ satisfies the linear (variational) system with non-zero initial condition. Thus both components of the solution cannot be equal to zero. To prove 3) we write $\frac{dJ}{dt}|_{\psi=\psi^F} = [\det(\dot{\mathrm{X}}, \dot{\mathrm{X}}_\psi) + \det(\ddot{\mathrm{X}}, \mathrm{X}_\psi)]_{\psi=\psi^{\mathrm{F}}} =$ using 4.3 $= \det(\dot{\mathrm{X}}, \mathrm{P} < \nabla\frac{\mathrm{C}^2(\mathrm{X})}{\mathrm{C}_0}, \mathrm{X}_\psi >)|_{\psi=\psi^{\mathrm{F}}} + \det(\dot{\mathrm{X}}, (\frac{\mathrm{P}_\psi \mathrm{C}^2(\mathrm{X})}{\mathrm{C}_0}))|_{\psi=\psi^{\mathrm{F}}} = \frac{\mathrm{C}_{\mathrm{F}}^2(\mathrm{X})}{\mathrm{C}_0}\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)|_{\psi=\psi^{\mathrm{F}}}$. $\square$

Corollary. *In the focal point 1)* $\frac{dJ}{dt} = \frac{C_F^2}{C_0}\det(\dot{\mathrm{X}}, \mathrm{P}_\psi) = \pm|\frac{\mathrm{C}^2(\mathrm{X}^{\mathrm{F}}}{\mathrm{C}_0}\dot{\mathrm{X}}|\,|\mathrm{P}_\psi|(\psi^{\mathrm{F}}, \mathrm{t}) \neq 0$; *2) during the passage through the focal point the Jacobian J changes its sign from - to + if* $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)|_{\psi=\psi^{\mathrm{F}}} > 0$ *and from + to - if* $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)|_{\psi=\psi^{\mathrm{F}}} < 0$; *3) There exists t_{cr} such that $J(\psi, t) > 0$ for $0 < t < t_{cr}$.*

PROOF. To prove 3) it is enough to note that $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)|_{\mathrm{t}=0} = \mathrm{C}(0)\det(\mathbf{n}(\psi), \mathbf{n}(\psi)_\perp) = \mathrm{C}(0)$. $\square$

5.2 The Maslov and Morse indices.

As we said before the front γ_t as well as the curve Γ_t is partitioned into arcs with the focal points at their ends and it is formed by the ends of trajectories having the same topological structure. This means that they have similar crossing (on M_t^2) with the focal points and the same topological characteristic, i.e. the Maslov index. But the Maslov index coincides with the Morse index for the considered situation (see subsection (4.3.1)). Let us prove this proposition.

Let us remind some necessary definitions and constructions. It is needless to say that there exist several definitions of the Maslov index. The original definition [16] is based on calculation of inertia indices of the matrices $\frac{\partial(x_1,p_2)}{\partial(x_1,x_2)}|_{M_t^2}$, $\frac{\partial(p_1,x_2)}{\partial(x_1,x_2)}|_{M_t^2}$, $\frac{\partial(p_1,x_2)}{\partial(x_1,p_2)}|_{M_t^2}$ etc. It is not very convenient in practical calculation. Thus we want to present below a definition[17, 26, 28] which, from our point of view, is more pragmatic for computer calculations. We already pointed out that the Maslov index of the points on $x \in \gamma_t$ is the index of the *nonsingular* point $\mathbf{r}(\psi, t) = (P(\psi, t), X(\psi, t))$ on the Lagrangian band M_t^2. According to the procedure from [18, 26, 28] one needs to fix the index m^0 in some marked nonsingular point $p = P(\psi_0, \zeta), x = X(\psi_0, \zeta)$ on M_0^2 and then to find the change of the argument of the determinant of the 2×2 matrix $\mathbb{C}_\varepsilon^{(1,2)} = (\mathcal{C} - i\varepsilon\mathcal{B}) \equiv (\dot{X} - i\varepsilon\dot{P}, X_\psi - i\varepsilon P_\psi)$ along one of the paths described below joining the marked point $p = P(\psi_0, \zeta)$, $x = X(\psi_0, \zeta)$ with the given nonsingular point $p = P(\psi, t), x = X(\psi, t)$, more precisely

$$m(\psi, t) = m(\psi_0, t_0) + \Delta m, \quad \Delta m = \frac{1}{\pi}\lim_{\varepsilon\to+0}\operatorname{Arg}\det(\dot{\mathrm{X}} - \mathrm{i}\varepsilon\dot{\mathrm{P}}, \mathrm{X}_\psi - \mathrm{i}\varepsilon\mathrm{P}_\psi)(\psi, \mathrm{t})|_{\psi_0,\mathrm{t}_0}^{\psi,\mathrm{t}}. \tag{5.2}$$

From the definition (5.2) it follows the fact that we used before: the index can change (jump) only crossing a focal point. In fact, if the point (ψ, t_ψ) is a regular point then $det(\dot{X}, X_\psi)$ is different from zero so the increment of the argument of the determinant goes to zero when ε goes to zero, otherwise, if the determinant of $(\dot{X}, X_\psi) = 0$, as it happens in a focal point, then the increment of the determinant in (5.2) is different from zero when ε goes to zero. We know (see Corollary from Lemma 4) that the Jacobian $J = \det\mathcal{C}(\psi, \zeta) > 0$ for small enough positive ζ. Thus all the points on the front γ_ζ are nonsingular. So we choose one of the point $(P(\psi_0, \zeta), X(\psi_0, \zeta))$ and put the Maslov index $m(\psi_0, \zeta) = 0$. It is possible (and natural) to use one of the following paths. 1) To move first along the trajectory $P(\psi_0,\tau), X(\psi_0,\tau)$ starting from $\tau=\zeta$ until $\tau=t$, then to move from the point $P(\psi_0,t), X(\psi_0,t)$ along the curve Γ_t to the point $P(\psi,t), X(\psi,t)$ changing the angle ψ. As we will see below this choice is not very convenient from the point of view of computer realization. The choice 2) is to move first from the point with the angle ψ_0 to the point with the angle ψ along the curve Γ_ζ, and then to move along the trajectory with the angle ψ, changing time from ζ to t.

It could happen that during the motion along some closed path on M_t^2 one can get nontrivial increments of the argument of the determinant of the matrix $\mathbb{C}_\varepsilon^{(1,2)}=(\mathcal{C}-i\varepsilon\mathcal{B})\equiv(\dot{X}-i\varepsilon\dot{P}, X_\psi-i\varepsilon P_\psi)$ and nontrivial Maslov index of this closed path. The following lemma shows that it is not so.

Lemma 5. *The increment of argument of the determinant of the matrix* $\mathbb{C}_\varepsilon^{(1,2)}=(\mathcal{C}-i\varepsilon\mathcal{B})\equiv(\dot{X}-i\varepsilon\dot{P}, X_\psi-i\varepsilon P_\psi)$ *along any closed path on* M_t^2 *is equal to zero. Thus the Maslov index of any closed path on* M_t^2 *is also equal to zero.*

PROOF First let us show that Lemma is true for $t=0$. Obviously all closed paths on M_0^2 are homotopic one to another. Let us choose as a path the curve Γ_ζ, $\delta>\zeta>0$. According to Corollary from Lemma 4 the Jacobian $J=\det\mathcal{C}(\psi,\zeta)>0$ for ζ small enough. Thus the increment of the argument of the determinant of the matrix $\mathcal{C}-i\varepsilon\mathcal{B}$ over Γ_ζ is zero and Maslov index of this path is also zero. Hence, according to the topological property of Maslov index, it is equal to zero for any other closed path on M_0^2. Any path on M_t^2 can be obtained from some path from M_0^2 by means of the canonical transformation (the flow) $g^t_{\mathcal{H}}$. But this transformation preserves the Maslov index of a closed path. □

Taking into account this Lemma, choosing the second way for the calculation of the Maslov index and putting $m(\psi_0,\zeta)=0$ for some small positive $\zeta<\delta$ and some fixed fixed ψ_0, (and hence for *any* ψ) we can write for the index $m(\psi,t)$:

$$m(\psi,t)=\Delta m,\quad \Delta m=\frac{1}{\pi}\lim_{\varepsilon\to+0}\mathrm{Arg}\det\big(\dot{\mathrm{X}}-\mathrm{i}\varepsilon\dot{\mathrm{P}}, \mathrm{X}_\psi-\mathrm{i}\varepsilon\mathrm{P}_\psi\big)(\psi,\mathrm{t})\big|_{\psi,\zeta}^{\psi,\mathrm{t}}. \tag{5.3}$$

