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Effective mass of nonrelativistic quantum electrodynamics

Fumio Hiroshima*and K. R. Ito†

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Abstract

The effective mass $m_{\rm eff}$ of the nonrelativistic quantum electrodynamics with spin 1/2 is investigated. Let $m_{\rm eff}/m=1+a_1(\Lambda/m)e^2+a_2(\Lambda/m)e^4+\mathcal{O}(e^6)$, where m denotes the bare mass. $a_1(\Lambda/m)\sim\log(\Lambda/m)$ as $\Lambda\to\infty$ is well known. Also $a_2(\Lambda/m)\sim\sqrt{\Lambda/m}$ is established for a spinless case. It is shown that $a_2(\Lambda/m)\sim(\Lambda/m)^2$ in the case including spin 1/2.

1 Introduction

1.1 Quantum electrodynamics

In this review we study an translation-invariant Hamiltonian minimally coupled to a quantized radiation field in the nonrelativistic quantum electrodynamics. Before going to discuss our problem, we informally derive our Hamiltonian from physical point of view. The conventional quantum electrodynamics is investigated through the Lagrangian density:

$$\mathcal{L}_{\rm QED}(\mathbf{x})=\overline{\psi}(\mathbf{x})(i\gamma^\mu\partial_\mu-m)\psi(\mathbf{x})-\frac{1}{4}F_{\mu\nu}(\mathbf{x})F^{\mu\nu}(\mathbf{x})-e\overline{\psi}(\mathbf{x})\gamma^\mu\psi(\mathbf{x})A_\mu(\mathbf{x}),$$

where $\mathbf{x}=(x_0,x)\in\mathbf{R}\times\mathbf{R}^3$, γ^μ, $\mu=0,1,2,3$, denotes 4×4 gamma matrices ψ the spinor given by $\psi=(\psi_0,\psi_1,\psi_2,\psi_3)^{\rm T}$ and $\overline{\psi}=(\bar\psi_0,\bar\psi_1,\bar\psi_2,\psi_3)\gamma^0$, $A_\mu(\mathbf{x})$ a radiation field with $F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu$, and m,e are the mass and the charge of an electron, respectively. The effective mass $m_{\rm eff}$ is given through two point function $\int_{\mathbf{R}^4}(\Psi,T[\psi(\mathbf{x})\overline{\psi}(0)]\Psi)e^{i(x^0p^0-x\cdot p)}dx$ and the effective charge through the two point function $\int_{\mathbf{R}^4}(\Psi,T[A_\mu(\mathbf{x})A_\nu(0)]\Psi)e^{i(x^0p^0-x\cdot p)}dx$, where Ψ denotes the ground state of the Hamiltonian derived from $\mathcal{L}_{\rm QED}$ and T the time ordered product. In the perturbative quantum electrodynamics, Feynman diagrammatically, the leading term of the effective mass is computed from the self-energy of electron, e.g.,

Figure 1: Electron self-energy

*Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan.
†Department of Math. and Phys. Setsunan University

and the effective charge from the self-energy of photon, e.g.,

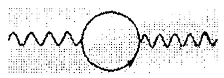

Figure 2: Photon self-energy

One can interpret the photon self-energy diagram as the emission of the pairs of virtual electrons and positrons. All these argument is successive from the physical point of view, but perturbative and implicit divergences includes.

1.2 Informal derivation of nonrelativistic quantum electrodynamics

In this note we want to discuss the quantum electrodynamics nonperturbatically, but we assume that **(1) an electron is in low energy, (2) we take the Coulomb gauge, and (3) we introduce a form factor φ of an electron.**

(1) implies that no emission of pairs of virtual electrons and positrons such as in Fig.2, then in our model the effective charge equals to the bare charge and the number of electrons is fixed. From (2) the theory is not relatively covariant. From (3) it follows that the density of the electron charge is smoothly localised around the position of the electron and the ultraviolet divergence does not exist. Taking into account of (1)-(3), we modify the quantum electrodynamics as follows. Let $E(t,x), B(t,x)$, $(t,x) \in \mathbf{R} \times \mathbf{R}^3$, be an electric field and a magnetic field respectively, and $q(t)$ the position of an electron at time $t \in \mathbf{R}$. The Maxwell equation with form factor φ is given by

$$\begin{aligned} \dot{B} &= -\nabla \times E, \\ \nabla \cdot B &= 0, \\ \dot{E} &= \nabla \times B - e\varphi(\cdot - q(t))\dot{q}(t), \\ \nabla \cdot E &= e\varphi(\cdot - q(t)). \end{aligned}$$

Here $\dot{X} = dX/dt$. Let $(J(t,x), \rho(t,x)) = (e\varphi(x-q(t))\dot{q}(t), e\varphi(x-q(t)))$. Then the Lagrangian density of the nonrelativistic quantum electrodynamics under consideration is given by

$$\mathcal{L}_{\mathrm{NRQED}}(t,x) = \frac{1}{2}m\dot{q}(t)^2 + \frac{1}{2}(E(t,x)^2 - B(t,x)^2) + J(t,x) \cdot A(t,x) - \rho(t,x)\phi(t,x),$$

where A and ϕ are a vector potential and a scalar potential related to E and B such as

$$E = -\dot{A} - \nabla_x \phi, \qquad B = \nabla_x \times A.$$

Let $L_{\mathrm{NRQED}} = \int \mathcal{L}_{\mathrm{NRQED}}(t,x)dx$. Then the conjugate momenta are given

$$p(t) := \frac{\partial L_{\mathrm{NRQED}}}{\partial \dot{q}} = m\dot{q}(t) + e\int A(t,x)\varphi(x-q(t))dx, \quad \Pi(t,x) := \frac{\delta L_{\mathrm{NRQED}}}{\delta \dot{A}} = \dot{A}(t,x).$$

Then the Hamiltonian is given through the Legendre transformation as

$$\begin{aligned} H_{\mathrm{NRQED}} &= p(t)\cdot \dot q(t) + \int \dot A(t,x)\Pi(t,x)dx - L_{\mathrm{NRQED}} \\ &= \frac{1}{2m}\left(p(t) - e\int A(t,x)\varphi(x-q(t))dx\right)^2 + V(q) + \frac{1}{2}\int \left\{\dot A(t,x)^2 + (\nabla \times A(t,x))^2\right\} dx \end{aligned}$$

where V is a smeared external potential given by

$$V(q) := \frac{1}{2}e^2 \int \frac{\varphi(q-y)\varphi(q-y')}{4\pi|y-y'|} dy dy'.$$

In the next subsection we quantize H_{NRQED} with spin 1/2 and total momentum $p \in \mathbf{R}^3$, which is denoted by H and is called the Pauli-Fierz Hamiltonian, in the rigorous way from mathematical point of view.

1.3 Non-relativistic quantum electrodynamics

Let $\mathcal{F}$ be the boson Fock space given by $\mathcal{F} \equiv \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbf{R}^3 \times \{1,2\})\right]$, where $\otimes_s^n$ denotes the n-fold symmetric tensor product with $\otimes_s^0 L^2(\mathbf{R}^3 \times \{1,2\}) \equiv \mathbf{C}$. The Fock vacuum $\Omega \in \mathcal{F}$ is defined by $\Omega \equiv \{1,0,0,..\}$. Let $a(f)$ be the creation operator and $a^*(f)$ the annihilation operator on $\mathcal{F}$ defined by

$$(a^*(f)\Psi)^{(n+1)} \equiv \sqrt{n+1}S_{n+1}(f \otimes \Psi^{(n)}), \quad f \in L^2(\mathbf{R}^3 \times \{1,2\}),$$

and $a(f) = [a^*(\bar f)]^*$, where S_n denotes the symmetrizer. The scalar product on $\mathcal{K}$ is denoted by $(f,g)_{\mathcal{K}}$ which is linear in g and anti-linear in f. They satisfy canonical commutation relations:

$$[a(f), a^*(g)] = (\bar f, g)_{L^2(\mathbf{R}^3 \times \{1,2\})}, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0.$$

We write as $\sum_{j=1,2} \int a^\sharp(k,j) f(k,j) dk$ for $a^\sharp(f)$ with a formal kernel $a^\sharp(k,j)$. Let T be a self-adjoint operator on $L^2(\mathbf{R}^3)$. We define $\Gamma(e^{itT})a^*(f_1)\cdots a^*(f_n)\Omega \equiv a^*(e^{itT}f_1)\cdots a^*(e^{itT}f_n)\Omega$. Thus $\Gamma(e^{itT})$ turns out to be a strongly continuous one-parameter unitary group in t, which implies that there exists a self-adjoint operator $d\Gamma(T)$ on $\mathcal{F}$ such that $\Gamma(e^{itT}) = e^{itd\Gamma(T)}$ for $t \in \mathbf{R}$. We define a Hilbert space $\mathcal{H}$ by $\mathcal{H} \equiv \mathbf{C}^2 \otimes \mathcal{F}$. The Pauli-Fierz Hamiltonian with total momentum $p = (p_1, p_2, p_3) \in \mathbf{R}^3$ is given by a symmetric operator on $\mathcal{H}$:

$$H(p) \equiv \frac{1}{2m}\left\{\sum_{\mu=1}^{3} \sigma_\mu \otimes (p_\mu - P_{\mathrm{f}\mu} - eA_{\hat\varphi_\mu})\right\}^2 + 1 \otimes H_{\mathrm{f}},$$

where $m > 0$ and $e \in \mathbf{R}$ denote the mass and the charge of an electron, respectively, $\sigma \equiv (\sigma_1, \sigma_2, \sigma_3)$ the 2×2 Pauli-matrices given by $\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$,

