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# Validity of dimensional reduction in the random field O($N$) spin model for sufficiently large $N$

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**Abstract**

We study the critical phenomena of a random field O($N$) spin model near the lower critical dimension, by means of the renormalization group method and the $1/N$ expansion. We treat the O($N$) nonlinear $\sigma$ model including a random field and all the random anisotropy terms, and calculate the one-loop beta function for a linear combination of them in $d = 4+\epsilon$ under the assumption of replica symmetry. At first, we obtain all fixed points for the one-loop beta function in the large $N$ limit, and discuss their stability. We find that the fixed point yielding dimensional reduction is singly unstable, and others are the fixed points with many relevant modes, or unphysical fixed point. Therefore, in the large $N$ limit, the critical phenomena in $4 + \epsilon$ dimensions is found to be governed by the fixed point which gives the result of dimensional reduction. Next, we investigate the $1/N$ correction to the fixed point yielding dimensional reduction. Careful analysis of the eigenvalue equation for the infinitesimal deviation from the fixed point is done at order $1/N$. In practice, the fixed point yielding dimensional reduction is found to be singly unstable. Thus, we conclude that the dimensional reduction holds for sufficiently large $N$.

## 1 Introduction

The critical phenomena in the random field O($N$) spin model is worth studying from the viewpoint of quenched disorder and spin correlations. Dimensional reduction [1] is one key to clarify the nature of this model. Dimensional reduction claims that the critical behavior of the $d$-dimensional random field O($N$) spin model is the same as of the $(d-2)$-dimensional pure O($N$) spin model, where $d$ is the spatial dimension. Dimensional reduction can predict the known upper critical dimension 6 for the random field O($N$) spin models ($N \geq 2$), since the upper critical dimension of the pure system are 4. It is thus natural to ask whether the dimensional reduction holds more precisely from six dimensions down to four. The strong version of dimensional reduction claims that all critical exponents of the random field spin model in $d$ dimensions are identical to those of the corresponding pure model in $d-2$ dimension. In some papers [2, 3, 4], however, the breakdown of the dimensional reduction has been reported.

Since several rigorous results for the random field Ising model ($N = 1$ case) indicated the failure of the dimensional reduction to predict the lower critical dimensions [5, 6, 7], people discussed the breakdown of dimensional reduction with some approximation methods for random field models. Fisher pointed out the breakdown of dimensional reduction due to the appearance of the infinite number of relevant operators near four dimensions [2]. He showed the existence of a fixed point yielding dimensional reduction for $N \geq 18$, but this fixed point is unstable as far as the number of spin components $N$ is finite. Therefore, he concluded that the dimensional

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reduction was not valid near four dimensions. Mézard and Young also suggested breakdown of dimensional reduction by replica symmetry breaking [3]. Now, many researchers believe that the dimensional reduction is incorrect in dimensions less than 6.

Recently, Tarjus and Tissier study the critical phenomena of this model in any dimension and for any value of $N$ by using the nonperturbative renormalization group method and the replica method [4]. They show that there is a critical $N_c$ the dimensional reduction $\eta_{\rm random}(d) = \eta_{\rm pure}(d-2)$ is valid for $N \geq N_c$. Since they show $N_c = 18$ near the lower critical dimension, their result does not agree with that of the references [2, 3].

To understand consistency of their works, we reexamine the one-loop beta function obtained by Fisher employing the $1/N$-expansion method. Since Fisher did not solve the eigenvalue problem for the stability around the fixed point, we solve this problem in a $1/N$ expansion. First, in the large $N$ limit, we calculate all the fixed points including nonanalytic fixed points as well as the fixed point yielding dimensional reduction. Then we investigate their stability by solving the eigenvalue equation. We find that the only nontrivial stable fixed point yields the dimensional reduction. Next, we calculate the subleading correction to the fixed point, and investigate the stability by solving the eigenvalue equation. We find that the unstable mode pointed by Fisher is fictitious, and that the fixed point yielding the dimensional reduction is practically singly unstable in a coupling constant space of the given model with large $N$. This result agrees with that by Tarjus and Tissier and a simple $1/N$-expansion. Thus, we conclude that the dimensional reduction holds for sufficiently large $N$. In this note, we review these results obtained in our recent study [8].

This note is organized as follows. In Sec. 2, we briefly review the renormalization group analysis for the random field $\mathrm{O}(N)$ spin model in $4+\epsilon$ dimensions, based on the reference [2]. In Sec. 3, we carefully reexamine the critical phenomena of $(4+\epsilon)$-dimensional random field $\mathrm{O}(N)$ spin model in the large $N$ limit. As a result, in the large $N$ limit, the critical phenomena in $4+\epsilon$ dimensions is shown to be governed by the fixed point which gives the result of dimensional reduction. In Sec. 4, we investigate the $1/N$ correction to the fixed point yielding dimensional reduction. We show that the fixed point yielding dimensional reduction is singly unstable. Thus, we conclude that the dimensional reduction holds for sufficiently large $N$. Sec. 5 contains conclusions.

## 2 Review of renormalization group analysis for random field $\mathrm{O}(N)$ spin model in $4+\epsilon$ dimensions

In this section, we briefly review the renormalization group analysis for the random field $\mathrm{O}(N)$ spin model in $4+\epsilon$ dimensions, based on the reference [2]. We deal with the random field and random anisotropy $\mathrm{O}(N)$ nonlinear $\sigma$ model which is known as an effective field theoretical model for the random field $\mathrm{O}(N)$ spin model near the lower critical dimension. We derive the one-loop beta functions for the temperature and a general anisotropy including the random field and random anisotropy terms, and obtain the fixed points of $O(\epsilon)$. We calculate the critical exponents $\eta$ and $\bar{\eta}$ for connected and disconnected correlation function. The stability of the fixed points is discussed.