The Corollary of Lemma 3 allows us (in the considered problem) to simplify the definition (5.2). Let us analyze the determinant

$$\begin{aligned}&\det\big(\dot{\mathrm{X}}-\mathrm{i}\varepsilon\dot{\mathrm{P}}, \mathrm{X}_\psi-\mathrm{i}\varepsilon\mathrm{P}_\psi\big)(\mathrm{t},\psi)\\ &=\ \det\big(\dot{\mathrm{X}},\mathrm{X}_\psi\big)-\mathrm{i}\varepsilon\det\big(\dot{\mathrm{X}},\mathrm{P}_\psi\big)-\mathrm{i}\varepsilon\det\big(\dot{\mathrm{P}},\mathrm{X}_\psi\big)-\varepsilon^2\det\big(\dot{\mathrm{P}},\mathrm{P}_\psi\big).\end{aligned} \tag{5.4}$$

Since $\det(\dot{\mathrm{X}},\mathrm{P}_\psi)$ is not equal to zero and the term $\det(\dot{\mathrm{P}},\mathrm{X}_\psi)=0$ in the focal point, then the third term of equation (5.4) can be omitted. Similarly taking into account the fact that the jump of the index is an integer number it easy to show that the term $\varepsilon^2\det(\dot{\mathrm{P}},\mathrm{P}_\psi)$ does not play any role in the calculation of Δm and also can be omitted. Thus instead of the determinant $\det(\dot{\mathrm{X}}-\mathrm{i}\varepsilon\dot{\mathrm{P}}, \mathrm{X}_\psi-\mathrm{i}\varepsilon\mathrm{P}_\psi)(\mathrm{t},\psi)$ we can use the determinant $\det(\dot{\mathrm{X}},\mathrm{X}_\psi-\mathrm{i}\varepsilon\mathrm{P}_\psi)(\psi,\mathrm{t})$.

Thus one needs to found the jumps of m during the crossing of the focal points and find

$$\Delta m=\frac{1}{\pi}\lim_{\varepsilon\to+0}\mathrm{Arg}\det\big(\dot{\mathrm{X}},\mathrm{X}_\psi-\mathrm{i}\varepsilon\mathrm{P}_\psi\big)(\psi,\tau)\big|_{\tau=\mathrm{t}_\psi-\Delta\mathrm{t}}^{\tau=\mathrm{t}_\psi+\Delta\mathrm{t}}, \tag{5.5}$$

where $\Delta t>0$ is small enough, and t_ψ is the time at which the trajectory crosses the focal point (with coordinates ψ, t_ψ)with the angle ψ. There may exist several such t_ψ, but all of them, according to the point 1 of the Corollary of Lemma 3, are isolated with respect to t. In fact the derivative of the Jacobian is different from 0 in a focal point so the Jacobian has an isolated zero and so those zeros cannot accumulate in one point. Now we use again this Corollary. Obviously we can change τ with t^F in the right hand side of (5.5). But the term $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)(\psi^{\mathrm{F}}, \mathrm{t}^{\mathrm{F}})$ characterizes the increasing or decreasing of the first term. Hence if $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)(\psi^{\mathrm{F}}, \mathrm{t}^{\mathrm{F}}) < 0$, then the argument of the complex vector in (5.5) changes on the upper half plane from $O(\varepsilon)$ to $\pi + O(\varepsilon)$ and $\Delta m = 1$. If $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi)(\psi^{\mathrm{F}}, \mathrm{t}^{\mathrm{F}}) > 0$ then the argument of the vector in (5.5) changes from $\pi + O(\varepsilon)$ to $2\pi + O(\varepsilon)$ and *again* $\Delta m = 1$. Thus we obtain the following important result.

Lemma 6. *The Maslov index $m(\psi, t)$ of any nonsingular point $(p = P(\psi, t), x = X(\psi, t)) \in \Gamma_t$ with the projection $x = X(\psi, t)$ on the front is equal to number of focal points lying on the trajectory $P(\psi, \zeta), X(\psi, \zeta), \zeta \in (+0, t)$, i.e. it coincides with the Morse index of this trajectory.*

5.3 The behavior of the front near the focal points.

For the future developments it is useful to have the description of the wave front γ_t in a neighborhood of the focal points. The focal points which are ends of arcs γ_t^j of the wavefront belong to the *caustics*, a well known concept in geometrical optics and in the space-time wave theory. It is an important fact that in our problem the caustics do not depend on the time because the family of manifolds (bands) M_t^2 is invariant with respect to the phase shift $g_{\mathcal{H}}^t$. One may distinguish two types of sets organized by the focal points with stable and unstable structures with respect to small changes of the Lagrangian manifold M_t^2 (or of the functions $X(\psi, t), P(\psi, t)$). The first type is under the so-called "general position", there exist only a finite numbers of them and in the considered 2-D situation namely there are only two types: the so-called *fold* and *cusp*[25]. Sometimes there exist also different focal sets with unstable structure. For instance the circle $\Gamma_0 = p = \mathbf{n}(\psi), x = 0$ on M_0^2, has the point $x = 0$ as the projection Γ_0 from M_T^2 (or $\mathbb{R}_{p,x}^4$) to $\mathbb{R}_x^2$. Rotating a little the coordinate system in $\mathbb{R}_{p,x}^4$ one can obtain a small ellipse on $\mathbb{R}_x^2$ instead of the point $x = 0$. We show below that namely this "unstable singular" circle determines the localized functions in the asymptotic constructions.

Taking into account the smoothness of the vector-functions $X(\psi, t)$ one can easily describe the behavior of the wavefront near the focal points. Namely let us fix the time t and let $\psi^F = \psi^F(t)$ define the angle (coordinate on Γ_t) of the focal point $X^F = X(\psi^F, t)$. We put $y = \Delta\psi = \psi - \psi^F$. Let $n \geq 2$ be the minimum degree of the Taylor expansion of the function X around ψ with increment y. We say that the focal point X^F is not completely degenerate if $n \neq \infty$. From the point of view of this definition the point $p = P_0 \equiv \mathbf{n}(\psi), x = X_0 = 0$ is a complete degenerate one.

Shifting the origin into the focal point and rotating the coordinates we can "kill" the second component of the n-th derivative vector $X_\psi^{(n)}(\psi^F(t), t)$ and write $x_1' = ay^n + O(y^{n+1})$, $x_2' = by^k + O(y^{k+1})$. Here $a \neq 0$, $b \neq 0$ are Taylor coefficients, the integer $k > n$ and the prime indicates the new coordinates; actually a and b depend on the time t, but now for us it is not important. It is clear that the previous Lemmas are true also in the new coordinates.

Lemma 7. *In the non degenerate case only one opportunity is possible: $k = n + 1$.*

PROOF. In new coordinates

$$X' = \begin{pmatrix} ay^n + O(y^{n+1})) \\ by^k + O(y^{k+1}) \end{pmatrix}, \quad X'_\psi = \begin{pmatrix} any^{n-1} + O(y^n) \\ bky^{k-1} + O(y^k) \end{pmatrix},$$

Since the vector P' is orthogonal to X'_ψ everywhere, and, according to the conservation of the Hamiltonian $\mathcal{H}$, $P' \neq 0$, then we can write

$$P' = q\begin{pmatrix} -bky^{k-n} + O(y^{k-n+1})) \\ an + \tilde{a}y + O(y^2) \end{pmatrix} \text{and} \qquad P'_\psi = q\begin{pmatrix} -bk(k-n)y^{k-n-1} + O(y^{k-n})) \\ \tilde{a} + O(y) \end{pmatrix},$$

where the factor $q(t) \neq 0$, $q\tilde{a}(t)$ is proportional to the Taylor coefficient just after $qa(t)$. Taking into account the point 1) of Lemma 4 we immediately find $\tilde{a}(t) = 0$. But from this it follows that if $n, k \neq \infty$ and $k > n+1$ then in the focal point $P'_\psi = 0$ which contradicts the point 2) of Lemma 4. □.

Corollary. *Let $n \neq \infty$ then in the neighborhood of the focal point $x = X^F$ in new coordinates*

$$X' = \begin{pmatrix} ay^n + O(y^{n+1})) \\ by^{n+1} + O(y^{n+2}) \end{pmatrix}, \quad X'_\psi = \begin{pmatrix} any^{n-1} + O(y^n) \\ b(n+1)y^n + O(y^{n+1}) \end{pmatrix},$$

$$P' = q\begin{pmatrix} -b(n+1)y + O(y^2)) \\ an + O(y^2) \end{pmatrix}, \qquad P'_\psi = q\begin{pmatrix} -b(n+1) + O(y)) \\ O(y) \end{pmatrix}, \tag{5.6}$$

and

$$\det(\dot{X}, X_\psi) = -\frac{qan}{|q|}C(X^F)y^{n-1} + O(y^n), \qquad \det(\dot{X}, P_\psi) = |q|b(n+1)C_F + O(y), \tag{5.7}$$

where it was before $C_F = C(X^F)$. The last two equalities do not depend on the choice of the coordinates.