$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the free Hamiltonian $H_{\rm f}$, the momentum operator $P_{\rm f}$ and quantum radiation field $A_{\hat\varphi_\mu}$ are given by

$$H_{\rm f} \equiv d\Gamma(\omega), \quad P_{{\rm f}\mu} \equiv d\Gamma(k_\mu),$$
$$A_{\hat\varphi_\mu} \equiv \frac{1}{\sqrt{2}} \sum_{j=1,2} \int \frac{\hat\varphi(k)}{\sqrt{\omega(k)}} e_\mu(k,j)(a^*(k,j) + a(k,j))dk, \quad \mu = 1,2,3.$$

Here $e(k,j)$, $j = 1,2$, denotes polarization vectors such that $|e(k,j)| = 1$, $e(k,1)\cdot e(k,2) = 0$, and $e(k,1) \times e(k,2) = k/|k|$. We omit the tensor notation $\otimes$ in what follows. Then

$$H(p) = \frac{1}{2m}(p - P_{\rm f} - eA_{\hat\varphi})^2 + H_{\rm f} - \frac{e}{2m}\sigma B_{\hat\varphi}, \quad p \in \mathbf{R}^3,$$

where B_μ denotes the quantum magnetic field given by

$$B_{\hat\varphi_\mu} \equiv \frac{i}{\sqrt{2}} \sum_{j=1,2} \int \frac{\hat\varphi(k)}{\sqrt{\omega(k)}} (k \times e(k,j))_\mu (a^*(k,j) - a(k,j))dk.$$

Note that $[A_{\hat\varphi_\mu}, B_{\hat\varphi_\nu}] = 0$ for $\mu, \nu = 1,2,3$.

2 Mass renormalization

2.1 Main theorems

Let

$$T_m(e,p) \equiv \frac{1}{2}(p - P_{\rm f} - eA_{\hat\varphi_m})^2 + H_{\rm f} - \frac{e}{2}\sigma B_{\hat\varphi_m}, \quad p \in \mathbf{R}^3,$$

where

$$\hat\varphi_m(k) \equiv \hat\varphi(mk) = \begin{cases} 0, & |k| < \kappa/m, \\ 1/\sqrt{(2\pi)^3}, & \kappa/m \le |k| \le \Lambda/m, \\ 0, & |k| > \Lambda/m. \end{cases} \tag{1}$$

It is established in [1] that $T_m(e,p)$ is self-adjoint on $D(P_{\rm f}^2) \cap D(H_{\rm f})$ for arbitrary $\Lambda > 0, m > 0, p \in \mathbf{R}^3, e \in \mathbf{R}$. Since $A_{\hat\varphi} \cong mA_{\hat\varphi_m}$, $B_{\hat\varphi} \cong mB_{\hat\varphi_m}$, $H_{\rm f} \cong mH_{\rm f}$ and $P_{\rm f} \cong mP_{\rm f}$, where $X \cong Y$ denotes the unitary equivalence, we have $H_m(e,p) \cong mT_m(e,(|p|/m)n_z)$, where $n_z = (0,0,1)$. Let $:X:$ be the Wick product of X. We define

$$H(e,\epsilon) \equiv :T_m(e,\epsilon n_z):, \quad \epsilon \in \mathbf{R}.$$

Set $E(e,\epsilon) \equiv \inf\sigma(H(e,\epsilon))$. It is established in [2] that there exist constants $e_0 > 0$ and $\epsilon_0 > 0$ such that for $(e,\epsilon) \in \mathcal{D}_a \equiv \{(e,\epsilon) \in \mathbf{R}^2 | |e| < e_0, |\epsilon| < \epsilon_0\}$, (1) the dimension of $\mathrm{Ker}(H(e,\epsilon) - E(e,\epsilon))$ is two, (2) $E(e,\epsilon)$ is an analytic function of e^2 and ϵ^2 on $\mathcal{D}_a$, (3) there exists a strongly analytic ground state of $H(e,\epsilon)$. The effective mass $m_{\rm eff}$ is defined by

$$\frac{m}{m_{\rm eff}} = \partial_\epsilon^2 E(e,\epsilon)\lceil_{\epsilon=0}.$$

From this it immediately follows that effective mass m_{eff} is an analytic function of e^2 on $\{e \in \mathbf{R}^3 || e| < e_*\}$ with some $e_* > 0$. Set

$$\frac{m}{m_{\text{eff}}} = \sum_{n=0}^{\infty} a_n(\Lambda/m) e^{2n}.$$

It is known and easily derived that

$$a_1(\Lambda/m) = \frac{8}{3\pi}\frac{1}{4\pi}\Big(\int_{\kappa/m}^{\Lambda/m} \frac{1}{r+2}dr + \int_{\kappa/m}^{\Lambda/m} \frac{r^2}{(r+2)^3}dr\Big).$$

Our next issue is to study $a_2(\Lambda/m)$.

Theorem 2.1 *There exist positive constants $c_2 < c_1$ such that* $-c_1 \le \lim_{\Lambda\to\infty} \frac{a_2(\Lambda/m)}{(\Lambda/m)^2} \le -c_2$.

2.2 Expansions

We set $A \equiv A_{\hat{\varphi}_m}$ and $B \equiv B_{\hat{\varphi}_m}$. Let us define H, E and φ_{g} by

$$H \equiv H(e,0) = H_0 + eH_{\text{I}} + \frac{e^2}{2}H_{\text{II}}, \quad E \equiv E(e,0) = \sum_{n=0}^{\infty}\frac{e^n}{n!}E_{(n)}, \quad \varphi_{\text{g}} \equiv \varphi_{\text{g}}(e,0) = \sum_{n=0}^{\infty}\frac{e^n}{n!}\varphi_{(n)},$$

where $H_0 \equiv H_{\text{f}} + \frac{1}{2}P_{\text{f}}^2$, $H_{\text{I}} \equiv H_{\text{I}}^{(1)} + H_{\text{I}}^{(2)}$, $H_{\text{I}}^{(1)} = AP_{\text{f}}$, $H_{\text{I}}^{(2)} \equiv -\frac{1}{2}\sigma B$, and $H_{\text{II}} \equiv: AA :=$ $A^+A^+ + 2A^+A^- + A^-A^-$. Here we put

$$A^+ \equiv \frac{1}{\sqrt{2}}\sum_{j=1,2}\int \frac{\hat{\varphi}_m(k)}{\sqrt{\omega(k)}}e(k,j)a^*(k,j)dk, \quad A^- \equiv \frac{1}{\sqrt{2}}\sum_{j=1,2}\int \frac{\hat{\varphi}_m(k)}{\sqrt{\omega(k)}}e(k,j)a(k,j)dk.$$

We can see that $E_{(0)} = E_{(2n+1)} = 0$, $n = 0,1,2,3,\ldots$, and

$$\frac{1}{2}E_{(2)} = (\varphi_{(0)}, H_{\text{I}}^{(2)}\varphi_{(1)})_{\mathcal{H}} = -(\varphi_{(0)}, \left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)})_{\mathcal{H}} \neq 0. \tag{2}$$

Note that $E_{(2)} \sim (\Lambda/m)^2$ as $\Lambda \to \infty$. Moreover

$$\begin{aligned}
\varphi_{(0)} &= \begin{pmatrix}1\\0\end{pmatrix}\otimes\Omega,\\
\varphi_{(1)} &= -\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)},\\
\varphi_{(2)} &= \frac{1}{H_0}(-H_{\text{II}})\varphi_{(0)} + 2\frac{1}{H_0}(P_{\text{f}}\cdot A)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)}\\
&\quad +2\frac{1}{H_0}\left\{\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right) - \left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)},\\
\varphi_{(3)} &= -3\frac{1}{H_0}(-H_{\text{II}})\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)} + 3\frac{1}{H_0}(P_{\text{f}}\cdot A)\frac{1}{H_0}(-H_{\text{II}})\varphi_{(0)}
\end{aligned}$$

$$+6\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)}$$
$$+6\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\left\{\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)-\left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)}$$
$$+3\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}(-H_{\rm II})\varphi_{(0)}+6\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\varphi_{(0)}$$
$$+6\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left\{\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)-\left(-\frac{E_{(2)}}{2}\right)\right\}\varphi_{(0)}.$$

Although formula

$$\frac{m}{m_{\rm eff}}=1-\frac{2}{3}\sum_{\mu=1}^{3}\frac{((P_{\rm f}+eA)_\mu\varphi_{\rm g},(H-E)^{-1}(P_{\rm f}+eA)_\mu\varphi_{\rm g})_{\mathcal H}}{(\varphi_{\rm g},\varphi_{\rm g})_{\mathcal H}} \tag{3}$$

is well known, it is not useful for our task, since expansion of $(H-E)^{-1}$ in e leads us to a complicated operator domain argument. Then, instead of (3), it is established in [3] that $\varphi_{\rm g}'\equiv {\rm s}-\partial\varphi_{\rm g}(e,\epsilon)/\partial\epsilon\lceil_{\epsilon=0}$ satisfies that $(P_{\rm f}+eA)_3\varphi_{\rm g}\in D((H-E)^{-1})$ with $\varphi_{\rm g}'=(H-E)^{-1}(P_{\rm f}+eA)_3\varphi_{\rm g}$ and

$$\frac{m}{m_{\rm eff}}=1-2\frac{(\varphi_{\rm g},(P_{\rm f}+eA)_3\varphi_{\rm g}')_{\mathcal H}}{(\varphi_{\rm g},\varphi_{\rm g})_{\mathcal H}}. \tag{4}$$