### 2.1 Model

We consider $\mathrm{O}(N)$ classical spins $S(x)$ with a fixed-length constraint $S(x)^2 = 1$. To take the average over the random field, one introduces replicas $S^\alpha(x)$, $\alpha = 1 \ldots n$. We start from a nonlinear $\sigma$ model of the following replica partition function and action:

$$\mathcal{Z} = \int \prod_{\alpha=1}^{n} \mathcal{D}S^\alpha \delta(S^{\alpha 2} - 1) e^{-\beta H_{\rm rep}},$$

$$\beta H_{\rm rep} = \frac{a^{2-d}}{2T} \int d^d x \sum_{\alpha=1}^{n} \sum_{\mu=1}^{d} (\partial_\mu S^\alpha)^2 - \frac{a^{-d}}{2T^2} \int d^d x \sum_{\alpha,\beta}^{n} R(S^\alpha \cdot S^\beta), \tag{2.1}$$

where $a$ is the ultraviolet cutoff, and the parameter $T$ denotes the dimensionless temperature. The function $R(S^\alpha \cdot S^\beta)$ represents general anisotropy including the random field and all the random anisotropies, and is given by

$$R(S^\alpha \cdot S^\beta) = \sum_{\mu=1}^{\infty} \Delta_\mu (S^\alpha \cdot S^\beta)^\mu, \tag{2.2}$$

where $\Delta_\mu$ denotes the strength of the random field and the $\mu$-th rank random anisotropy ($\mu = 1$ is the random field, and $\mu \geq 2$ is the second and higher-rank random anisotropy).

## 2.2 Renormalization group

We use the method obtained by Polyakov [9] for the pure system in $2+\epsilon$ dimensions. We express each replica $S^\alpha(x)$ of the magnetization as a combination of fast fields $\varphi_i^\alpha(x)$, $i = 1, \ldots, N-1$ and a slow field $n_0^\alpha(x)$ of the unit length. We use the representation

$$S^\alpha = n_0^\alpha \sqrt{1 - \varphi^{\alpha 2}} + \varphi^\alpha, \quad \varphi^\alpha = \sum_{i=1}^{N-1} \varphi_i^\alpha e_i^\alpha, \tag{2.3}$$

where the unit vectors $e_i^\alpha(x)$ are perpendicular to each other and also to the vector $n_0^\alpha(x)$. Substituting the equation (2.3) into the Hamiltonian (2.1) and selecting quadratic terms in $\varphi^\alpha(x)$, we have

$$\beta H_{\rm rep} = \beta H_{\rm unpert.} + \beta H_{\rm int} + \beta H_0, \tag{2.4}$$

$$\beta H_{\rm unpert.} = \frac{a^{2-d}}{2T} \int d^d x \sum_{\alpha=1}^{n} \sum_{\mu=1}^{d} (\partial_\mu n_0^\alpha)^2 - \frac{a^{-d}}{2T^2} \int d^d x \sum_{\alpha,\beta}^{n} R(n_0^\alpha \cdot n_0^\beta), \tag{2.5}$$

$$\begin{aligned} \beta H_{\rm int} = {} & \frac{a^{2-d}}{2T} \int d^d x \sum_{\alpha=1}^{n} \sum_{\mu=1}^{d} \left\{ (\partial_\mu n_0^\alpha)^2 \cdot \left( - \sum_{i=1}^{N-1} \varphi_i^{\alpha 2} \right) + \sum_{i,j}^{N-1} c_{\mu i}^\alpha c_{\mu j}^\alpha \varphi_i^\alpha \varphi_j^\alpha \right\} \\ & - \frac{a^{-d}}{2T^2} \int d^d x \sum_{\alpha,\beta}^{n} \left\{ A^{\alpha\beta} \sum_{i=1}^{N-1} \varphi_i^{\alpha 2} + \sum_{i,j}^{N-1} B_{ij}^{\alpha\beta} \varphi_i^\alpha \varphi_j^\alpha + \sum_{i,j}^{N-1} C_{ij}^{\alpha\beta} \varphi_i^\alpha \varphi_j^\beta \right\}, \end{aligned} \tag{2.6}$$

$$\beta H_0 = \frac{a^{2-d}}{2T} \int d^d x \sum_{\alpha=1}^{n} \sum_{\mu=1}^{d} \sum_{i=1}^{N-1} \left( \partial_\mu \varphi_i^\alpha - \sum_{j=1}^{N-1} f_{\mu ij}^\alpha \varphi_j^\alpha \right)^2, \tag{2.7}$$

where

$$c_{\mu i}^\alpha = (\partial_\mu n_0^\alpha) \cdot e_i^\alpha, \tag{2.8}$$

$$A^{\alpha\beta} = -(n_0^\alpha \cdot n_0^\beta) R'(n_0^\alpha \cdot n_0^\beta), \tag{2.9}$$

$$B_{ij}^{\alpha\beta} = (n_0^\beta \cdot e_i^\alpha)(n_0^\beta \cdot e_j^\alpha) R''(n_0^\alpha \cdot n_0^\beta), \tag{2.10}$$

$$C_{ij}^{\alpha\beta} = (e_i^\alpha \cdot e_j^\beta) R'(n_0^\alpha \cdot n_0^\beta) + (n_0^\beta \cdot e_i^\alpha)(n_0^\alpha \cdot e_j^\beta) R''(n_0^\alpha \cdot n_0^\beta), \tag{2.11}$$

$$f_{\mu ij}^\alpha = (\partial_\mu e_i^\alpha) \cdot e_j^\alpha. \tag{2.12}$$

Here we put $f^{\alpha}_{\mu ij}(x) = 0$.