Thus in agreement with the Lemma 3 the determinant $\det(\dot{X}, P_\psi)$ does not change its sign in a neighborhood of a non degenerate focal point.

Finally in the non completely degenerate case, in a neighborhood of the focal point, we have $X'_1 = ay^n + O(y^{n+1})$, $X'_2 = by^{n+1} + O(y^{n+2})$, $P'_1 = -qb(n+1)y + O(y^2)$, $P'_2 = qan + O(y^2)$.

Omitting the higher corrections we find the equation for the part of the front γ_t in a neighborhood of the focal point x^F: $x'_1 = ay^n$, $x'_2 = by^{n+1}$. The sign of ab for odd n and the sign of a for even n defines the direction of the passage from higher to lower leaves for odd n and from left to right leaves for even n. Let us note also that in the general case n is equal to 2 or 3 only [25].

It is convenient to express the coefficients a, b, q via P, X and their derivatives in the focal point $\psi^F(t)$. Putting in formulas (5.6) $y = 0$ we find

$$P'_1 = 0, \qquad P'_2 = qan, \qquad P'_{1\psi} = -bq(n+1), \text{ for } \psi = \psi^F(t),$$

and

$$a = \frac{X_1'^{(n)}}{n!} \equiv \frac{1}{n!}\frac{\partial^n X'_1}{\partial \psi^n}, \quad b = -\frac{P'_{1\psi}X_1'^{(n)}}{(n-1)!(n+1)P'_2}, \quad q = \frac{P'_2(n-1)!}{X_1'^{(n)}} \text{ for } \psi = \psi^F(t).$$

The directions of the vectors P and $\dot{X}$ coincide in the focal points. Thus we see that the coordinates with index "prime" could be chosen to be the coordinates introduced in (4.11). This gives: $P'_2 = |P^F| \equiv \frac{C_0}{C_F}$, $P'_1 = \langle \mathbf{k}_1, P^F_\psi \rangle \equiv -\det(\dot{X}^F, P^F_\psi)/C_F \equiv -\tilde{J}_F/C_F$, $X_1'^{(n)} = \langle \mathbf{k}_1, X^{(n)F} \rangle \equiv -\det(\dot{X}^F, X^{(n)F})/C_F \equiv -J_F^{(n)}/C_F$ and

$$a = -\frac{J_F^{(n)}}{n!C_F}, \quad b = -\frac{n\tilde{J}_F J_F^{(n)}}{(n+1)!C_F C_0}, \quad q = -\frac{C_0(n-1)!}{J_F^{(n)}}. \tag{5.8}$$

5.4 The jumps of the Maslov index along the front.

Let us find the jumps Δm of the Maslov index during the passage through the focal points along the front. We fix the time $t > 0$ and consider the path to cross non degenerate focal points (studied above) starting from the angle $\psi_0 - \delta$ and ending at the angle $\psi_0 + \delta$.

Lemma 8. *The following equalities are true : for odd n $\Delta m = 0$, for even n and* $\mathrm{sign}(\tilde{\mathrm{J}}_{\mathrm{F}}\mathrm{J}_{\mathrm{F}}^{(\mathrm{n})}) = \pm 1$ $\Delta \mathrm{m} = \pm 1$.

PROOF. Similarly to the proof of Lemma 4, taking into account the inequality $\det(\dot{\mathrm{X}}, \mathrm{P}_\psi) \neq 0$ instead of (5.2) we can write:

$$\Delta m = \frac{1}{\pi} \lim_{\varepsilon \to +0} \mathrm{Arg} \det(\dot{\mathrm{X}}, \mathrm{X}_\psi - i\varepsilon \mathrm{P}_\psi)(\psi, \mathrm{t})|_{\psi_0-\delta,\mathrm{t}}^{\psi_0+\delta,\mathrm{t}} = \tag{5.9}$$

$$\frac{1}{\pi} \lim_{\varepsilon \to +0} \mathrm{Arg}\,[-\mathrm{qanC}(\mathrm{X}^{\mathrm{F}})\mathrm{y}^{\mathrm{n}-1} + \mathrm{O}(\mathrm{y}^{\mathrm{n}}) - i\varepsilon(\mathrm{q}^2\mathrm{C}(\mathrm{X}^{\mathrm{F}})\mathrm{b}(\mathrm{n}+1) + \mathrm{O}(\mathrm{y}))]|_{\mathrm{y}=-\delta}^{\mathrm{y}=\delta} = \tag{5.10}$$

$$\frac{1}{\pi} \lim_{\varepsilon \to +0} \mathrm{Arg}\,[-\mathrm{y}^{\mathrm{n}-1} - i\varepsilon(\frac{\mathrm{qb}(\mathrm{n}+1)}{\mathrm{an}})]|_{\mathrm{y}=-\delta}^{\mathrm{y}=\delta}. \tag{5.11}$$

We see now that the complex vector-function $-y^{n-1} - i\varepsilon(\frac{qb(n+1)}{a})$ lies in one half plane for even values of n for each y and in one quadrant for odd values of n. So for n even one has $\Delta m = -1$ if $qab > 0$ and $\Delta m = 1$ if $qab < 0$ while for n odd $\Delta m = 0$. To finish the proof it is enough to take into account formulas (5.8). □

Coming back to the original variables we can make the following conclusion.

Lemma 9. *During the motion along the front γ_t*

1) the Maslov index does not change if the path does not cross the focal points or if the Jacobian $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)$ does not change the sign after the passage through the focal point;

2)let the Jacobian $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)$ change sign after the passage through the focal point then $\Delta m = 1$ if the signs of $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)$ and $\tilde{J} = \det(\dot{\mathrm{X}}, \mathrm{P}_\psi)$ coincide in the end of the path and $\Delta m = -1$ if the signs of $J = \det(\dot{\mathrm{X}}, \mathrm{X}_\psi)$ and $\tilde{J} = \det(\dot{\mathrm{X}}, \mathrm{P}_\psi)$ are different.

5.5 Canonical planes in the phase space, nonsingular and singular maps.

To construct the asymptotic solution of problem (1.1), (1.2) in the neighborhood of the focal points we need some additional constructions, related with the fronts, maps covering Lagrangian bands M_t^2, indices of these maps etc. Let us describe also them briefly, using the notations introduced in [28].

The 2-D planes with the *focal* coordinates $x^{(1,2)} = (x_1, x_2)$, $x^{(1,0)} = (x_1, p_2)$, $x^{(0,2)} = (p_1, x_2)$, $x^{(0,0)} = (p_1, p_2)$ in the phase space $\mathbb{R}^4_{p,x}$ are called symplectic canonical planes. It is convenient to introduce the multi-indices $I = (1, 2)$ corresponding to the canonical plane (x_1, x_2), $I = (1, 0)$ to (x_1, p_2), $I = (0, 2)$ to (p_1, x_2), $I = (0, 0)$ to (p_1, p_2). [2] We denote also $p^{(1,2)} = (p_1, p_2)$, $p^{(1,0)} = (p_1, -x_2)$, $p^{(0,2)} = (-x_1, p_2)$, $p^{(0,0)} = (-x_1, -x_2)$. We call I the index of singularity. It is convenient to mark the canonical plane by the corresponding index I and write $\mathcal{R}_I^2$.

[2]These multi-indices indicate the replacement of the coordinate x_j corresponding to the zero entry of the pair (a_1, a_2) by the momentum p_j (*with the same number* j).

According to a general property of Lagrangian manifolds one can cover M_t^2 by the maps $\Omega_j^{I_j}$ with the numbers j such that there exists a one-to-one map from $\Omega_j^{I_j}$ to its projection to the canonical plane $\mathbb{R}^2_{I_j}$. This means the following. Along with the matrices $\mathcal{B}^{(1,2)} = \mathcal{B}$, $\mathcal{C}^{(1,2)} = \mathcal{C}$ it is convenient to introduce the matrices

$$\mathcal{B}^{(0,2)}(\psi,\tau) = \begin{pmatrix} -\frac{\partial X_1}{\partial t} & -\frac{\partial X_1}{\partial \psi} \\ \frac{\partial P_2}{\partial t} & \frac{\partial P_2}{\partial \psi} \end{pmatrix}, \quad \mathcal{C}^{(0,2)}(\psi,\tau) = \begin{pmatrix} \frac{\partial P_1}{\partial t} & \frac{\partial P_1}{\partial \psi} \\ \frac{\partial X_2}{\partial t} & \frac{\partial X_2}{\partial \psi} \end{pmatrix},$$
$$\mathcal{B}^{(1,0)}(\psi,\tau) = \begin{pmatrix} \frac{\partial P_1}{\partial t} & \frac{\partial P_1}{\partial \psi} \\ -\frac{\partial X_2}{\partial t} & -\frac{\partial X_2}{\partial \psi} \end{pmatrix}, \quad \mathcal{C}^{(1,0)}(\psi,\tau) = \begin{pmatrix} \frac{\partial X_1}{\partial t} & \frac{\partial X_1}{\partial \psi} \\ \frac{\partial P_2}{\partial t} & \frac{\partial P_2}{\partial \psi} \end{pmatrix}. \tag{5.12}$$

$$\mathcal{B}^{(0,0)}(\psi,\tau) = -\mathcal{C} = \begin{pmatrix} -\frac{\partial X_1}{\partial t} & -\frac{\partial X_1}{\partial \psi} \\ -\frac{\partial X_2}{\partial t} & -\frac{\partial X_2}{\partial \psi} \end{pmatrix}, \quad \mathcal{C}^{(0,0)}(\psi,\tau) = \mathcal{B} = \begin{pmatrix} \frac{\partial P_1}{\partial t} & \frac{\partial P_1}{\partial \psi} \\ \frac{\partial P_2}{\partial t} & \frac{\partial P_2}{\partial \psi} \end{pmatrix}. \tag{5.13}$$