Using (4) in [2] it is proven that the effective mass is expanded as

$$\frac{m}{m_{\rm eff}}=1-\frac{2}{3}c_1(\Lambda/m)e^2-\frac{2}{3}c_2(\Lambda/m)e^4+\mathcal{O}(e^6), \tag{5}$$

or

$$\frac{m_{\rm eff}}{m}=1+\frac{2}{3}c_1(\Lambda/m)e^2+\left(\frac{2}{3}c_2(\Lambda/m)+\left(\frac{2}{3}\right)^2c_1(\Lambda/m)^2\right)e^4+\mathcal{O}(e^6), \tag{6}$$

where

$$c_1(\Lambda/m)\equiv\sum_{\mu=1}^{3}(\Psi_1^\mu,\tilde H_0\Psi_1^\mu)_{\mathcal H},$$
$$c_2(\Lambda/m)\equiv\sum_{\mu=1}^{3}\Big\{(\Psi_1^\mu,\tilde H_2\Psi_1^\mu)_{\mathcal H}-(\Psi_1^\mu,\tilde H_0\Psi_1^\mu)_{\mathcal H}(\varphi_{(1)},\varphi_{(1)})_{\mathcal H}+2\Re(\Psi_2^\mu,\tilde H_1\Psi_1^\mu)_{\mathcal H}$$
$$+(\Psi_2^\mu,\tilde H_0\Psi_2^\mu)_{\mathcal H}+2\Re(\Psi_3^\mu,\tilde H_0\Psi_1^\mu)_{\mathcal H}\Big\}. \tag{7}$$

Here

$$\Psi_n^\mu\equiv\frac{1}{(n-1)!}A_\mu\varphi_{(n-1)}+\frac{1}{n!}P_{{\rm f}\mu}\varphi_{(n)},\quad n=1,2,3,\quad \mu=1,2,3,$$

and $\tilde H_0\equiv H_1$, $\tilde H_1\equiv H_2+H_3$, $\tilde H_2\equiv H_4+H_5+H_6+H_7+H_8$, where we put

$$H_1=\frac{1}{H_0},\quad H_2=-\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0},\quad H_3=-\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0},$$

$$H_4=\frac{1}{2}\frac{1}{H_0}(-H_{\rm II})\frac{1}{H_0},\quad H_5=\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0},$$
$$H_6=\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0},\quad H_7=H_6{}^*=\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0},$$
$$H_8=\frac{1}{H_0}\left\{\left(-\frac{\sigma B}{2}\right)\frac{1}{H_0}\left(-\frac{\sigma B}{2}\right)-\left(-\frac{E_{(2)}}{2}\right)\right\}\frac{1}{H_0}.$$

From above expressions of $\varphi_{(1)},\varphi_{(2)},\varphi_{(3)}$, it follows that for $\mu=1,2,3$,

$$\Psi_1^\mu\equiv\Phi_1^\mu+\Phi_2^\mu,\quad \Psi_2^\mu\equiv\sum_{i=2}^{6}\Phi_i^\mu,\quad \Psi_3^\mu\equiv\sum_{i=7}^{16}\Phi_i^\mu,$$

where

$$\Phi_1^\mu=A_\mu^+\varphi_{(0)}\qquad \Phi_2^\mu=\frac{1}{2}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}\qquad \Phi_3^\mu=\frac{1}{2}A_\mu\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$
$$\Phi_4^\mu=-\frac{1}{2}P_{{\rm f}\mu}\frac{1}{H_0}A^+A^+\varphi_{(0)}\qquad \Phi_5^\mu=-\frac{1}{2}P_{{\rm f}\mu}\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$
$$\Phi_6^\mu=\frac{1}{4}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}\qquad \Phi_7^\mu=-\frac{1}{2}A_\mu\frac{1}{H_0}A^+A^+\varphi_{(0)}$$
$$\Phi_8^\mu=-\frac{1}{2}A_\mu\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\sigma B^+\varphi_{(0)}\qquad \Phi_9^\mu=\frac{1}{4}A_\mu\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$
$$\Phi_{10}^\mu=-\frac{1}{4}P_{{\rm f}\mu}\frac{1}{H_0}:AA:\frac{1}{H_0}\sigma B^+\varphi_{(0)}\qquad \Phi_{11}^\mu=-\frac{1}{2}P_{{\rm f}\mu}\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}A^+A^+\varphi_{(0)}$$
$$\Phi_{12}^\mu=-\frac{1}{2}P_{{\rm f}\mu}\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\sigma B^+\varphi_{(0)}\qquad \Phi_{13}^\mu=\frac{1}{4}P_{{\rm f}\mu}\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$
$$\Phi_{14}^\mu=\frac{1}{4}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B\frac{1}{H_0}A^+A^+\varphi_{(0)}\qquad \Phi_{15}^\mu=\frac{1}{4}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B\frac{1}{H_0}(P_{\rm f}\cdot A)\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$
$$\Phi_{16}^\mu=-\frac{1}{8}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}$$

Substituting $H_1,...,H_8$ and $\Phi_1^\mu,...,\Phi_{16}^\mu$ into (7), we see that

$$c_2(\Lambda/m)=\sum_{\mu=1}^{3}\left\{(\sum_{i=1}^{2}\Phi_i^\mu,\sum_{\ell=4}^{8}H_\ell\sum_{i=1}^{2}\Phi_i^\mu)_{\mathcal H}+(\sum_{i=1}^{2}\Phi_i^\mu,H_1\sum_{i=1}^{2}\Phi_i^\mu)_{\mathcal H}\right.$$
$$\left.(\sum_{i=3}^{6}\Phi_i^\mu,(H_2+H_3)\sum_{i=1}^{2}\Phi_i^\mu)_{\mathcal H}+(\sum_{i=3}^{6}\Phi_i^\mu,H_1\sum_{i=3}^{6}\Phi_i^\mu)_{\mathcal H}+(\sum_{i=7}^{16}\Phi_i^\mu,H_1\sum_{i=1}^{2}\Phi_i^\mu)_{\mathcal H}\right\}.\qquad(8)$$

From (8) it follows that $c_2(\Lambda/m)$ is decomposed into 76 terms. Fortunately it is, however, enough to consider terms containing even number of σB's, since the terms with odd number of σB vanishes by a symmetry. See Fig. 3-7.

No.	Term	σB	P_f	$\frac{1}{H_0}$	Order
(1)	$-(\Phi_1^\mu, H_1\Phi_1^\mu)(\varphi_{(1)}, \varphi_{(1)})_{\mathcal{H}}$	0	0	1	$[\log(\Lambda/m)]^2$
(2)	$-(\Phi_2^\mu, H_1\Phi_2^\mu)(\varphi_{(1)}, \varphi_{(1)})_{\mathcal{H}}$	2	2	3	$[\log(\Lambda/m)]^2$

Figure 3: $(\sum_{i=1}^2 \Phi_i^\mu, H_1 \sum_{i=1}^2 \Phi_i^\mu)_{\mathcal{H}}$

No.	Term	σB	P_f	$\frac{1}{H_0}$	Order
(3)	$(\Phi_3^\mu, H_1\Phi_3^\mu)$	2	0	3	$[\log(\Lambda/m)]^2$
(4)	$(\Phi_5^\mu, H_1\Phi_3^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(5)	$(\Phi_3^\mu, H_1\Phi_5^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(6)	$(\Phi_4^\mu, H_1\Phi_4^\mu)$	0	2	3	$+\sqrt{\Lambda/m}$
(7)	$(\Phi_6^\mu, H_1\Phi_4^\mu)$	2	2	4	$-\sqrt{\Lambda/m}$
(8)	$(\Phi_4^\mu, H_1\Phi_6^\mu)$	2	2	4	$-\sqrt{\Lambda/m}$
(9)	$(\Phi_5^\mu, H_1\Phi_5^\mu)$	2	4	5	$+\sqrt{\Lambda/m}$
(10)	$(\Phi_6^\mu, H_1\Phi_6^\mu)$	4	2	5	$[\log(\Lambda/m)]^2$

Figure 4: $(\sum_{i=3}^6 \Phi_i^\mu, H_1 \sum_{i=3}^6 \Phi_i^\mu)_{\mathcal{H}}$

No.	Term	σB	P_f	$\frac{1}{H_0}$	Order
(11)	$(\Phi_1^\mu, H_4\Phi_1^\mu)$	0	0	2	$[\log(\Lambda/m)]^2$
(12)	$(\Phi_2^\mu, H_4\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(13)	$(\Phi_1^\mu, H_5\Phi_1^\mu)$	0	2	3	$\log(\Lambda/m)$
(14)	$(\Phi_2^\mu, H_5\Phi_2^\mu)$	2	4	5	$[\log(\Lambda/m)]^2$
(15)	$(\Phi_2^\mu, H_6\Phi_1^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(16)	$(\Phi_1^\mu, H_6\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(17)	$(\Phi_2^\mu, H_7\Phi_1^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(18)	$(\Phi_1^\mu, H_7\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(19)	$(\Phi_1^\mu, H_8\Phi_1^\mu)$	2	0	3	$[\log(\Lambda/m)]^2$
(20)	$(\Phi_2^\mu, H_8\Phi_2^\mu)$	4	2	5	$[\log(\Lambda/m)]^2$