We turn to the perturbative renormalization group transformation. Representing the new replicated Hamiltonian by $\beta H'_{\rm rep}$, we have the following expression for $\beta H'_{\rm rep}$ up to the second order of the perturbation expansion:

$$\beta H'_{\rm rep} \simeq \beta H_{\rm unpert.} + \langle \beta H_{\rm int} \rangle_0 - \frac{1}{2!}\langle \beta H_{\rm int}; \beta H_{\rm int} \rangle_0, \tag{2.13}$$

where $\langle A; B\rangle_0 = \langle AB\rangle_0 - \langle A\rangle_0 \langle B\rangle_0$, and $\langle\cdots\rangle_0$ denotes the average defined by

$$\langle \cdots \rangle_0 = \frac{\int \prod_{\alpha=1}^{n} \mathcal{D}\varphi^{\alpha}(\cdots)e^{-\beta H_0}}{\int \prod_{\alpha=1}^{n} \mathcal{D}\varphi^{\alpha} e^{-\beta H_0}}. \tag{2.14}$$

To calculate the renormalization group beta function, we use a free propagator of the fluctuation field

$$\langle \varphi^{\alpha}_i(x)\varphi^{\beta}_j(y)\rangle_0 = Ta^{d-2}G_0(x-y)\delta_{\alpha\beta}\delta_{ij}, \tag{2.15}$$

$$G_0(x-y) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot(x-y)}}{p^2}, \tag{2.16}$$

$$\begin{aligned} G_0(0) &= \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2} \\ &= \frac{S_d}{(2\pi)^d}\int_{b^{-1}}^{a^{-1}} dp p^{d-3} = \frac{S_d}{(2\pi)^d}\frac{a^{d-2}}{d-2}\left\{1-\left(\frac{b}{a}\right)^{2-d}\right\}, \end{aligned} \tag{2.17}$$

where $b$ is the ultraviolet cutoff; $b > a$, and $S_d = 2\pi^{\frac{d}{2}}/\Gamma(d/2)$. At one-loop level, we have the expressions for $\langle\beta H_{\rm int}\rangle_0$ and $\langle\beta H_{\rm int};\beta H_{\rm int}\rangle_0$:

$$\begin{aligned} \langle\beta H_{\rm int}\rangle_0 \simeq\ & \frac{a^{2-d}}{2T}\{-(N-2)(Ta^{d-2})G_0(0)\}\int d^dx\sum_{\alpha=1}^{n}\sum_{\mu=1}^{d}(\partial_\mu \boldsymbol{n}_0^{\alpha})^2 \\ & -\frac{a^{-d}}{2T^2}(Ta^{d-2})G_0(0)\int d^dx\sum_{\alpha,\beta}^{n}\Big\{-(N-1)zR'(z)+(1-z^2)R''(z)\Big\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} -\frac{1}{2!}\langle\beta H_{\rm int};\beta H_{\rm int}\rangle_0 \simeq\ & \frac{a^{2-d}}{2T}\left(a^{d-4}\int dy G_0(y)^2\right)\int d^dx\sum_{\alpha=1}^{n}\sum_{\mu=1}^{d}\Big\{-(N-2)R'(1)(\partial_\mu \boldsymbol{n}_0^{\alpha})^2\Big\} \\ & -\frac{a^{-d}}{2T^2}\left(\frac{a^{d-4}}{2}\int dy G_0(y)^2\right)\int d^dx\sum_{\alpha,\beta}^{n}\Big\{(N-2+z^2)[R'(z)]^2 \\ & -2z(1-z^2)R'(z)R''(z)+(1-z^2)^2[R''(z)]^2-2(N-1)zR'(1)R'(z) \\ & +2(1-z^2)R'(1)R''(z)\Big\}. \end{aligned} \tag{2.19}$$

where we put $z = \boldsymbol{n}_0^{\alpha}\cdot\boldsymbol{n}_0^{\beta}$ for simplicity. If we define the new coupling constants by

$$\beta H'_{\rm rep} = \frac{b^{2-d}}{2T'}\int d^dx\sum_{\alpha=1}^{n}\sum_{\mu=1}^{d}(\partial_\mu \boldsymbol{n}_0^{\alpha})^2 - \frac{b^{-d}}{2T'^2}\int d^dx\sum_{\alpha,\beta}^{n}\tilde{R}(z), \tag{2.20}$$

we have the one-loop beta functions

$$\frac{dT}{dt} \equiv \partial_t T = (2-d)T + A(N-2)T^2 + A(N-2)TR'(1), \tag{2.21}$$

$$\begin{aligned}
\frac{dR(z)}{dt} &\equiv \partial_t R(z) \\
&= (4-d)R(z) + AT\{2(N-2)R(z) - (N-1)zR'(z) + (1-z^2)R''(z)\} \\
&\quad + A\Big(2(N-2)R'(1)R(z) - (N-1)zR'(1)R'(z) + (1-z^2)R'(1)R''(z) \\
&\quad + \frac{1}{2}[R'(z)]^2(N-2+z^2) - R'(z)R''(z)z(1-z^2) + \frac{1}{2}[R''(z)]^2(1-z^2)^2\Big), \qquad (2.22)
\end{aligned}$$

where $t = \ln(b/a)$, and $A = S_d/(2\pi)^d$.