The matrices $\mathcal{C}^I$ give the Jacobians $J^I(\psi,\tau) = \det \mathcal{C}^I$. Then in each map $\Omega_j^{I_j}(\tau)$ $J^{I_j} \neq 0$. The maps with the indices $I_j = (1,2)$ are nonsingular ones, all others are singular ones with the focal coordinates x^{I_j}. Note that *for practical applications sometimes it is useful to choose some rotated coordinates* (x_1', x_2') *and* (p_1', p_2') *in some maps* $\Omega_j^{I_j}$. It is important to remember, that *the Jacobians* $J = J^{(1,2)}$ *and* $J^{(0,0)}$ *are invariant with respect to rotations, but the Jacobians* $J^{(0,2)}$, $J^{(1,0)}$ *are not.* Actually in the considered problem the maps with index $I = (0,0)$ are not needed.

Lemma 10. *For any time t there exists a finite covering of the neighborhood of Γ_t from Lagrangian manifold M_t^2 by the maps $\Omega_j^{I_j}$ with the indices $I_j = (1,2)$, $I_j = (1,0)$, $I_j = (0,2)$.*

PROOF. It is enough to prove that at least one of the Jacobians $J^{(0,2)}$ or $J^{(1,0)}$ is not equal to zero in each focal point. Assume that both $J^{(0,2)} = 0$ and $J^{(1,0)} = 0$ in the point ψ^F, t^F. Since $X_\psi = 0$ in the focal point, this means that $\dot{X}_1 P_{2\psi} = \dot{X}_2 P_{1\psi} = 0$ and $\det(\dot{X}, P_\psi)$=0. But according to Lemmas 2,3 the vector $P_\psi \neq 0$ in the focal point and P_ψ is orthogonal to $\dot{X}$, which is nonzero everywhere. This contradiction proves this lemma. □

5.6 The Maslov index of a singular map.

The last object we need is the Maslov index of chains of maps $\{\Omega_j^{I_j}(t)\}$. To find it one has to fix some nonsingular point $\mathbf{r}(\tilde{\psi}, \tilde{\tau})$ in the corresponding map $\Omega_j^{I_j}$ and construct one of the following matrices $\mathbb{C}_\varepsilon^{(1,0)}, \mathbb{C}_\varepsilon^{(0,2)}$ with the elements of the matrices $\mathcal{B}, \mathcal{C}$ defined in (5.1):

$$\mathbb{C}_\varepsilon^{(1,2)} = \mathcal{C} - i\varepsilon\mathcal{B} = \begin{pmatrix} \mathcal{C}_{11} - i\varepsilon\mathcal{B}_{11} & \mathcal{C}_{12} - i\varepsilon\mathcal{B}_{12} \\ \mathcal{C}_{21} - i\varepsilon\mathcal{B}_{21} & \mathcal{C}_{22} - i\varepsilon\mathcal{B}_{22} \end{pmatrix}$$

$$\mathbb{C}_\varepsilon^{(1,0)} = \begin{pmatrix} \mathcal{C}_{11} - i\varepsilon\mathcal{B}_{11} & \mathcal{C}_{12} - i\varepsilon\mathcal{B}_{12} \\ (\mathcal{C}_{21} - i\varepsilon\mathcal{B}_{21})\cos\eta + (\mathcal{B}_{21} + i\varepsilon\mathcal{C}_{21})\sin\eta & (\mathcal{C}_{22} - i\varepsilon\mathcal{B}_{22})\cos\eta + (\mathcal{B}_{22} + i\varepsilon\mathcal{C}_{22})\sin\eta \end{pmatrix}$$

$$\mathbb{C}_\varepsilon^{(0,2)} = \begin{pmatrix} (\mathcal{C}_{11} - i\varepsilon\mathcal{B}_{11})\cos\eta + (\mathcal{B}_{11} + i\varepsilon\mathcal{C}_{11})\sin\eta & (\mathcal{C}_{12} - i\varepsilon\mathcal{B}_{12})\cos\eta + (\mathcal{B}_{12} + i\varepsilon\mathcal{C}_{12})\sin\eta \\ \mathcal{C}_{21} - i\varepsilon\mathcal{B}_{21} & \mathcal{C}_{22} - i\varepsilon\mathcal{B}_{22} \end{pmatrix}$$

$$\mathbb{C}_\varepsilon^{(0,0)} = \begin{pmatrix} (\mathcal{C}_{11} - i\varepsilon\mathcal{B}_{11})\cos\eta + (\mathcal{B}_{11} + i\varepsilon\mathcal{C}_{11})\sin\eta & (\mathcal{C}_{12} - i\varepsilon\mathcal{B}_{12})\cos\eta + (\mathcal{B}_{12} + i\varepsilon\mathcal{C}_{12})\sin\eta \\ (\mathcal{C}_{21} - i\varepsilon\mathcal{B}_{21})\cos\eta + (\mathcal{B}_{21} + i\varepsilon\mathcal{C}_{21})\sin\eta & (\mathcal{C}_{22} - i\varepsilon\mathcal{B}_{22})\cos\eta + (\mathcal{B}_{22} + i\varepsilon\mathcal{C}_{22})\sin\eta \end{pmatrix}.$$

These matrices are not degenerate, for any $\eta \in [0, \pi/2]$ and for any positive ε, in the maps with indexes $(1,0)$, $(0,2)$ and $(0,0)$ respectively [3]. Obviously $\mathbb{C}_\varepsilon^I|_{\eta=0} = \mathbb{C}_\varepsilon^{(1,2)} \equiv \mathcal{C} - i\varepsilon\mathcal{B}$ and $\mathbb{C}_\varepsilon^I|_{\eta=0} = \mathcal{C}^I - i\varepsilon\mathcal{B}^I$. These matrices determine a continuous non degenerate transition from the matrix $\mathbb{C}_\varepsilon^{(1,2)}$ to the matrix $\mathcal{C}^{(1,2)} - i\varepsilon\mathcal{B}^{(1,2)}$. The corresponding determinants $\mathbb{J}_\varepsilon^{(1,0)} = \det\mathbb{C}_\varepsilon^{(1,0)}$ or $\mathbb{J}_\varepsilon^{(0,1)} = \det\mathbb{C}_\varepsilon^{(0,2)}$ are not equal to zero. Let $m(\tilde{\psi}, \tilde{\tau})$ be the Maslov index of the point $\mathbf{r}(\tilde{\psi}, \tilde{\tau})$. Then the index $\mathbf{m}(\Omega_j^{I_j})$ of the map $\Omega_j^{I_j}$ is

$$\mathbf{m}(\Omega_j^{I_j}) = m(\tilde{\psi}, \tilde{\tau}) + \frac{1}{\pi} \lim_{\varepsilon \to +0} \mathrm{Arg}\mathbb{J}_\varepsilon^{\mathrm{I}}|_{\eta=0}^{\eta=\pi/2}. \tag{5.14}$$

This definition *does not depend* mod 4 *on the choice of the point* $(\tilde{\psi}, \tilde{\tau})$ *in a given map.* The calculation of the index $\mathbf{m}(\Omega_j^{I_j})$ could be technically complicated even in quite simple situations. But taking into account the fact that we can restrict ourselves to maps with $I = (1,0)$ and $I = (0,2)$ it is possible to simplify the application of this formula.