Figure 5: $(\sum_{i=1}^2 \Phi_i^\mu, \sum_{\ell=4}^8 H_\ell \sum_{i=1}^2 \Phi_i^\mu)_{\mathcal{H}}$

No.	Term	σB	P_f	$\frac{1}{H_0}$	Order
(21)	$(\Phi_4^\mu, H_2\Phi_1^\mu)$	0	2	3	$\log(\Lambda/m)$
(22)	$(\Phi_6^\mu, H_2\Phi_1^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(23)	$(\Phi_3^\mu, H_2\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(24)	$(\Phi_5^\mu, H_2\Phi_2^\mu)$	2	4	5	$\sqrt{\Lambda/m}$
(25)	$(\Phi_3^\mu, H_3\Phi_1^\mu)$	2	0	4	$[\log(\Lambda/m)]^2$
(26)	$(\Phi_5^\mu, H_3\Phi_1^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(27)	$(\Phi_4^\mu, H_3\Phi_2^\mu)$	2	2	4	$\sqrt{\Lambda/m}$
(28)	$(\Phi_6^\mu, H_3\Phi_2^\mu)$	4	2	5	$-\Lambda/m$

Figure 6: $(\sum_{i=3}^{6}\Phi_i^\mu, (H_2+H_3)(\sum_{i=1}^{2}\Phi_i^\mu))_{\mathcal{H}}$

No.	Term	σB	P_f	$\frac{1}{H_0}$	Order
(29)	$(\Phi_7^\mu, H_1\Phi_1^\mu)$	0	0	2	$[\log(\Lambda/m)]^2$
(30)	$(\Phi_9^\mu, H_1\Phi_1^\mu)$	2	0	3	$[\log(\Lambda/m)]^2$
(31)	$(\Phi_{11}^\mu, H_1\Phi_1^\mu)$	0	2	3	$=0$
(32)	$(\Phi_{13}^\mu, H_1\Phi_1^\mu)$	2	2	4	$=0$
(33)	$(\Phi_{15}^\mu, H_1\Phi_1^\mu)$	2	2	4	$=0$
(34)	$(\Phi_8^\mu, H_1\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(35)	$(\Phi_{10}^\mu, H_1\Phi_2^\mu)$	2	2	4	$\sqrt{\Lambda/m}$
(36)	$(\Phi_{12}^\mu, H_1\Phi_2^\mu)$	2	4	5	$[\log(\Lambda/m)]^2$
(37)	$(\Phi_{14}^\mu, H_1\Phi_2^\mu)$	2	2	4	$[\log(\Lambda/m)]^2$
(38)	$(\Phi_{16}^\mu, H_1\Phi_2^\mu)$	4	2	5	$-(\Lambda/m)^2$

Figure 7: $(\sum_{i=7}^{16}\Phi_i^\mu, H_1\sum_{i=1}^{2}\Phi_i^\mu)_{\mathcal{H}}$

2.3 Feynman diagrams

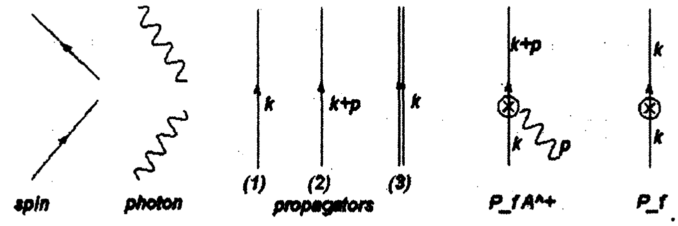

Figure 8: Items of diagrams

The 38 terms in Fig. 3-7 can be represented by Feynman diagrams. The items of diagrams are in Fig. 8.

$$\begin{aligned}
&\text{spin} = \sigma B^+ = \frac{1}{\sqrt{2}}\frac{\sigma(k,j)}{\sqrt{(2\pi)^3\omega(k)}}, \quad \text{photon} = A_\mu = \frac{1}{\sqrt{2}}\frac{e_\mu(k,j)}{\sqrt{(2\pi)^3\omega(k)}} \\
&\text{propagator}(1) = \frac{1}{H_0} = \frac{1}{\omega(k_i) + |k_i|^2/2} \equiv \frac{1}{E_i} \\
&\text{propagator}(2) = \frac{1}{H_0} = \frac{1}{\omega(k_1) + \omega(k_2) + |k_1 + k_2|^2/2} \equiv \frac{1}{E_{12}} \\
&\text{propagator}(3) = (\frac{1}{H_0})^2 = (\frac{1}{\omega(k) + |k|^2/2})^2 \\
&(P_f \cdot A^+) = k \cdot e(p,j), \quad P_{f\mu} = k_\mu,
\end{aligned}$$

where $\sigma(k,j) = \sigma \cdot (k \times e(k,j))$.

Example 2.2 *We compute $(\Phi_5^\mu, H_1\Phi_5^\mu)$ as an example. Since*

$$(\Phi_5^\mu, H_1\Phi_5^\mu) = \frac{1}{4}(P_{f\mu}\frac{1}{H_0}(P_f \cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{f\mu}\frac{1}{H_0}(P_f \cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}),$$

its diagram is given as in Fig. 9.

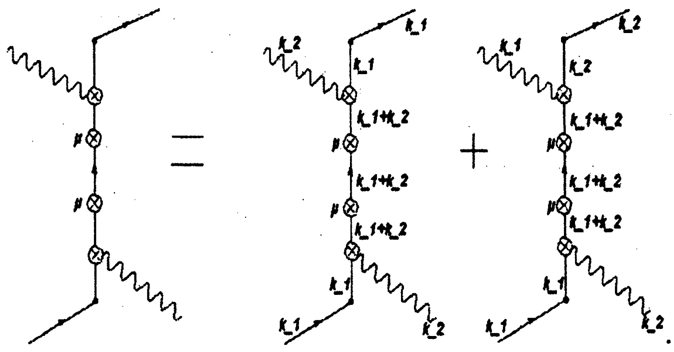

Figure 9: Example

Then

$$(\Phi_5^\mu, H_1\Phi_5^\mu) = \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}$$
$$\times\langle \sigma_1 \frac{1}{E_1}(k_1\cdot e_2)\frac{1}{E_{12}}(k_1+k_2)_\mu \frac{1}{E_{12}}(k_1+k_2)_\mu \frac{1}{E_{12}}\Big((k_1+k_2)\cdot e_2\frac{1}{E_1}\sigma_1 + (k_1+k_2)\cdot e_1 \frac{1}{E_2}\sigma_2\Big)\rangle$$
$$= \int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{|k_1+k_2|^2}{E_{12}^3}\Big(\frac{\langle\sigma_1\sigma_1\rangle(k_1\cdot e_2)(k_1\cdot e_2)}{E_1^2} + \frac{\langle\sigma_1\sigma_2\rangle(k_1\cdot e_2)(k_2\cdot e_1)}{E_1E_2}\Big)$$

Here and in what follows we set $e_1 = e(k_1, j)$, $e_2 = e(k_2, j')$, $\sigma_1 = \sigma(k_1, j)$, $\sigma_2 = \sigma(k_2, j')$ *and* $\langle X\rangle$ *denotes the expectation value of* X*:* $\langle X\rangle = \left(\begin{pmatrix}1\\0\end{pmatrix}, X\begin{pmatrix}1\\0\end{pmatrix}\right)_{\mathbb{C}^2}$.

The diagrams consists of three kinds of diagrams; $AA \to AA$ type, $\sigma A \to \sigma A$ type and $\sigma\sigma \to \sigma\sigma$ type. In particular $AA \to AA$ type corresponds to the spinless model discussed in [3].