## 2.3 Critical phenomena in $4+\epsilon$ dimensions

In $4+\epsilon$ dimensions, the one-loop beta functions $\partial_t T$ and $\partial_t R(z)$ become

$$\begin{aligned}
\partial_t T &= -(2+\epsilon)T + A(N-2)T^2 + A(N-2)TR'(1), \qquad (2.23) \\
\partial_t R(z) &= -\epsilon R(z) + AT\{2(N-2)R(z) - (N-1)zR'(z) + (1-z^2)R''(z)\} \\
&\quad + A\Big(2(N-2)R'(1)R(z) - (N-1)zR'(1)R'(z) + (1-z^2)R'(1)R''(z) \\
&\quad + \frac{1}{2}[R'(z)]^2(N-2+z^2) - R'(z)R''(z)z(1-z^2) + \frac{1}{2}[R''(z)]^2(1-z^2)^2\Big). \qquad (2.24)
\end{aligned}$$

Solving the fixed-point equation $\partial_t T = 0$, we find that there is no nontrivial fixed point for $T$ of $O(\epsilon)$. Thus, we have only trivial fixed point $T = 0$. Linearizing $\partial_t T$ around $T = 0$, we have

$$\left.\frac{\partial(\partial_t T)}{\partial T}\right|_{T=0} = -2 - \epsilon + A(N-2)R'(1). \qquad (2.25)$$

Since $R'(1)$ is at most of order $\epsilon$, $\partial(\partial_t T)/\partial T|_{T=0}$ is negative. Thus, the renormalization group flows around a zero temperature go into the fixed point $T = 0$. We put $T = 0$ below. Then the one-loop beta function $\partial_t R(z)$ becomes as follows:

$$\begin{aligned}
\partial_t R(z) &= -\epsilon R(z) + A\Big(2(N-2)R'(1)R(z) - (N-1)zR'(1)R'(z) + (1-z^2)R'(1)R''(z) \\
&\quad + \frac{1}{2}[R'(z)]^2(N-2+z^2) - R'(z)R''(z)z(1-z^2) + \frac{1}{2}[R''(z)]^2(1-z^2)^2\Big). \qquad (2.26)
\end{aligned}$$

Expanding $R(z)$ about the aligned state with $z = 1$ for all $\alpha$, $\beta$, we obtain the one-loop beta functions for $R'(1)$, $R''(1)$ at zero-temperature fixed point:

$$\begin{aligned}
\partial_t R'(1) &= -\epsilon R'(1) + A(N-2)R'(1)^2, \qquad (2.27) \\
\partial_t R''(1) &= -\epsilon R''(1) + A[6R'(1)R''(1) + (N+7)R''(1)^2 + R'(1)^2]. \qquad (2.28)
\end{aligned}$$

The beta functions (2.27) and (2.28) have two nontrivial fixed points:

$$\begin{aligned}
(R'(1), R''_+(1)) &= \left(\frac{\epsilon}{A(N-2)}, \frac{(N-8)+\sqrt{(N-2)(N-18)}}{2A(N-2)(N+7)}\epsilon\right), \qquad (2.29) \\
(R'(1), R''_-(1)) &= \left(\frac{\epsilon}{A(N-2)}, \frac{(N-8)-\sqrt{(N-2)(N-18)}}{2A(N-2)(N+7)}\epsilon\right). \qquad (2.30)
\end{aligned}$$

The formulas for the critical exponents $\eta$ and $\bar{\eta}$

$$\eta = AR'(1), \quad \bar{\eta} = A(N-1)R'(1) - \epsilon, \qquad (2.31)$$

enable us to obtain

$$\bar{\eta} = \eta = \frac{\epsilon}{N-2}. \tag{2.32}$$

This result of $\eta$ is consistent with that of a pure system in $d = 2 + \epsilon$ up to order $\epsilon$. The result $\bar{\eta} = \eta$ confirms the dimensional reduction. From the fixed point (2.29) and (2.30), we find that these results are applicable only for $N \geq 18$. Feldman carefully reexamined the one-loop beta function, and found nonanalytic fixed points which control the critical phenomena instead of the fixed point (2.29) and (2.30) [10]. He calculated the exponents $\eta$ and $\bar{\eta}$ for $N = 3, 4, 5$ in $4 + \epsilon$ dimensions numerically:

$$\begin{array}{lll} \eta = 5.5\epsilon, & \bar{\eta} = 10.1\epsilon, & \text{for } N = 3 \\ \eta = 0.79\epsilon, & \bar{\eta} = 1.4\epsilon, & \text{for } N = 4 \\ \eta = 0.42\epsilon, & \bar{\eta} = 0.70\epsilon, & \text{for } N = 5 \end{array} \tag{2.33}$$

Then he concluded that dimensional reduction breaks down near four dimensions for several finite $N$.

The eigenvalues of the scaling matrix at the fixed points (2.29) and (2.30) are

$$\lambda_1 = +\epsilon, \tag{2.34}$$

$$\lambda_2^{\pm} = \pm\epsilon\sqrt{\frac{N-18}{N-2}}. \tag{2.35}$$

Thus, the fixed point (2.29) is unstable. The fixed point (2.30) seems to be stable for $N \geq 18$. However, Fisher showed that the fixed point (2.30) is also unstable [2]. His statement is as follows. Expanding $R(z)$ about the aligned state with $z = 1$ up to the $k$-th order, we have the one-loop beta function for $R^{(k)}(1)$. Substituting the fixed point $(R'(1)^*, R''_{-}(1)^*, \ldots, R^{(k-1)}(1)^*)$ into the beta function for $R^{(k)}(1)$, we can obtain the nontrivial fixed point $R^{(k)}(1)^*$ at $\mathrm{O}(\epsilon)$. The eigenvalue at the fixed point is given by

$$\begin{aligned} \lambda_k &= \epsilon\left(\frac{2k^2 - k(N-1) + 2N - 4}{N-2} - 1\right) \\ &\simeq \epsilon\left((1-k)\epsilon + \frac{2k^2 - k}{N}\right), \end{aligned} \tag{2.36}$$

for $k \geq 3$. The eigenvalue is found to be positive for large $k$. Then Fisher concluded that there is no singly unstable fixed point, and the dimensional reduction breaks down near four dimensions. In Sec. 4, we show that the infinitely many relevant modes pointed out by Fisher are unphysical modes.