Lemma 11. *One can always find a nonsingular point* $\mathbf{r}(\tilde{\psi}, \tilde{\tau})$ *in the map* $\Omega_j^{I_j}$ *with* $I_j = (1,0)$ *or* $(0,2)$ *such that the sign of the Jacobian* $J(\tilde{\psi}, \tilde{\tau})$ *coincides with the sign of the Jacobian* $J^{I_j}(\psi, \tau)$ *in this map. Then the second term in* (5.14) *is equal to zero and*

$$\mathbf{m}(\Omega_j^{I_j}) = m(\tilde{\psi}, \tilde{\tau}).$$

Since the sign of the Jacobian J^{I_j} *does not depend on the point in the* $\Omega_j^{I_j}$, *then one can evaluate it in any point, for instance in a focal one.*

PROOF. It is obvious that the second term in (5.14) can be equal to $0, 1$ or -1 only. Consider for ε the Jacobian $\mathbb{J}_0^I$. A simple calculation gives $\mathbb{J}_0^I = J\cos\eta + J^I\sin\eta$. In the interval $[0, \pi/2]$ this function has no zero if J and J^I have the same signs and one zero in the opposite situation. Hence including the parameter ε gives only the rule of bypass of the zero point on the complex plane. It is not necessary to use this rule in the case when $JJ^I > 0$. Thus one obtains $\lim_{\varepsilon\to+0}\mathrm{Arg}\mathbb{J}_\varepsilon^{\mathrm{I}}|_{\eta=0}^{\eta=\pi/2} = 0$ if a point $\mathbf{r}(\tilde{\psi}, \tilde{\tau})$ is chosen in the way prescribed in the Lemma. From the other side, according to Lemma 3 and its Corollary, the existence of the focal point in any focal map means

[3] The objects with singular index $(0,0)$ are not needed in the considered problem we present $\mathbb{C}_\varepsilon^{(0,0)}$ for completeness.

the existence of nonsingular points with positive and negative signs of the Jacobian J. □

Corollary. *The index* $\mathbf{m}(\Omega_j^{I_j}(t))$ *of the singular map* $\Omega_j^{I_j}$ *coincides with the index of any nonsingular point* $m(\psi,t)$ *on the front* Γ_t *where the Jacobians* J *and* J^{I_j} *have the same sign.*

5.7 Germs of Lagrangian manifolds and their properties.

One can meet in many asymptotical problems having fast oscillating solution the geometrical objects described above . We see that the solution of the problem under consideration decays quite rapidly outside of some neighborhood of the front γ_t. This gives the opportunity to use the ideas of the boundary layer expansions [31] and the "complex germ theory" [19]. Their geometrical realizations can be accomplished in strip (germ) by linearizing the Lagrangian manifold M_t^2 near the curve Γ_t.

Definition 1. *For each fixed t we call the linear germ corresponding to the manifold* M_T^2 *a vector fiber bundle in the phase space with base coinciding with the front* $\Gamma_t = (x = X(\psi,t), p = P(\psi,t))$ *and fibers generated by the vectors* $\dot{X}, \dot{P}$.

Denote $\alpha \in \mathbb{R}$ the coordinate on the bundle (which is the linear analogue of the proper time τ), then we can define a family of manifolds Λ_t^2 as a strip in a neighborhood of the front Γ_t in the phase space $\mathbb{R}^4_{p,x}$

$$\Lambda_t^2 = \{p = \mathrm{P}(\psi,t,\alpha) \equiv P(\psi,t) + \dot{P}(\psi,t)\alpha, \quad x = \mathrm{X}(\psi,t,\alpha) \equiv X(\psi,t) + \dot{X}(\psi,t)\alpha\}, \tag{5.15}$$

where $\psi \in S^1 = [0,2\pi]$, $|\alpha| < \alpha_0$ are the coordinates in Λ_t^2. M_t^2 can be approximated by Λ_t^2, the parameter α is used for linearizing the functions $X(\psi,t+\alpha), P(\psi,t+\alpha)$. α defines a shift of the time near the front Γ_t. Taking into account these facts it is easy to prove the following proposition.

Lemma 12. *1) With an error of the order* $O(\alpha^2)$ *the manifold* Λ_t^2 *is obtained from* Λ_0^2 *by means of a shift of time t along the trajectories of the phase flow with the Hamiltonian* $\mathcal{H} = H|p|C(x)$. *2) For the matrices* $\mathrm{B}(\psi,t,\alpha) = \frac{\partial \mathrm{P}}{\partial(\alpha,\psi)}$, $\mathrm{C}(\psi,t,\alpha) = \frac{\partial \mathrm{X}}{\partial(\alpha,\psi)}$ *the following equalities are true* $\mathrm{B} = \mathcal{B} + O(\alpha), \mathrm{C} = \mathcal{C} + O(\alpha);\ {}^t\mathrm{CB} = {}^t\mathrm{BC} + O(\alpha)$, *where as before* $\mathcal{B} = \frac{\partial P}{\partial(t,\psi)}$, $\mathcal{C} = \frac{\partial X}{\partial(t,\psi)}$. *The last equality means that the manifold (band)* Λ_t^2 *is (almost) Lagrangian mod* $O(\alpha)$, *the statement 1) means that it is (almost) invariant mod* $O(\alpha)$.

PROOF. Consider the Hamilton equations $\dot{x} = \mathcal{H}_p, \dot{p} = -\mathcal{H}_x$. We expand the derivatives of the Hamiltonian $\mathcal{H}$ around the point $X(\psi,t)$, $P(\psi,t)$ and use the variational system for the evolution of (x,p). This gives

$$\dot{X} + \alpha\ddot{X} - (\mathcal{H}_p(X(\psi,t), P(\psi,t)) + \mathcal{H}_{pp}\alpha\dot{P} + \mathcal{H}_{px}\alpha\dot{X}) = O(\alpha^2)$$
$$\dot{P} + \alpha\ddot{P} + \mathcal{H}_x(X(\psi,t), P(\psi,t)) + \mathcal{H}_{xp}\alpha\dot{P} + \mathcal{H}_{xx}\alpha\dot{X} = O(\alpha^2)$$

and we get the result. □

Since the germ Λ_t^2 is an approximation of M_t^2, almost all the previous propositions, geometrical definitions and constructions (like Maslov and Morse index) related to the band M_t^2 are true for the band (germ) Λ_t^2. From the other side obviously one does not need any additional objects besides the family of curves (fronts in the phase space)

Γ_t to construct both the Lagrangian bands M_t^2 and their germs Λ_t^2. It also follows from formulas (4.4),(4.7),(4.14) that the leading term of the solution η is based only on these objects. Nevertheless the proof of (4.4),(4.7),(4.14) needs something more, and it seems that sometimes, for technical reasons, instead of the germ Λ_t^2, it is convenient to consider also another germ (fiberbundle) namely

$$\widetilde{\Lambda}_t^2 = \{p = \widetilde{\mathrm{P}}(\psi,t,\alpha) \equiv P(\psi,t)+(\dot{P}(\psi,t)-\lambda(\psi)P)\alpha, \quad x = \widetilde{\mathrm{X}}(\psi,t,\alpha) \equiv X(\psi,t)+\dot{X}(\psi,t)\alpha\} \tag{5.16}$$

where (see Lemma 2) $\lambda = \langle\frac{\partial C}{\partial x}(0), \mathbf{n}(\psi)\rangle$. This germ is also associated to the matrices $\widetilde{\mathrm{B}}(\psi,t,\alpha) = \frac{\partial \mathrm{P}}{\partial(\alpha,\psi)}$, $\widetilde{\mathrm{C}}(\psi,t,\alpha) = \mathrm{C}(\psi,t,\alpha) = \frac{\partial \mathrm{X}}{\partial(\alpha,\psi)}$. But now $\widetilde{\mathrm{B}} = \widetilde{\mathcal{B}} + O(\alpha)$, where $\widetilde{\mathcal{B}} = (\dot{P} - \lambda P, P_\psi)$. After these considerations it is possible to prove the following statement.

Lemma 13. *All previous proposition concerning the germ Λ_t^2 and matrices $\mathcal{B},\mathcal{C}$ are true for the germ $\widetilde{\Lambda_t^2}$ and matrices $\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}$.*

6 Geometrical asymptotic solution and the Maslov canonical operator.

The central mathematical result of our paper is the observation that the asymptotic solution of the problem (1.1) can be represented as an integral over $d\rho$ of the canonical Maslov operator $K^h_{\Lambda_t^2}$ with "semiclassical" parameter $h = \mu/\rho$, defined on the appropriate family of Lagrangian manifolds Λ_t^2, and acting on the function V (3.16) defining the initial localized perturbation. In some neighborhood of the front line γ_t the final formula has the form:

$$\eta = \mathrm{Re}\{(\sqrt{\frac{\mu \mathrm{C}_0}{2\pi \mathrm{i}}}\int_0^\infty \mathrm{K}^{\mu/\rho}_{\Lambda_t^2}(\sqrt{\rho}\tilde{\mathrm{V}}(\rho\mathbf{n}(\psi))\mathrm{d}\rho\} + \mathrm{o}(\mu), \tag{6.1}$$

and $\eta = o(\mu)$ outside of this neighborhood. The initial data (3.16), the representation of the asymptotic solution (4.4),(4.7) out of the neighborhood of the focal points as well as the future representation of the solution in a neighborhood of the focal points is only a realization of (6.1) in the corresponding domain of $\mathbb{R}^2_x$. As we said before the integral over parameter ρ in (6.1) plays a very important role: it implies the decay of the function η outside a neighborhood of the front and in turn allows one to simplify the objects and formulas appearing in the construction of the Maslov canonical operator. This simplification is based on the mentioned ideas of the "complex germ theory" [17, 19], but in a simpler "boundary layer" version. As we said before from the geometrical point of view it means that we can use the germs Λ_t^2 or $\widetilde{\Lambda}_t^2$ instead of the Lagrangian band M_t^2 in (6.1). In next subsections we shall describe the functions and other objects determining the operator $K^\mu_{\Lambda_t^2}$.