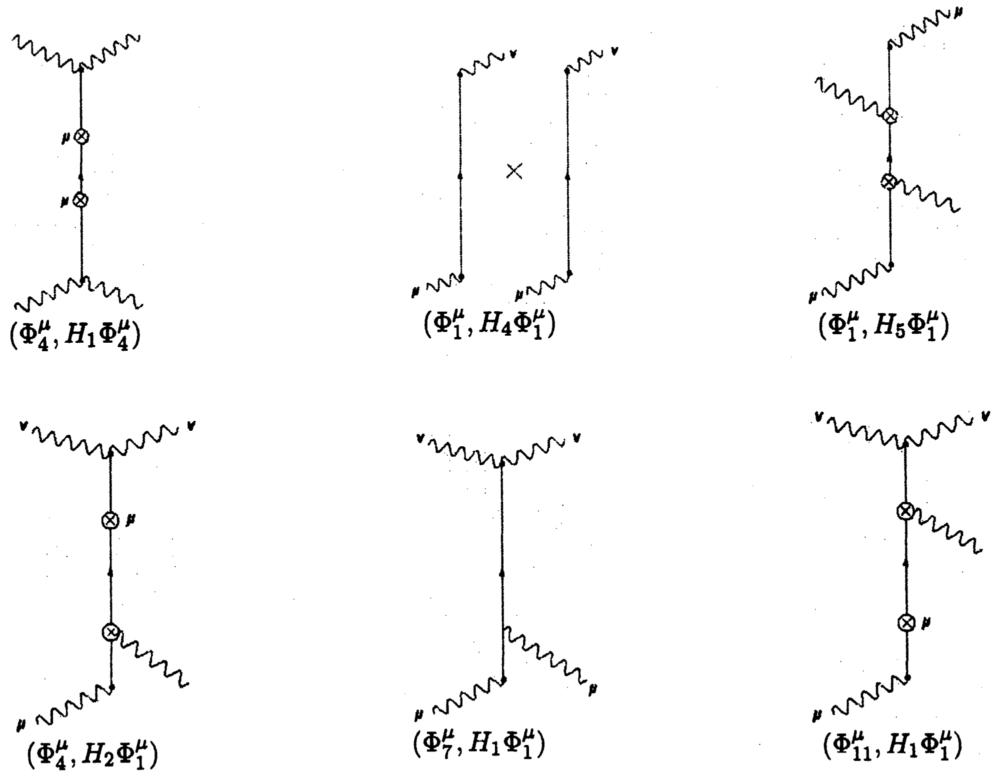

Figure 10: $AA \to AA$ type Feynman diagrams

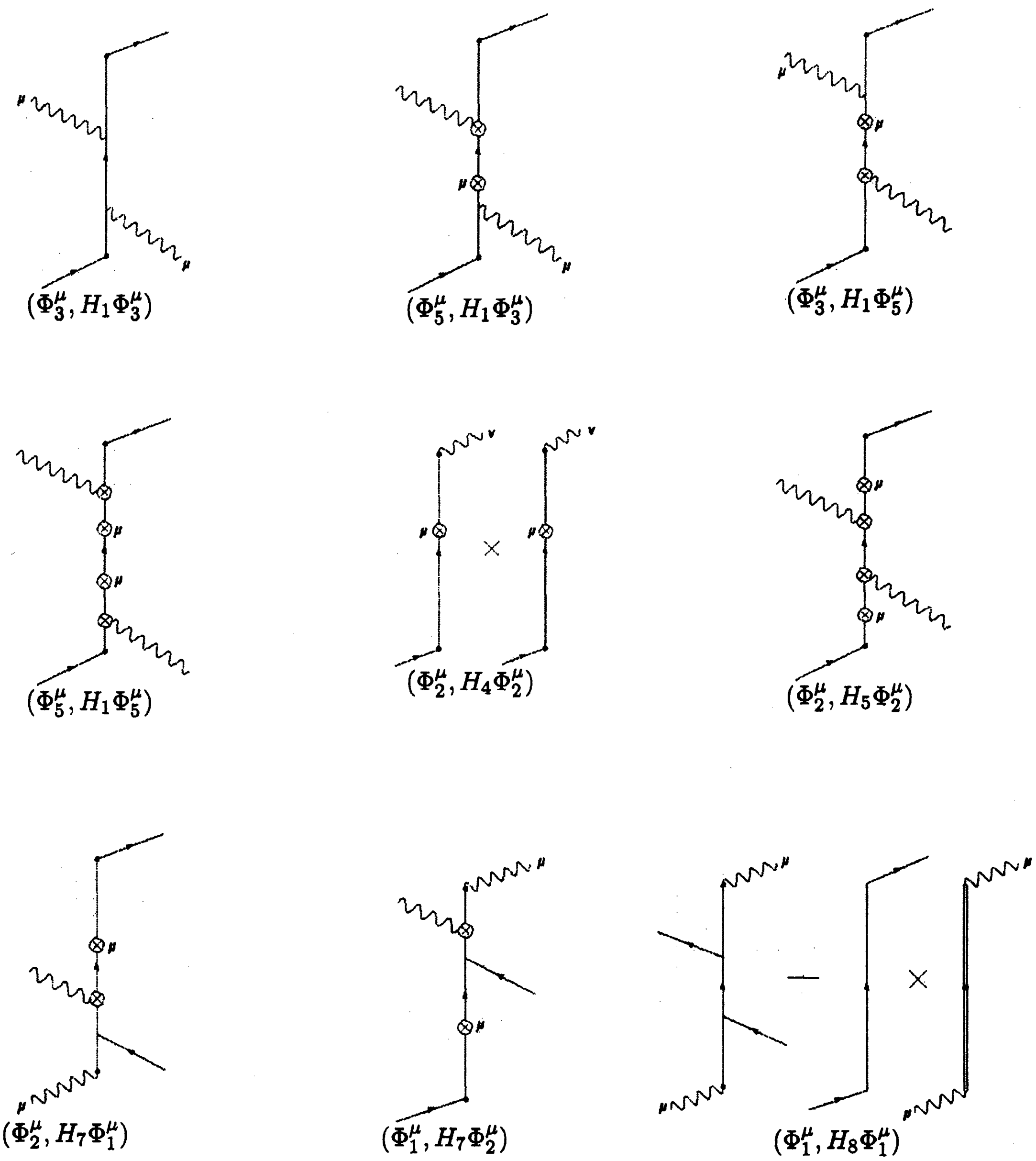

Figure 11: $A\sigma \to A\sigma$ type Feynman diagrams

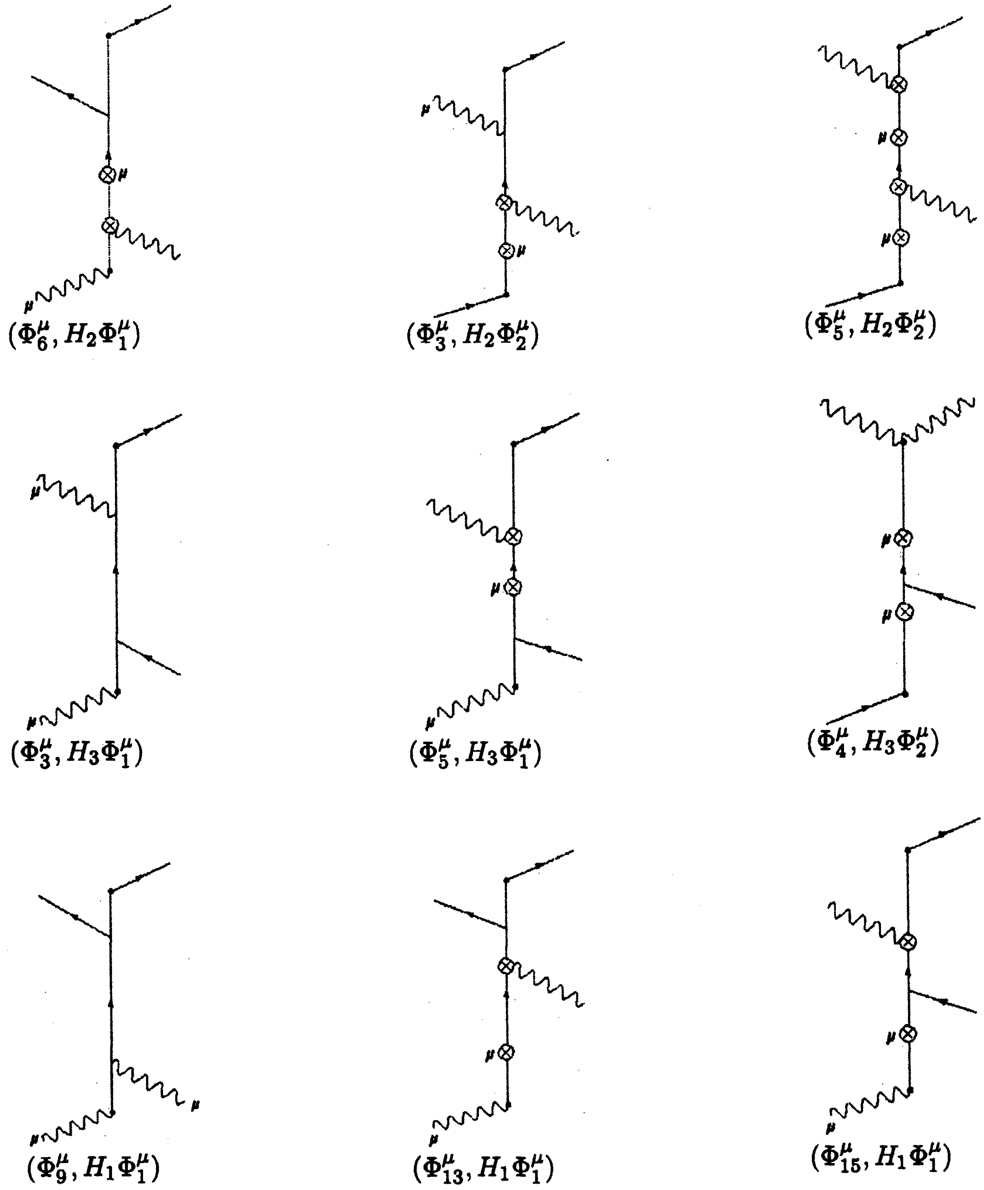

Figure 12: $A\sigma \rightarrow A\sigma$ type Feynman diagrams

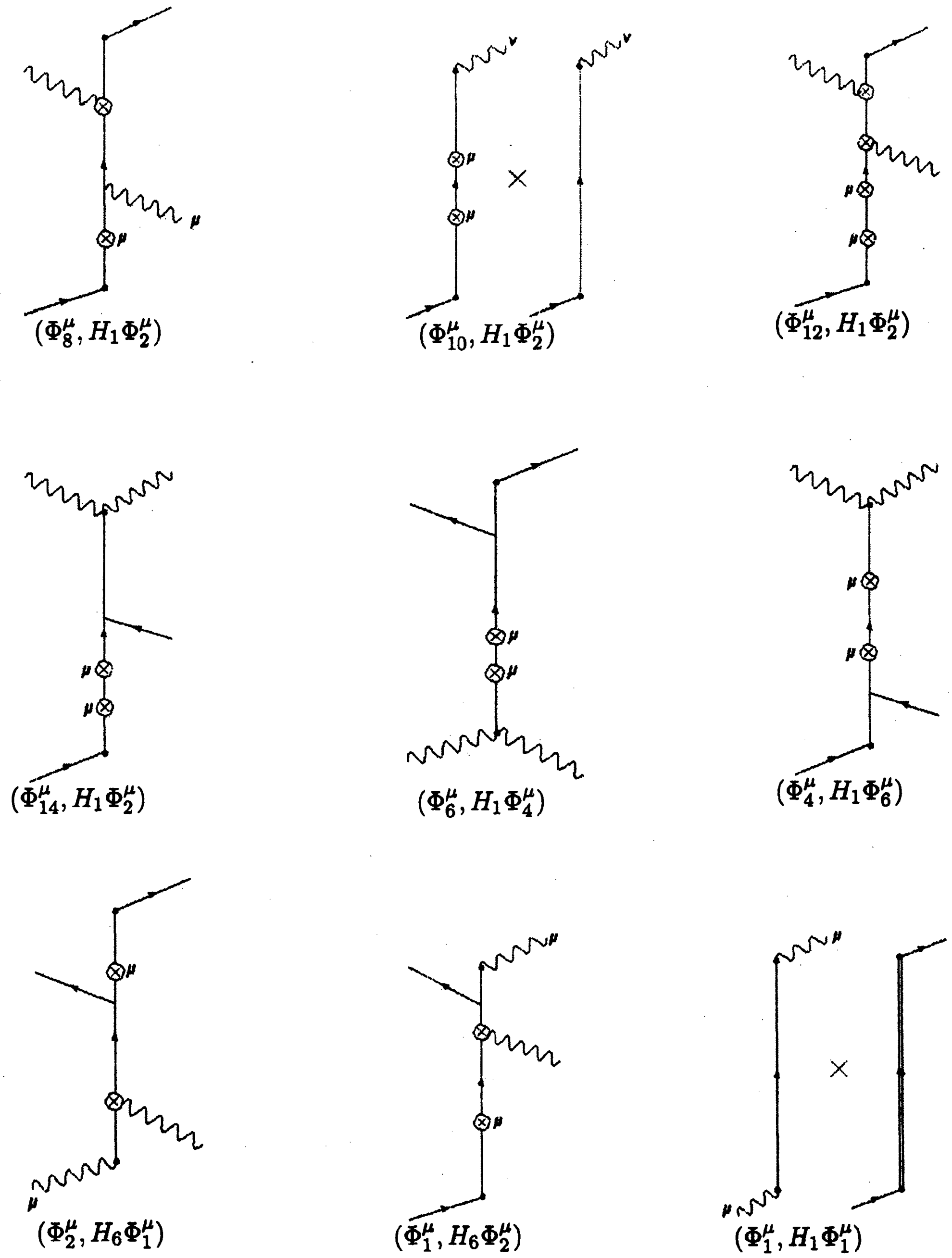

Figure 13: $A\sigma \to A\sigma$ type Feynman diagrams

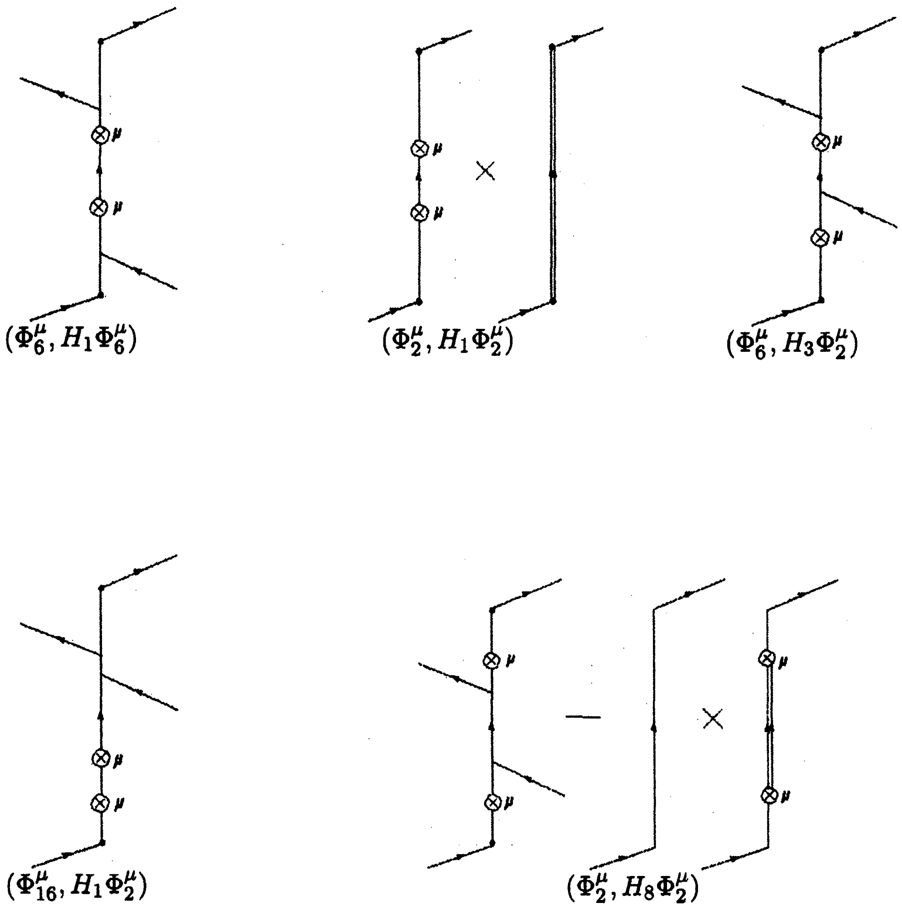

Figure 14: $\sigma\sigma \to \sigma\sigma$ type Feynman diagrams

2.4 Proof of Theorem 2.1

We shall prove Theorem 2.1. Note that the following formulas are useful.

Lemma 2.3

$$\begin{aligned}
e_i \cdot e_i &= 2 && (9)\\
(e_1\cdot e_2)(e_1\cdot e_2) &= 1 + (\hat{k}_1, \hat{k}_2)^2 && (10)\\
(k_2\cdot e_1)(k_2\cdot e_1) &= |k_2|^2(1 - (\hat{k}_1, \hat{k}_2)^2) && (11)\\
(k_1\cdot e_2)(e_2\cdot e_1)(e_1\cdot k_2) &= -(k_1, k_2)(1 - (\hat{k}_1, \hat{k}_2)^2) && (12)\\
\langle \sigma_i \sigma_i \rangle &= 2|k_i|^2 && (13)
\end{aligned}$$