In the next section, we carefully reexamine the critical phenomena of $(4 + \epsilon)$-dimensional random field $\mathrm{O}(N)$ spin model in the large $N$ limit.

## 3 Large $N$ limit

We take the large $N$ limit with $NR(z)$ finite, and redefine $R(z)$ as follows: $NR(z) \to R(z)$. Thus, the one-loop beta function for $R(z)$ becomes

$$\partial_t R(z) = -\epsilon R(z) + A\left(2R'(1)R(z) - zR'(1)R'(z) + \frac{1}{2}[R'(z)]^2\right). \tag{3.1}$$

### 3.1 Fixed points

Following the method given by Balents and Fisher [11], we consider the flow equation for $R'(z)$ instead of that for $R(z)$. Differentiating the one-loop beta function with respect to $z$, we have

$$\partial_t R'(z) = -\epsilon R'(z) + A\Big(R'(1)R'(z) - zR'(1)R''(z) + R'(z)R''(z)\Big). \tag{3.2}$$

We redefine the parameters as follows:

$$R'(z) \equiv \frac{\epsilon}{A}u(z), \quad t' \equiv \epsilon t, \quad u(1) \equiv a. \tag{3.3}$$

The one-loop beta function becomes

$$\partial_{t'} u(z) = (a-1)u(z) - zau'(z) + u(z)u'(z). \tag{3.4}$$

Here, we consider the fixed-point equation

$$0 = (a-1)u(z) - zau'(z) + u(z)u'(z). \tag{3.5}$$

Substituting $z = 1$ into the equation (3.5), we have two fixed points

$$a = 0, 1. \tag{3.6}$$

Solving the differential equation (3.5) under the condition $a = 1$, we have two nontrivial solutions:

$$u(z) = 1, z. \tag{3.7}$$

In the case of $u(z) = 1$, we have

$$R(z) = \frac{\epsilon}{A}\left(z - \frac{1}{2}\right). \tag{3.8}$$

It indicates that

$$(\Delta_1, \Delta_2) = \left(\frac{\epsilon}{A}, 0\right), \tag{3.9}$$

$$(R'(1), R''(1)) = \left(\frac{\epsilon}{A}, 0\right). \tag{3.10}$$

Thus, the solution (3.8) is the "random field solution". In the case of $u(z) = 1$, we have

$$R(z) = \frac{\epsilon}{2A}z^2. \tag{3.11}$$

It indicates that

$$(\Delta_1, \Delta_2) = \left(0, \frac{\epsilon}{2A}\right), \tag{3.12}$$

$$(R'(1), R''(1)) = \left(\frac{\epsilon}{A}, \frac{\epsilon}{A}\right). \tag{3.13}$$

Thus, the solution (3.11) is not the "random field solution" but the "random anisotropy solution".

If we solve the differential equation (3.5) under the condition $a = 0$, the nontrivial solution is obtained as follows:

$$u(z) = z - 1. \tag{3.14}$$

From the solution (3.14), we have

$$R(z) = \frac{\epsilon}{2A}(z-1)^2. \tag{3.15}$$

It indicates that

$$(\Delta_1, \Delta_2) = \left(-\frac{\epsilon}{A}, \frac{\epsilon}{2A}\right), \tag{3.16}$$

$$(R'(1), R''(1)) = \left(0, \frac{\epsilon}{A}\right). \tag{3.17}$$

Thus, the solution (3.15) is unphysical.

We turn to the general $a$. If $a \neq 0, 1$,

$$\frac{du(z)}{dz} = \frac{(a-1)u(z)}{za - u(z)}. \tag{3.18}$$

Taking the inversion, we regard $z$ as a function of $u$. One gets

$$\frac{dz(u)}{du} = \frac{a}{a-1}\frac{z(u)}{u} - \frac{1}{a-1}, \tag{3.19}$$

which is easily integrated. The result is

$$z(u) = C|u|^{\frac{a}{a-1}} + u. \tag{3.20}$$

The constant $C$ is fixed by putting $z = 1$ as follows:

$$C = (1-a)|a|^{-\frac{a}{a-1}}. \tag{3.21}$$

Then, we have

$$z = u - (a-1)\left|\frac{u}{a}\right|^{\frac{a}{a-1}}. \tag{3.22}$$

Now we revert (3.22) to the solution $u(z)$ for (3.18). Because $z(u)$ takes the maximum value 1 at $u = a$, $u(z)$ is double valued as we show in Fig. 3.1. It is seen from (3.18) that $du/dz$ is ill defined on $u = az$. Therefore the lower branch terminates at the origin, so that it should be continued to the region $-1 \leq z < 0$. This is possible only if $a/(a-1)$ is a positive integer.