6.1 The functions on Lagrangian bands M_t^2.

a.The action-function. The Lagrangian property allows one to define on the family M_t^2, a function $s(\psi,t,\alpha)$ satisfying the equation $ds = \langle P, dX\rangle\big|_{M_t^2}$.

Lemma 14. *The phase $s(\psi,\tau)$ on M_t^2 is equal to $C_0\alpha$.*

PROOF. First let us find s on M_0^2. Then the coordinate α is the proper time: $\tau = \alpha$. But in this case we have

$$\int_{(0,0)}^{(\psi,\tau)} \langle P, dX\rangle = \int_{(0,0)}^{(0,\tau)} \langle P, \dot{X}\rangle dt + \int_{(0,\tau)}^{(\psi,\tau)} \langle P, X_\psi\rangle d\psi.$$

The second term in the last expression is equal to zero according to Lemma 3. Changing $\dot{X}$ by the right hand side from system (4.2) and using the integral of motion (4.3), we find (using the proper time τ) $s(\psi,\tau) = C_0\tau = C_0\alpha$. According to [16, 18, 28] to construct the action on the band M_t^2 one has to add $\int_0^t \mathcal{L}dt$ to $C_0\alpha$, where the Lagrangian $\mathcal{L} = \langle p, \mathcal{H}\rangle - \mathcal{H}$. But for the wave equation $\mathcal{L} = 0$, which gives the Lemma.□

The phase s is used for defining the phases associated with the projection in the singular maps Ω_j^I:

$$s^{(1,0)}(\psi,\tau) = C_0\tau - P_2(\psi,\tau)X_2(\psi,\tau), s^{(0,2)}(\psi,\tau) = C_0\tau - P_1(\psi,\tau)X_1(\psi,\tau).$$

b.The projection of the source function on the Lagrangian bands. The initial source function $V(y)$ defines on M_t^2 the function (more precisely the family of functions depending on parameter $\rho \in (0,\infty)$)

$$f(\rho,\psi) = \sqrt{\rho}\tilde{V}(\rho\mathbf{n}(\psi)). \tag{6.2}$$

We need also the smooth cut off function $\mathbf{e}_0(\alpha)$, $\mathbf{e}_0(\alpha) = 1$ for $|\alpha| < \alpha_0$, and $\mathbf{e}_0(\alpha) = 0$ for $|\alpha| > 2\alpha_0$ where α_0 is some small enough positive number. The product $f(\rho,\psi)\mathbf{e}_0(\alpha)$ defines a finite function on the band M_0^2. We continue it on all the family M_t^2 assuming that it does not depend on time t.

c. Functions in the maps Ω_j^I. In each map Ω_j^I the Jacobians J^I are not equal to zero. This means that one can construct the smooth solutions $(\psi_j^I(x^I), \tau_j^I(x^I)) \in \Omega^{I_j}$ of the system of equations

$$X^I(\psi,\tau) = x^I. \tag{6.3}$$

Let us emphasize that there could exist several angles $\psi_j^I(x^I)$ corresponding to one vector x^I. Although the manifold M_t^2 is invariant with respect to the shift $g_{\mathcal{H}}^t$ it does not mean that there is no dynamics on M_t^2. It only means that turning on the time dependence we transform the objects related to M_t^2 in a special way. Namely let us use from the beginning the coordinate α instead of the proper time τ. We have already mentioned that the points $\mathbf{r}(\psi,\alpha) \in M_T^2$ after the action of the transformation $g_{\mathcal{H}}^t$ pass to the points $\mathbf{r}(\psi,\alpha) \in M_t^2$ with shifted coordinate $\tau = \alpha + t$, but with the same angle ψ. Thus the equations (6.3) are changed by the equations

$$X^I(\psi,\alpha+t) = x^I; \tag{6.4}$$

The following trivial proposition is very useful.

Lemma 15. *Let $\psi_j(x^I), \tau_j(x^I)$ be the solution of the equations* (6.3) *in the map $\Omega_j^{I_j}(t)$ and let the point $\mathbf{r}(\psi,\alpha+t) \in M_t^2$ with coordinates $\psi, \alpha+t$ belong to the same map $\Omega_j^{I_j}$. Then the angle component ψ_j of the solution of the equation* (6.4) *does not depend on t: $\psi_j = \psi_j(x^{I_j})$ and $\alpha_j = \alpha_j(x^{I_j},t)$ is*

$$\psi_j(x^{I_j},t) = \psi_j(x^{I_j}), \quad \alpha_j(x^{I_j},t) = \tau_j(x^{I_j}) - t, \tag{6.5}$$

PROOF is obvious $\Box$.

Using (6.4) and (6.5) we can rewrite the action-functions and the Jacobians in the coordinates x^I. The action functions on M_t^2 have different behavior with respect to the shift $g_{\mathcal{H}}^t$. Namely the function $s = \alpha$, and the functions f,e are constant, this means that for each t in the coordinates ψ, α they have the same form. On the contrary the functions s^I and all the Jacobians [4] J^I take the forms:

$$\begin{aligned} s^{(1,0)}(\psi,\alpha,t) &= C_0\alpha - P_2(\psi,\alpha+t)X_2(\psi,\alpha+t), \\ s^{(0,2)}(\psi,\alpha+t) &= C_0\alpha - P_1(\psi,\alpha+t)X_1(\psi,\alpha+t), \end{aligned} \tag{6.6}$$

$$J^I = J^I(\psi,\alpha+t). \tag{6.7}$$

Now in the maps $\{\Omega_j^{I_j}(t)\}$ we want to pass from the coordinates ψ, α to the coordinates x^{I_j}. This gives us the actions, Jacobians etc. in the coordinates x^{I_j}:

$$\begin{aligned} S_j^{(1,2)}(x_1,x_2) &= \tau_j(x_1,x_2) - C_0 t, \\ S_j^{(1,0)}(x_1,p_2) &= \tau_j(x_1,p_2) - p_2 X_2(\psi_j(x_1,p_2),\tau_j(x_1,p_2)) - C_0 t, \\ S_j^{(0,2)}(p_1,x_2) &= \tau_j(p_1,x_2) - p_1 X_1(\psi_j(p_1,x_2),\tau_j(p_1,x_2)) - C_0 t, \\ J_j^{I_j}(x^{I_j}) &= J^{I_j}(\psi_j(x^{I_j}),\tau_j(x^{I_j})), \\ \mathbf{e}^t &= \mathbf{e}(\tau_j(x^{I_j}) - t). \end{aligned} \tag{6.8}$$

Let us emphasize that the complicated notations only reflect the situation: each map has it own number j and index of singularity I_j.

Finally we need to introduce the partition of unity with the maps $\Omega_j^{I_j}(t)$ covering Γ_t: the set of smooth functions $\mathbf{e}_j(\psi,t)$ associated with the covering $\{\Omega_j^{I_j}(t)\}$: $\mathrm{supp}\mathbf{e}_j(\psi) \in \Omega_j^{I_j}(t)$, $\sum_j \mathbf{e}_j(\psi) = 1$.