$$\begin{aligned}\Re\langle\sigma_1\sigma_2\rangle(e_1\cdot k_2)(k_1\cdot e_2) &= |k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1) &(14)\\ \Re\langle\sigma_1\sigma_2\rangle(e_1\cdot e_2) &= 2|k_1||k_2|(\hat{k}_1,\hat{k}_2) &(15)\\ \langle\sigma_1\sigma_2\sigma_2\sigma_1\rangle &= 4|k_1|^2|k_2|^2 &(16)\\ \Re\langle\sigma_1\sigma_2\sigma_1\sigma_2\rangle &= -2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2) &(17)\end{aligned}$$

Proof: Note that

$$e_\mu(k,j)e_\nu(k,j) = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}), \quad (k\times e(k,j))_\mu e_\nu(k,j) = -\epsilon^{\mu\nu\alpha}k_\alpha, \quad \Re\langle\sigma_\mu\sigma_\nu\rangle = \delta_{\mu\nu}.$$

We see that

(1) $e_i\cdot e_i = e_\mu(k_i,j)e_\mu(k_i,j') = 2$,

(2) $(e_1\cdot e_2)(e_1\cdot e_2) = e_\mu(k_1,j)e_\mu(k_2,j')e_\nu(k_1,j)e_\nu(k_2,j')$
$= (\delta_{\mu\nu} - \frac{k_{1\mu}k_{1\nu}}{|k_1|^2})(\delta_{\mu\nu} - \frac{k_{2\mu}k_{2\nu}}{|k_2|^2}) = (3-1-1+\frac{(\hat{k}_1,\hat{k}_2)^2}{|k_1|^2|k_2|^2}) = 1+(\hat{k}_1,\hat{k}_2)^2$,