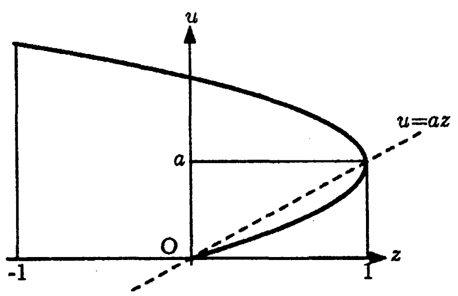


Figure 3.1: A schematic graph of $u(z)$. Since the derivative of $u$ is ill defined on $u = az$, the solution terminates on this line. The above graph represents two solutions meeting at $(1, a)$.

Expanding $u$ around $a$, we have

$$\begin{aligned} z &= u - (a-1)\left|\frac{u}{a}\right|^{\frac{a}{a-1}} \\ &= a + (u-a) - (a-1)\left(1 + \frac{u-a}{a}\right)^{\frac{a}{a-1}} \\ &\simeq 1 - \frac{1}{2a(a-1)}(u-a)^2. \end{aligned} \tag{3.23}$$

Since $-1 \leq z \leq 1$, we have

$$1 - z \simeq \frac{(u-a)^2}{2a(a-1)} \geq 0. \tag{3.24}$$

Thus, we find that the fixed point $a$ must be $a \geq 1$. In practice, in the case of $0 \leq a < 1$, the critical exponent $\bar{\eta}$ becomes negative. In the case of $a > 1$, the equation (3.24) is rewritten as follows:

$$u(z) \simeq a \pm \sqrt{2a(a-1)(1-z)}. \tag{3.25}$$

Note that the plus (minus) sign in front of the square root means to take the upper (lower) branch. Differentiating the above equation by $z$, we have

$$u'(z) \simeq \mp\sqrt{\frac{a(a-1)}{2}}(1-z)^{-1/2}. \tag{3.26}$$

We find that $u'(z)$ diverges as $z \to 1$. Thus, the fixed point $a > 1$ is called the nonanalytic fixed point. In contrast to it, the fixed points (3.8) and (3.11) are called the analytic fixed points.

## 3.2 Stabilities of the fixed-point solutions

Next, we study the stability of the fixed points. Let $u(z)^*$ be a fixed point solution:

$$0 = u(z)^*(a^* - 1) + u'(z)^*(u(z)^* - a^*z). \tag{3.27}$$

We put $u(z)^* \to u(z)^* + v(z)$ and $a^* \to a^* + b$, and study the behavior of the first order in $v(z)$ and $b$:

$$v(z)(a-1) + u(z)b + v'(z)(u(z) - az) + u'(z)(v(z) - bz) = \lambda v(z). \tag{3.28}$$

Here, we omit the asterisk $*$ for brevity. $\lambda$ denotes the eigenvalue. The negative eigenvalue $\lambda < 0$ indicates that the fixed-point solution is stable, and the positive eigenvalue $\lambda > 0$ indicates that the fixed-point solution is unstable. Normalizing $v(z)$ appropriately, we can take $v(1) = 0$ or $v(1) = 1$.

### 3.2.1 $R(z) = \epsilon(z - 1/2)/A$

For $a = 1$ and $u(z) = 1$, the equation (3.28) becomes

$$b + v'(z)(1-z) = \lambda v(z), \tag{3.29}$$

where $b$ represents $v(1)$ taking 0 or 1. When $b = 0$, the solution is

$$v(z) = C(1-z)^{-\lambda}, \tag{3.30}$$

where $\lambda < 0$ because of the initial condition $b = v(1) = 0$. On the other hand, when $b = 1$, a general solution is

$$v(z) = \begin{cases} \lambda^{-1} + c(1-z)^{-\lambda} & (\lambda \neq 0), \\ \ln|1-z| & (\lambda = 0). \end{cases} \tag{3.31}$$

Here the condition $b = 1$ requires that $\lambda = 1$ and $c = 0$. In conclusion, the allowed value of $\lambda$ is $\lambda < 0$ or $\lambda = 1$. This shows that the fixed-point solution is singly unstable.

### 3.2.2 $R(z) = \epsilon z^2/(2A)$

For $a = 1$ and $u(z) = z$, the equation (3.28) becomes

$$v(z) = \lambda v(z). \tag{3.32}$$

Then, $\lambda = 1$, and the fixed point is fully unstable.

### 3.2.3 $R(z) = 0$

Since $a = 0$ and $u = 0$ in this case, the equation (3.28) is $-v(z) = \lambda v(z)$, which means $\lambda = -1$ for any $v(z)$; thus the trivial fixed point is fully stable.

### 3.2.4 Nonanalytic case

Next we turn to the nonanalytic case. Using the identity

$$v'(z) = \frac{dv(u)}{du}\frac{du}{dz}, \tag{3.33}$$

the equation (3.28) is rewritten as follows:

$$\frac{dv}{du} + f(u)v = g(u), \tag{3.34}$$

$$f(u) = -\left(\frac{1}{u} + \frac{1}{az-u} - \frac{\lambda}{(a-1)u}\right), \tag{3.35}$$

$$g(u) = \left(\frac{1}{a-1} - \frac{z}{az-u}\right)b. \tag{3.36}$$

In the case of $b = 0$, the equation (3.34) becomes

$$\frac{dv}{du} + f(u)v = 0. \tag{3.37}$$

Solving the above differential equation, we have

$$\begin{aligned} v(u) &= C\exp\left(-\int f(u)du\right) \\ &= C\frac{|u(z)|^{\frac{-\lambda+a}{a-1}}}{|a^{\frac{1}{a-1}} - u(z)^{\frac{1}{a-1}}|}. \end{aligned} \tag{3.38}$$

According to the condition $v(u(1)) = b = 0$, the constant $C$ is fixed as $C = 0$; $v(z) = 0$. Hence, there are no nontrivial solutions satisfying $b = 0$.