6.2 The generalized time-dependent Maslov canonical operator on the manifold M_t^2.

Now everything is ready to determine the Maslov canonical operator $K_{M_t^2}^h$, acting on the function $f(\rho,\psi)\mathbf{e}(\alpha)$, which is constant on the trajectories of the system (4.2) and depending on the parameter $h > 0$. It means that this function is the same in all points $P(\psi,t+\alpha), X(\psi,t+\alpha)$. Let $\{\Omega_j^{(I_j)}\}$ be a covering of the curve Γ_t. Let us divide the set of indices $\{j\}$ into three parts $\{\mathrm{j}(1,2),\mathrm{j}(1,0),\mathrm{j}(0,2)\}$ corresponding to

[4]To simplify notation we do not introduce a new symbol for time-shifted Jacobian.

the maps with indices of singularity $(1,2)$,$(1,0)$ and $(0,2)$ respectively. We put

$$\Psi(\rho,x_1,x_2,t) = K^h_{M^2_t}(f\mathbf{e})$$

$$\equiv \sum_{j\in \mathbf{j}(1,2)} e^{-\frac{i\pi}{2}m(\psi_j(x_1,x_2))}\frac{\exp\frac{iS_j^{(1,2)}(x_1,x_2,t)}{h}}{\sqrt{|J_j^{(1,2)}(x_1,x_2)|}}f(\rho,\psi)$$
$$\times \mathbf{e}_j(\psi)|_{\psi=\psi_j(x_1,x_2)}\mathbf{e}(\tau_j(x_1,x_2,t)-t)$$
$$+\sum_{j\in\mathbf{j}(1,0)} e^{-\frac{i\pi}{2}\mathbf{m}(\Omega_j^{(1,0)})}\frac{i}{\sqrt{2\pi h}}\int_{-\infty}^{+\infty}\frac{\exp\frac{i(S_j^{(1,0)}(x_1,p_2,t)+x_2p_2)}{h}}{\sqrt{|J_j^{(1,0)}(x_1,p_2)|}}f(\rho,\psi)$$
$$\times \mathbf{e}_j(\psi)|_{\psi=\psi_j(x_1,p_2)}\mathbf{e}(\tau_j(x_1,p_2,t)-t)dp_2$$
$$+\sum_{j\in\mathbf{j}(0,2)} e^{-\frac{i\pi}{2}\mathbf{m}(\Omega_j^{(0,2)})}\frac{i}{\sqrt{2\pi h}}\int_{-\infty}^{+\infty}\frac{\exp\frac{i(S_j^{(0,2)}(p_1,x_2,t)+x_1p_1)}{h}}{\sqrt{|J_j^{(0,2)}(p_1,x_2)|}}f(\rho,\psi)$$
$$\times \mathbf{e}_j(\psi)|_{\psi=\psi_j(p_1,x_2)}\mathbf{e}(\tau_j(p_1,x_2,t)-t)dp_1 \qquad (6.9)$$

where S^I_j and J^I_j are defined in (6.8). Now we can construct the asymptotic solution η to the problem (1.1). We put in the last formula $h=\mu/\rho$.

Theorem 4. *1) For any T independent of $\mu = l/L$, the solution η to the problem* (1.1) *in the interval $t\in[0,T]$ has the form:*

$$\eta = \eta_{\mathrm{as}} + o(\mu), \quad \eta_{\mathrm{as}} = \sqrt{\frac{\mu C_0}{2\pi}}\mathrm{Re}(\mathrm{e}^{-\frac{i\pi}{4}}\int_0^\infty \Psi(\rho,\mathrm{x}_1,\mathrm{x}_2))\mathrm{d}\rho. \qquad (6.10)$$

This asymptotic, apart from terms of the order $O(\mu)$, does not depend on the choice of the covering $\{\Omega_j^{I_j}\}$, and functions $\mathbf{e}_j^{\dagger}$, $\mathbf{e}$.

2) For each time $t\in[O,T]$ the function η is localized in a neighborhood of the front: the function η is equal $o(\mu)$ outside some neighborhood of the front γ_t.

Sketch of Proof. Using the results [16, 17, 28, 15] one can show that the function (6.9) is a leading term of some asymptotic solution Ψ^k $\mathrm{mod}O(\mathrm{h}^k)$ to the original equation (1.1), where k is an arbitrary large but fixed integer number. We introduce the smooth cut off function $\mathbf{g}(y)$: $\mathbf{g}(y)=0$ for $y\le 1/2$ and $\mathbf{g}(y)=1$ for $y\ge 1$. Multiplying Ψ^k by $\mathbf{g}(\rho/\mu)$ and integrating the product by $d\rho$ we obtain that the result is an asymptotic solution of (1.1) $\mathrm{mod}O(\mu^2)$. Then as in [15] we show that the contribution of the term $\int_0^1(1-\mathbf{g}(\rho/\mu))\Psi^k|_{t=0}d\rho$ to the solution (1.1) is $o(\mu)$, and hence the function η_{as} from (6.10) is a leading term of some asymptotic solution of (1.1). Now we need to check the conditions (1.2). But it is better to do after a simplification of the function (6.10) and we shall do it in the next subsection.

6.3 The germ Λ_t^2 of the manifold M_t^2 and the simplification of the asymptotic.

Since the function 6.10 decays quite rapidly when the point x goes away from the front, it is possible to change the functions $S^{I_j}, |J_j^{I_j}$ in a neighborhood of the front by their Taylor expansions. The nice fact is that one does not change the accuracy $O(\mu)$ in formula 6.10 using only the zero order , first order terms and sometimes second order terms of the Taylor expansions of the phases, and zero order terms in the other functions. All these expansions are expressed via the vector functions $(P(\psi,t), X(\psi,t))$ and matrices $\mathcal{B}(\psi,t), \mathcal{C}(\psi,t)$.

We need the Taylor expansion in the following form. Let the equations $(y_1, y_2) = (Y_1(\psi), Y_1(\psi))$ determine a smooth curve Υ in some domain in R^2_y and $\Phi(y_1, y_2)$ be a smooth function in some neighborhood $\mathcal{D}$ of Υ. Let $q(\psi)$ be the smooth family of nonzero vectors with components (q_1, q_2) transversal to the curve Υ.

This means that the vectors $q(\psi)$ and $Y_\psi(\psi)$ are not parallel. The parameter ψ on Υ and the family of vectors $q(\psi)$ define the curvilinear system in some neighborhood of Υ: each point y in this neighborhood can be characterized by two values: $\psi(y)$ and the length (with the sign) $z = \langle y - Y(\psi(y)), q(\psi(y))\rangle/(q(\psi(y)))^2$ of the vector $y - Y(\psi(y))$. To find the value of $\psi(y)$ one has to solve the equation

$$\langle y - Y(\psi), q_\perp(\psi)\rangle \equiv (y_1 - Y_1(\psi))q_2(\psi) - (y_2 - Y_2(\psi))q_1(\psi) = 0. \tag{6.11}$$

Lemma 16. *The following expansion is valid:*

$$\Phi(y) = \Phi(Y) + \langle\frac{\partial\Phi}{\partial y}(Y), (y-Y)\rangle + \frac{1}{2}\langle(y-Y), \frac{\partial^2\Phi}{\partial y^2}(Y)(y-Y)\rangle|_{Y=Y(\psi(y))} + O((y-Y(\psi(y)))^3) \tag{6.12}$$

PROOF follows from the 1-D Taylor expansion of the function $\Phi(Y+qz)$ with respect to variable z $\square$.

Now we want to apply this lemma to the phases $S_j^{I_j}$ and Jacobians $J_j^{I_j}$ in (6.8) and (6.10). The variable y are x^I, the curve $\Upsilon = \{x^I = X^I(\psi,t)\}$, thus the solution ψ will depend also on time t. We need the first and second derivatives of S^I in the points $x^I = X^I(\psi,t)$. From the general theory of Hamilton-Jacobi equation [16, 18] it follows

$$\frac{\partial S^I}{\partial x^I} = P^I(\psi,t), \quad \frac{\partial^2 S^I}{\partial (x^I)^2} = \frac{\partial P^I}{\partial x^I}(\psi,t) \equiv \mathcal{B}^I(\psi,t)(\mathcal{C}^I(\psi,t))^{-1},$$

where the matrices $\mathcal{B}^I, \mathcal{C}^I$ are defined in (5.13). Now let us choose the vector q as following. In the case $I = (1,2)$ $q = \dot{X}(\psi,t) = (X_\psi)_\perp$, then equation (6.11) is equation. (6.4). In the case $I = (0,2)$ $q = {}^t(0,1)$, then equation (6.11) is

$$P_1(\psi,t) = p_1. \tag{6.13}$$

In the case $I = (1,0)$ $q = {}^t(1,0)$, then Eq.(6.11) is

$$P_2(\psi,t) = p_2. \tag{6.14}$$

We denote $\psi^j(x^{I_j},t)$ the solution of these equations in the map with the number j. [5] Then after some algebra we obtain the following formulas in the maps with numbers j

$$S_j^{I_j}(x^I,t) = \{\mathcal{S}_j^{I_j}(\psi,t) + O(x^{I_j} - X^{I_j}(\psi,t))^3\}|_{\psi=\psi^j(x,t)},$$
$$J_j^{I_j}(x^{I_j},t) = \{\mathcal{J}_j^{I_j}(\psi,t) + O(x^{I_j} - X^{I_j}(\psi,t))\}|_{\psi=\psi^j(x,t)}, \tag{6.15}$$

[5] These solutions are different from the solutions $\psi_j(x^{I_j})$ of equation (6.5), we use almost the same symbol to simplify the notation.