(3) $(k_2\cdot e_1)(k_2\cdot e_1) = k_{2\mu}e_\mu(k_1,j)k_{2\nu}e_\nu(k_1,j) = k_{2\mu}k_{2\nu}(\delta_{\mu\nu} - \frac{k_{1\mu}k_{1\nu}}{|k_1|^2}) = |k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)$,

(4) $(k_1\cdot e_2)(e_2\cdot e_1)(e_1\cdot k_2) = k_{2\mu}e_\mu(k_1,j)e_\lambda(k_1,j)e_\lambda(k_2,j')k_{1\nu}e_\nu(k_2,j')$
$= k_{2\mu}(\delta_{\mu\lambda} - \frac{k_{1\mu}k_{1\lambda}}{|k_1|^2})k_{1\nu}(\delta_{\lambda\nu} - \frac{k_{2\lambda}k_{2\nu}}{|k_2|^2}) = -(k_1,k_2)(1-(\hat{k}_1,\hat{k}_2)^2)$,

(5) $\Re\langle\sigma_1\sigma_1\rangle = \Re\langle\sigma_\mu\sigma_\nu\rangle(k_1\times e(k_1,j))_\mu(k_1\times e(k_1,j))_\nu = \delta_{\mu\nu}(k_1\times e(k_1,j))_\mu(k_1\times e(k_1,j))_\nu$
$= (k_1\times e(k_1,j))_\mu(k_1\times e(k_1,j))_\mu = |k_1|^2(|e(k_1,2)|^2+|e(k_1,3)|^2) = 2|k_1|^2$,

(6) $\Re\langle\sigma_1\sigma_2\rangle(e_1\cdot k_2)(k_1\cdot e_2) = \Re\langle\sigma_\mu\sigma_\nu\rangle(k_1\times e(k_1,j))_\mu(k_2\times e(k_2,j'))_\nu e_\alpha(k_1,j)k_{2\alpha}e_\beta(k_2,j')k_{1\beta}$
$= (k_1\times e(k_1,j))_\mu(k_2\times e(k_2,j'))_\mu e_\alpha(k_1,j)k_{2\alpha}e_\beta(k_2,j')k_{1\beta}$
$= (-\epsilon^{\mu\alpha\gamma}k_{1\gamma})(-\epsilon^{\mu\beta\delta}k_{2\delta})k_{2\alpha}k_{1\beta} = -|k_1\times k_2|^2 = -|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)$,

(7) $\Re\langle\sigma_1\sigma_2\rangle(e_1\cdot e_2) = \Re\langle\sigma_\mu\sigma_\nu\rangle(k_1\times e(k_1,j))_\mu(k_2\times e(k_2,j'))_\nu e_\alpha(k_1,j)e_\alpha(k_2,j')$
$= (-\epsilon^{\mu\alpha\beta}k_{1\beta})(-\epsilon^{\mu\alpha\gamma}k_{2\gamma}) = 2(k_1\cdot k_2)$,

(8) $\langle\sigma_1\sigma_2\sigma_2\sigma_1\rangle = \langle\sigma_1\sigma_\mu\sigma_\nu\sigma_1\rangle(k_2\times e(k_2,j'))_\mu(k_2\times e(k_2,j'))_\nu$
$= \langle\sigma_1\sigma_1\rangle(k_2\times e(k_2,j'))_\mu(k_2\times e(k_2,j'))_\mu$
$= |(k_2\times e(k_2,j'))|^2|(k_1\times e(k_1,j'))|^2 = 2|k_1|^2 2|k_2|^2 = 4|k_1|^2|k_2|^2$,

(9) $\Re\langle\sigma_1\sigma_2\sigma_1\sigma_2\rangle = -\langle\sigma_1\sigma_2\sigma_2\sigma_1\rangle + 2(k_1\times e(k_1,j))_\mu(k_2\times e(k_2,j'))_\mu\Re\langle\sigma_1\sigma_2\rangle$
$= -4|k_1|^2|k_2|^2 + 2(k_1\times e(k_1,j))_\mu(k_2\times e(k_2,j'))_\mu(k_1\times e(k_1,j))_\nu(k_2\times e(k_2,j'))_\nu$
$= -4|k_1|^2|k_2|^2 + 2(\delta_{\mu\nu} - \frac{k_{1\mu}k_{1\nu}}{|k_1|^2})(\delta_{\mu\nu} - \frac{k_{2\mu}k_{2\nu}}{|k_2|^2})$
$= -4|k_1|^2|k_2|^2 + 2|k_1|^2|k_2|^2(1+(\hat{k}_1,\hat{k}_2)^2) = -2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)$.

□

Using the diagrams presented in Fig.10-14 and (9)-(17), we can easily expressed 38 terms as integrals on

$$D = \{(k_1,k_2)\in\mathbf{R}^3\times\mathbf{R}^3 | \kappa/m \le k_1 \le \Lambda/m, \kappa/m \le k_2 \le \Lambda/m\}.$$

Note that imaginary part of $\langle\sigma_a\sigma_b\rangle$ does not contribute integrals. We show the results:

1.

$$
\begin{aligned}
&(\Phi_1^\mu, H_1\Phi_1^\mu)(\varphi_{(1)},\varphi_{(1)}) = \frac{1}{4}(A_\mu^+\varphi_{(0)}, \frac{1}{H_0}A_\mu^+\varphi_{(0)})(\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_2}e_1\cdot e_1\langle\sigma_2\sigma_2\rangle\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{4|k_2|^2}{E_1E_2^2}
\end{aligned}
$$

2.

$$
\begin{aligned}
&(\Phi_2^\mu, H_1\Phi_2^\mu)(\varphi_{(1)},\varphi_{(1)})\\
&= \frac{1}{4}(P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})(\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_1}\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_2}|k_1|^2\langle\sigma_1\sigma_1\rangle\langle\sigma_2\sigma_2\rangle\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{4|k_1|^4|k_2|^2}{E_1^3E_2^2}
\end{aligned}
$$

3.

$$
\begin{aligned}
&(\Phi_3^\mu, H_1\Phi_3^\mu) = \frac{1}{4}(A_\mu^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}A_\mu^+\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_2}\frac{1}{E_{12}}(\langle\sigma_2\sigma_2\rangle(e_1\cdot e_1)\frac{1}{E_2} + \langle\sigma_2\sigma_1\rangle(e_1\cdot e_2)\frac{1}{E_1})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_2}\frac{1}{E_{12}}(\frac{4|k_2|^2}{E_2} + \frac{2|k_1||k_2|(\hat{k}_1,\hat{k}_2)}{E_1})
\end{aligned}
$$

4.

$$
\begin{aligned}
&(\Phi_5^\mu, H_1\Phi_3^\mu) = -\frac{1}{4}(P_{\mathrm{f}\mu}\frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}A_\mu^+\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}((k_2\cdot e_1)(k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2} + (k_2\cdot e_1)(k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})\\
&= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(\frac{2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2} + \frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1})
\end{aligned}
$$

5.

$$
(\Phi_3^\mu, H_1\Phi_5^\mu) = \overline{(\Phi_5^\mu, H_1\Phi_3^\mu)}
$$

6.

$$
\begin{aligned}
&(\Phi_4^\mu, H_1\Phi_4^\mu) = [3,(3.32)] = \frac{1}{4}(P_{\mathrm{f}\mu}\frac{1}{H_0}(A^+A^+)\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}(A^+A^+)\varphi_{(0)})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}|k_1+k_2|^2(e_1\cdot e_2)(e_1\cdot e_2+e_2\cdot e_1)\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}|k_1+k_2|^2 2(1+(\hat{k}_1,\hat{k}_2)^2)
\end{aligned}
$$

7.

$$
\begin{aligned}
&(\Phi_6^\mu, H_1\Phi_4^\mu) = -\frac{1}{8}(P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}(A^+A^+)\varphi_{(0)})\\
&= -\frac{1}{8}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}|k_1+k_2|^2(\frac{\langle\sigma_1\sigma_2\rangle e_1\cdot e_2}{E_2}+\frac{\langle\sigma_2\sigma_1\rangle e_2\cdot e_1}{E_1})\\
&= -\frac{1}{8}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}|k_1+k_2|^2 2|k_1||k_2|(\hat{k}_1,\hat{k}_2)(\frac{1}{E_2}+\frac{1}{E_1})
\end{aligned}
$$

8.

$$
(\Phi_4^\mu, H_1\Phi_6^\mu) = \overline{(\Phi_6^\mu, H_1\Phi_4^\mu)}
$$

9.

$$
\begin{aligned}
&(\Phi_5^\mu, H_1\Phi_5^\mu) = \frac{1}{4}(P_{\mathrm{f}\mu}\frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}\frac{1}{E_2}|k_1+k_2|^2((k_2\cdot e_1)(k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2}+(k_2\cdot e_1)(k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})\\
&= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}\frac{1}{E_2}|k_1+k_2|^2(\frac{2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2}+\frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1})
\end{aligned}
$$

10.

$$
\begin{aligned}
&(\Phi_6^\mu, H_1\Phi_6^\mu) = \mathcal{E}_2 = \frac{1}{16}(P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{16}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}\frac{1}{E_2}|k_1+k_2|^2(\langle\sigma_2\sigma_1\sigma_1\sigma_2\rangle\frac{1}{E_2}+\langle\sigma_2\sigma_1\sigma_2\sigma_1\rangle\frac{1}{E_1})\\
&= \frac{1}{16}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^3}\frac{1}{E_2}|k_1+k_2|^2(\frac{4|k_1|^2|k_2|^2}{E_2}+\frac{-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1})
\end{aligned}
$$