Next, we consider the case of $b = 1$. The solution of the differential equation (3.34) is generally written as follows:

$$v(u) = \exp\left(-\int f(u)du\right)\left\{\int g(u)\exp\left(\int f(u)du\right)du + C\right\}. \tag{3.39}$$

Then, we concentrate on the calculation of the integration in the curly bracket. The integrand becomes

$$g(u)\exp\Big(\int f(u)du\Big) = \pm\frac{u^{\frac{\lambda}{a-1}-1}}{a(a-1)}. \tag{3.40}$$

Note that the plus sign is taken for the upper branch and the minus for the lower branch. Inserting this into (3.39), we get

$$v(u) = \begin{cases} -\dfrac{\hat{u}^{a/(a-1)}}{\lambda}\dfrac{1-\hat{u}^{-\lambda/(a-1)}}{1-\hat{u}^{1/(a-1)}} & (\lambda \neq 0), \\ \dfrac{\hat{u}^{a/(a-1)}\ln\hat{u}}{(1-a)(1-\hat{u}^{1/(a-1)})} & (\lambda = 0), \end{cases} \tag{3.41}$$

where $\hat{u} \equiv u/a$. Here the constant terms are chosen to satisfy $v(u(z)) \to 1$ as $z \to 1$. Thus, the deviation $v(u)$ from the upper branch is finite for any $\lambda$, because $\hat{u} \geq 1$. On the contrary, $v(u)$ from the lower branch may diverge at $u = 0$ and $-1$. We need a constraint on $\lambda$ for $v(u)$ to be finite. We find that the lower branch with $a = 3/2$ can be extended to $-1 \leq z \leq 0$, and that $v(u)$ remains finite for $\lambda = 1$ or negative integers; namely, the lower branch with $a = 3/2$ is singly unstable. However, this fixed-point solution is unphysical because it does not satisfy the Schwartz-Soffer inequality $2\eta \geq \bar{\eta}$ [12]. This inequality requires $a = 1+\mathrm{O}(1/N)$. Other physical lower-branch fixed points satisfying the Schwartz-Soffer inequality has many relevant modes of $\mathrm{O}(N)$.

# 4 Subleading corrections

## 4.1 The fixed point

Here, we calculate the subleading correction to the analytic fixed point $R(z) = (\epsilon/A)(z-1/2)$ and the eigenfunctions. We expand the fixed-point solution

$$R(z) = \frac{1}{N}R_0(z) + \frac{1}{N^2}R_1(z) + \mathrm{O}\left(\frac{1}{N^3}\right), \tag{4.1}$$

and calculate the subleading correction $R_1(z)$. Substituting this expansion into (2.26), we obtain

$$\begin{aligned}\partial_t R_1(z) &= -\epsilon R_1(z) + A\Big(2R_1'(1)R_0(z) + 2R_0'(1)R_1(z) - zR_1'(1)R_0'(z) - zR_0'(1)R_1'(z) \\ &\quad + R_1'(z)R_0'(z) - 4R_0'(1)R_0(z) + zR_0'(1)R_0'(z) + (1-z^2)R_0'(1)R_0''(z) \\ &\quad + \frac{1}{2}(z^2-2)R_0'(z)^2 - R_0'(z)R_0''(z)z(1-z^2) + \frac{1}{2}R_0''(z)^2(1-z^2)^2\Big).\end{aligned}$$

We substitute the unique singly unstable fixed-point solution

$$R_0(z) = \frac{\epsilon}{A}\Big(z - \frac{1}{2}\Big)$$

into the above equation; then we obtain a fixed-point equation for the corresponding correction $R_1(z)$,

$$(1-z)R_1'(z) + R_1(z) - (1-z)R_1'(1) + \frac{\epsilon}{A}\Big(\frac{1}{2}z^2 - 3z + 1\Big) = 0. \tag{4.2}$$

We obtain the following unique solution of this equation:

$$R_1(z) = \frac{\epsilon}{2A}(z^2 + 2z). \tag{4.3}$$

Fisher indicated that this fixed point exists for $N \geq 18$.

### 4.2 Stability of the analytic fixed point

We substitute the analytic fixed point expanded in $1/N$ into the eigenvalue equation for an infinitesimal deformation of the coupling function

$$(1-z)^2(1+z)v''(z)+(1-z)(N-4z-2)v'(z)+(2z-N\lambda)v(z)+(N-2)v(1)=0. \quad (4.4)$$

First, we study the equation for $v(1)=0$. Solutions of this equation have regular singular points $z=1$ and $-1$ for the interval $-1\leq z\leq 1$. Therefore, we can obtain the solutions in the following expansion forms around $z=1$:

$$v(z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}a_n(1-z)^n, \quad (4.5)$$

and around $z=-1$

$$v(z)=(1+z)^{\beta}\sum_{n=0}^{\infty}b_n(1+z)^n. \quad (4.6)$$

Substituting these forms into the eigenvalue equation, we require that the coefficient of the lowest order vanishes. This requirement gives the indicial equations for the exponents $\alpha$ and $\beta$

$$2\alpha^2+(N-4)\alpha+2-N\lambda=0, \quad \beta(2\beta+N)=0, \quad (4.7)$$

which have solutions

$$\alpha_\pm=\frac{4-N\pm\sqrt{N^2-8N+8N\lambda}}{4}, \quad \beta=-\frac{N}{2},\ 0. \quad (4.8)$$