here

$$
\begin{aligned}
\mathcal{S}_j^{(1,2)}(\psi,t) &= \langle P(\psi,t),(x-X(\psi,t)\rangle - \frac{1}{2}\langle P(\psi,t)), C_x(X(\psi,t))\rangle(x-X(\psi,t)^2, \\
\mathcal{J}_j^{(1,2)} &= \det(\dot{X},X_\psi)(\psi,t) \qquad (6.16) \\
\mathcal{S}_j^{(0,2)}(\psi,t) &= -P_1(\psi,t)X_1(\psi,t)+P_2(\psi,t)(x_2-X_2(\psi,t)) \\
&\quad +\frac{1}{2}(x_2-X_2(\psi,t))^2\frac{\dot{P}_1P_{2\psi}-\dot{P}_2P_{1\psi}}{\dot{P}_1X_{2\psi}-\dot{X}_2P_{1\psi}}, \\
\mathcal{J}^{(0,2)}(\psi,t) &= \det C^{(0,2)}(\psi,t) = (\dot{P}_1X_{2\psi}-\dot{X}_2P_{1\psi})(\psi,t) \qquad (6.17) \\
\mathcal{S}^{(1,0)}(\psi,t) &= -P_2(\psi,t)X_2(\psi,t)+P_1(\psi,t)(x_1-X_1(\psi,t)) \\
&\quad +\frac{1}{2}(x_1-X_1(\psi,t))^2\frac{\dot{P}_1P_{2\psi}-\dot{P}_2P_{1\psi}}{\dot{X}_1P_{2\psi}-\dot{P}_2X_{1\psi}} \\
\mathcal{J}^{(1,0)}(\psi,t) &= \det C^{(1,0)}(\psi,t) = (\dot{X}_1P_{2\psi}-\dot{P}_2X_{1\psi})(\psi,t),t) \qquad (6.18)
\end{aligned}
$$

Remark 1. *It is important that the last formulas do not depend on the choice of the vector q with the same accuracy they are valid.*

Theorem 5. *The proposition of Theorem 1 is valid if one changes in the formulas (6.9),(6.10) $S_j^{I_j}$ by $\mathcal{S}_j^{I_j}$, $J_j^{I_j}$, by $\mathcal{J}_j^{I_j}$, $\psi_j(x^{I_j})$ by $\psi^j(x^{I_j},t)$, $\mathbf{e}(\tau_j(x^{I_j},t)-t)$ by $\mathbf{e}(|x^{I_j}-X^{I_j}(\psi^j(x^{I_j},t)|)$. 2) In the singular maps in formulas (6.9) one can change the integration over $p_j\in(-\infty,\infty)$ by the integration over the angle $\psi\in\Omega_j^{I_j}$, putting $p_j=P_j(\psi,t)$ and $dp_j=\frac{\partial P_j}{\partial\psi}(\psi,t)d\psi$ adjusting the limits in the integral with these change.*

THE IDEA OF PROOF. The proof in regular maps is no more but the Taylor expansion of regular components in (6.9), (6.10) with respect to distance from γ_t. The proof in the focal maps is based on the Taylor expansions but also on the estimate of some rapidly oscillating integrals.

Theorems 1-3 and the examination of the initial data are no more but the realization of the last theorem in different maps. We have no space to explain all details (similar to [15, 30])and we will present them in another paper.

Acknowledgments

This work was supported by RFBI- 05-01-00968 and the scientific agreement among the Department of Physics of the University of Rome "La Sapienza" and the Institute for Problems in Mechanics of Russian Academy of Sciences, Moscow.

Bibliography

[1] González F.I. Tsunami! Predicting Destruction by Monster Waves, Sci. American, v.21, May 1999, pp.58-65.

[2] Whitmore P.M., Sokolowski T.J., Predicting tsunami amplitudes along the North American coast from tsunamis generated in the Northwest Pacific ocean during tsunami warnings. Science of Tsunami Hazards, v.4 (1996), N 3, 147-166.

[3] E.N.Pelinovski, Hydrodynamics of Tsunami Waves, Nizhnii Novgorod, 1996.

[4] Kowalik Z., Murty T., Numerical modeling of ocean dynamics, World Scientific, Singapore, 1993, pp.5-19.

[5] Lewis C.L., Adams W.M., Development of a tsunami-fluding model having versatile formulation of moving boundary conditions, The Tsunami Society Monograph Series, 1983, N 1.

[6] Shokin Yu.I., Chubarov L.B., Marchuk A.G., Simonov K.V., Numerical Experiment in Tsunami Problem."Nauka", Siberian Division, Novosibirsk, 1989.

[7] Tinti S., *Tsunami in the world.* -Kluver Acad.Press, 1993.

[8] Mei, C. C. The applied dynamics of ocean surface waves, World Scientific, Singapore, 1989.

[9] Stoker, J. J. Water waves , Interscience, New York, 1957.

[10] Whitham G.B., Linear and Nonlinear Waves, Wiley, New York, 1974.

[11] Kurkin A.A., Pelinovsky E.N., Freak waves: facts, theory and modelling, Nizhegorodski State Univ., 2004, Nizhnii Novgorod

[12] Borovikov V.A., Kelbert M.Ya., Fluid Dynamics, vol.31, N 4, 1996

[13] Dobrokhotov, S. Yu., Kuzmina V.M. and Zhevandrov, P. N. Asymptotic of the solution of the Cauchy-Poisson problem in a layer of nonconstant thickness. Math.Notes, v.53,N 6,1993, pp.657-660.

[14] Berry, New Journal of Physics, V 7, 129, 2005, pp. 1-18.

[15] Dobrokhotov S.Yu., Maslov V.P., Zhevandrov P.N., Shafarevich A.I. Asymptotic fast-decreasing solution of linear, strictly hyperbolic systems with variable coefficients. Math.Notes, v.49, N 4 (1991) pp.355-365

[16] V.P. Maslov, *Perturbation Theory and Asymptotic Methods*. Moscow: Moscow Univ. Publ. (1965) 549h pp. V.P. Maslov, *Théorie des perturbations et méthodes asymptotiques*. Paris: Dunod, Gauthier-Villars (1972) xii, 384 pp.

[17] V.P. Maslov, *Operational Methods. Transl. from the Russian*. Moscow: Mir Publ. (1973) 559 pp.

[18] V.P. Maslov and M.V. Fedoriuk, *Semi-Classical Approximation in Quantum Mechanics*. In: *Mathematical Physics and Applied Mathematics* 7, *Contemporary Mathematics* 5. Dordrecht etc.: D. Reidel Publishing Co. (1981) ix, 301 pp.

[19] Maslov, V. P., *The complex WKB method for nonlinear equations*, Moscow: Nauka, 1977, in Russian; English transl.: *The complex WKB method for nonlinear equations. I. Linear theory*, Basel–Boston–Berlin: Birkhäuser, (1994).

[20] Babich, V. M., Buldyrev, Short-Wavelength Diffraction Theory, Springer-Verlag, 1991

[21] Egorov, Yu. V., Shubin, M. A. Partial differential equations IV. Microlocal analysis and hyperbolic equations, Springer-Verlag. Berlin 1993

[22] Kravtsov Yu.A., Orlov Yu.i, Geometrical Optics of Inhomogeneous Media, Springer-Verlag, 1990

[23] Brekhovskikh, L. M., Godin O.A., Acoustics of layeard madia II, Point source and bounded beams, Springer-Verlag, 1992

[24] Kiselev A.P., Generation of modulated vibrations, in Generation of modulated vibrations in nonhomogeneous media, Mathematical quaestions of wave propogation theory, 11, Zapiski nauchnyh seminarov LOMI, v.104, pp.111–122

[25] Arnold, V. I. Singulariries of caustics and wavefronts, Kluwer Academic Publishers, 1990.

[26] Arnold, V. I. On the characteristic class entering the quantization condition, Functional. Anal. Appl. v. 1, (1967), N 1 pp. 1–14.

[27] S.Yu. Dobrokhotov, Maslov's methods in linearized theory of gravitational waves on the liquid surface. *Doklady Akad. Nauk SSSR* 269 (1983) 76–80.
S.Yu. Dobrokhotov, Maslov's methods in linearized theory of gravitational waves on the liquid surface. *Sov. Phys. –Doklady* 28 (1983) 229–231.

[28] Dobrokhotov S.Yu., Zhevandrov P.N., Asymptotic expansions and the Maslov canonical operator in the linear theory of water waves. I. Main constructions and equations for surface gravity waves. *Russ.J.Math.Phys.* v. 10, 2003, 1–31.

[29] Maslov V.P., Fedorjuk M.V., Logarithmic asymptotics of fast decaying solutions to Petrovskii tipe hyperbolic systems, Math.Notes, v.45, 1989, N 5, pp.50-62

[30] Dobrokhotov, S. Yu. and Zhevandrov, P. N. Nonstandard characteristics and the Maslov operator method in linear problems on time dependent waves in water. Funct. Anal. Appl. v. 19 N 4 (1985), pp. 285–295.

[31] Vishik M.I., Lusternik L.A., Regular degeneration and boundary layer for linear differential equations with small parameter, Uspekhi Mat.Nauk, v.12, 1957, N 5, pp. 3-122, Engl.translation: American Math. Society Transl. (2) v.20, 1962, 239-364