The coefficient of an arbitrary order satisfies the following recursion relation:

$$2k(k-\alpha_\pm+\alpha_\mp)a_k^\pm-(\alpha_\pm-k)(\alpha_\pm-k-1)a_{k-1}^\pm=0,$$

for $k=1,2,3,\ldots$. By solving this recursion relation, the expanded solution can be written in the Gaussian hypergeometric function as follows:

$$\sum_{n=0}^{\infty}a_n^\pm(1-z)^n=F\Big(1-\alpha_\pm,2-\alpha_\pm,3-2\alpha_\pm-\frac{N}{2};\frac{1-z}{2}\Big). \quad (4.9)$$

Solutions with $\alpha>0$ or $\beta<0$ diverge at $z=1$ or $-1$, and they are unphysical. To obtain a finite solution for the interval $-1\leq z\leq 1$, we construct a general solution as a linear combination of two solutions,

$$v(z)=C_+(1-z)^{-\alpha_+}\sum_{n=0}^{\infty}a_n^+(1-z)^n+C_-(1-z)^{-\alpha_-}\sum_{n=0}^{\infty}a_n^-(1-z)^n. \quad (4.10)$$

We can eliminate the divergent solution with $\beta=-N/2$ at $z=-1$ by choosing $C_\pm$ for a requirement $|v(-1)|<\infty$. Also the finiteness of $v(1)$ requires $\alpha_\pm<0$, then we obtain a condition on the eigenvalue

$$\lambda<\frac{2}{N}. \quad (4.11)$$

This condition on $\lambda$ implies the existence of slightly relevant modes at this analytic fixed point. In addition to these modes, we find one relevant mode for $v(1)\neq 0$ with $\lambda=1$ by solving the eigenvalue equation, as well as in the large $N$ limit. This fixed point yielding dimensional reduction seems to be unstable except in the large $N$ limit. There is no singly unstable fixed point generally. The only stable fixed point is the trivial fixed point. In a limited coupling constant space where $R''(1)$ is finite, however, the analytic fixed point is singly unstable in the following reason. The renormalization group flow for the couplings $R'(1)$ and $R''(1)$ is depicted in Fig.4.1. From Fig. 4.1, we find that, for a small initial value of $R''(1)$, the flow of $R''(1)$ stays in a compact area. If $R'(1)$ takes a critical value, the coupling $R(z)$ flows toward the analytic fixed point with a finite $R''(1)$. Then, the flow does not generate the relevant mode with an exponent $0 < \lambda < 2/N$ from an initial function with a finite $R''(1)$. This analytic fixed point controls the phase transition, and therefore the critical behavior obeys the dimensional reduction. Since this analytic fixed point exists for $N \geq 18$ as pointed out by Fisher [2], the dimensional reduction occurs for $N \geq 18$. In this case, the critical exponents of correlation function are given by (2.32). This result agrees with our simple $1/N$ expansion [8].

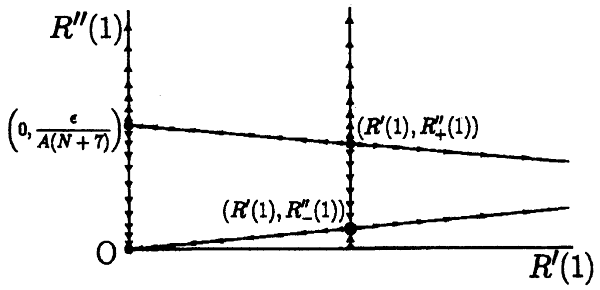


Figure 4.1: The renormalization group flow for the couplings $R'(1)$ and $R''(1)$.

Here we comment on the infinitely many relevant modes pointed out by Fisher [2]. They are included in the following series in our solution (4.10):

$$\alpha_- = 1 - k, \ (k = 3, 4, 5, \dots) \quad \text{and} \quad C_+ = 0.$$

These belong to the eigenvalues

$$\lambda_k = 1 - k + \frac{2k^2}{N} + O\left(\frac{1}{N^2}\right),$$

which are positive for sufficiently large $k$. These agree with the eigenvalues obtained by Fisher, although we should add a term $2nkP_2P_k$ missed in Eq. (C6) of his paper. Since these relevant modes diverge at $z = -1$, we have eliminated them as unphysical modes, as discussed above.

## 5 Conclusion

In this note, we have studied the critical phenomena of $(4+\epsilon)$-dimensional random field $O(N)$ spin model for sufficiently large $N$, by means of the renormalization group method. We find all fixed points which consist of analytic and nonanalytic ones in the large $N$ limit. On the other hand for $N < 18$, it is known that there are no nontrivial analytic fixed points [2]. By solving the eigenvalue problem for the infinitesimal deviation from the fixed point, we find that the nonanalytic fixed points are fully unstable. We search for consistent solutions of the renormalization group with the $1/N$ expansion. If the initial $R''(1)$ is finite, the nonanalytic relevant modes cannot be generated. In this case, the unique analytic fixed point practically behaves as a singly unstable fixed point, which gives the dimensional reduction. This result agrees with the stability of the replica-symmetric saddle-point solution in the $1/N$ expansion. Thus, we conclude dimensional reduction occurs.

Our result also agrees with a recent study of the random field O($N$) model by Tarjus and Tissier. They study the model by a nonperturbative renormalization group [4]. Although their work to obtain a full solution is in progress, they give a global picture in a $d$-$N$ phase diagram and discuss the consistency of their results with those by some perturbative results. They propose a scheme to fix a phase boundary of the phase where the dimensional reduction breaks down. Using an approximation method, they show that the phase is in a compact area on the $d$-$N$ plane.

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