11.

$$
\begin{aligned}
&(\Phi_1^\mu, H_4\Phi_1^\mu) = [3,(3.34)] = -(A_\nu^-\frac{1}{H_0}A_\mu^+\varphi_{(0)}, A_\nu^-\frac{1}{H_0}A_\mu^+\varphi_{(0)})\\
&= -\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2}(e_1\cdot e_2)(e_1\cdot e_2)\\
&= -\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2}(1+(\hat{k}_1,\hat{k}_2)^2)
\end{aligned}
$$

12.

$$
\begin{aligned}
&(\Phi_2^\mu, H_4\Phi_2^\mu)\\
&= -\frac{1}{8}(P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}(A^+A^+ + 2A^+A^- + A^-A^-)P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= -\frac{1}{4}(A_\nu^-\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, A_\nu^-\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_2}\langle\sigma_2\sigma_1\rangle(k_1\cdot k_2)(e_1\cdot e_2)\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{2|k_1|^2|k_2|^2(\hat{k}_1,\hat{k}_2)}{E_1^2E_2^2}
\end{aligned}
$$

13.

$$
\begin{aligned}
&(\Phi_1^\mu, H_5\Phi_1^\mu) = [3,(3.36)] = (A_\mu^+\varphi_{(0)}, \frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}A_\mu^+\varphi_{(0)})\\
&= \int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}(k_2\cdot e_1)((k_2\cdot e_1)(e_2\cdot e_2)\frac{1}{E_2} + (k_1\cdot e_2)(e_1\cdot e_2)\frac{1}{E_1})\\
&= \int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}(\frac{2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2} + \frac{-(k_1\cdot k_2)(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1})
\end{aligned}
$$

14.

$$
\begin{aligned}
&(\Phi_2^\mu, H_5\Phi_2^\mu) = \frac{1}{4}(\frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, (P_\mathrm{f}\cdot A^+)\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= \frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}(k_2\cdot e_1)((k_2\cdot e_1)|k_2|^2\langle\sigma_2\sigma_2\rangle\frac{1}{E_2^2} + (k_1\cdot e_2)(k_1\cdot k_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1^2})\\
&= \frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}(\frac{2|k_2|^6(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2^2} + \frac{(k_1\cdot k_2)|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1^2})
\end{aligned}
$$

15.

$$
\begin{aligned}
&(\Phi_2^\mu, H_6\Phi_1^\mu) = -\frac{1}{4}(\frac{1}{H_0}\sigma B\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, (P_\mathrm{f}\cdot A^+)\frac{1}{H_0}A_\mu^+\varphi_{(0)})\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}(k_2\cdot e_1)((k_2\cdot e_2)\langle\sigma_1\sigma_2\rangle\frac{1}{E_2}\frac{1}{E_2} + (k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1}\frac{1}{E_1})\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1^2}
\end{aligned}
$$

16.

$$
\begin{aligned}
&(\Phi_1^\mu, H_6\Phi_2^\mu) = -\frac{1}{4}(\sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)}, \frac{1}{H_0}(P_\mathrm{f}\cdot A^+)\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}((k_2\cdot e_1)(k_2\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_2} + (k_2\cdot e_1)(k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_1})\\
&= -\frac{1}{4}\int_D\frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}\frac{2|k_2|^4((\hat{k}_1,\hat{k}_2)^2-1)}{E_1}
\end{aligned}
$$

17.

$$(\Phi_2^\mu, H_7\Phi_1^\mu) = -\frac{1}{4}(\frac{1}{H_0}(P_{\mathrm{f}}\cdot A^+)\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}((k_2\cdot e_1)(k_2\cdot e_2)\langle\sigma_1\sigma_2\rangle\frac{1}{E_2}\frac{1}{E_2} + (k_1\cdot e_2)(k_1\cdot e_2)\langle\sigma_1\sigma_1\rangle\frac{1}{E_1}\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{2|k_1|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1^2}$$

18.

$$(\Phi_1^\mu, H_7\Phi_2^\mu) = -\frac{1}{4}(\sigma B^+\frac{1}{H_0}(P_{\mathrm{f}}\cdot A^+)\frac{1}{H_0}A_\mu^+\varphi_{(0)}, \frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}\langle\sigma_2\sigma_1\rangle((k_2\cdot e_1)(k_2\cdot e_2)\frac{1}{E_2} + (k_1\cdot e_2)(k_2\cdot e_1)\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}\frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1}$$

19.

$$(\Phi_1^\mu, H_8\Phi_1^\mu) = \mathcal{E}_0$$
$$= \frac{1}{4}(\sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$-\frac{1}{4}(\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\varphi_{(0)})(A_\mu^+\varphi_{(0)}, \frac{1}{H_0}\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}((e_2\cdot e_2)\langle\sigma_1\sigma_1\rangle\frac{1}{E_2} + (e_2\cdot e_1)\langle\sigma_1\sigma_2\rangle\frac{1}{E_1})$$
$$-\frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_2}\langle\sigma_1\sigma_1\rangle(e_2\cdot e_2)$$
$$= \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}(\frac{4|k_1|^2}{E_2^2} + \frac{2|k_1||k_2|(\hat{k}_1,\hat{k}_2)}{E_1E_2}) - \frac{1}{4}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2^2}4|k_1|^2$$

20.

$$(\Phi_2^\mu, H_8\Phi_2^\mu) = \mathcal{E}_3$$
$$= \frac{1}{16}(\sigma B^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$-\frac{1}{16}(\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\varphi_{(0)})(\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}P_{\mathrm{f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= \frac{1}{16}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}(\langle\sigma_2\sigma_1\sigma_1\sigma_2\rangle|k_2|^2\frac{1}{E_2}\frac{1}{E_2} + (k_1\cdot k_2)\langle\sigma_2\sigma_1\sigma_2\sigma_1\rangle\frac{1}{E_1}\frac{1}{E_1})$$
$$-\frac{1}{16}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}|k_2|^2\langle\sigma_1\sigma_1\rangle\langle\sigma_2\sigma_2\rangle$$
$$= \frac{1}{16}\int_D \frac{\mathrm{d}^3k_1\mathrm{d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}(\frac{4|k_1|^2|k_2|^4}{E_2^2} + \frac{(k_1\cdot k_2)(-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2))}{E_1^2})$$

$$-\frac{1}{16}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_1}\frac{1}{E_2^4}4|k_1|^2|k_2|^4$$

21.

$$(\Phi_4^\mu, H_2\Phi_1^\mu)(=[3,(3.33)]) = \frac{1}{2}(P_{f\mu}\frac{1}{H_0}A^+A^+\varphi_{(0)}, \frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= \frac{1}{2}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(k_2\cdot e_1)(k_1\cdot e_2)((e_1\cdot e_2)+(e_2\cdot e_1))$$
$$= \frac{1}{2}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(-2)(k_1\cdot k_2)(1-(\hat{k}_1,\hat{k}_2)^2)$$

22.

$$(\Phi_6^\mu, H_2\Phi_1^\mu) = -\frac{1}{4}(P_{f\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}((k_2\cdot e_1)(k_1\cdot e_2)\langle\sigma_1\sigma_2\rangle\frac{1}{E_2}+(k_1\cdot e_2)(k_2\cdot e_1)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)(\frac{1}{E_1}+\frac{1}{E_2})$$

23.

$$(\Phi_3^\mu, H_2\Phi_2^\mu) = -\frac{1}{4}(A_\mu^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}((k_2\cdot e_1)(k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2}+(k_2\cdot e_1)(k_2\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^3}2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)$$

24.

$$(\Phi_5^\mu, H_2\Phi_2^\mu) = \frac{1}{4}(P_{f\mu}\frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}\frac{1}{E_2}(k_2\cdot e_1)(k_1+k_2)\cdot k_2((k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2}+(k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2^2}k_2\cdot(k_1+k_2)(\frac{2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2}+\frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1})$$

25.

$$(\Phi_3^\mu, H_3\Phi_1^\mu) = \frac{1}{4}(A_\mu^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}(\langle\sigma_2\sigma_1\rangle(e_1\cdot e_2)\frac{1}{E_2}+\langle\sigma_1\sigma_1\rangle(e_2\cdot e_2)\frac{1}{E_1})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(\frac{2|k_1||k_2|(\hat{k}_1,\hat{k}_2)}{E_2}+\frac{4|k_1|^2}{E_1})$$

26.

$$(\Phi_5^\mu, H_3\Phi_1^\mu) = -\frac{1}{4}(P_{f\mu}\frac{1}{H_0}(P_f\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(k_1\cdot e_2)(\langle\sigma_1\sigma_2\rangle(k_2\cdot e_1)\frac{1}{E_2} + \langle\sigma_1\sigma_1\rangle(k_1\cdot e_2)\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}(\frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_2} + \frac{2|k_1|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1})$$

27.

$$(\Phi_4^\mu, H_3\Phi_2^\mu) = -\frac{1}{8}(P_{f\mu}\frac{1}{H_0}A^+A^+\varphi_{(0)}, \frac{1}{H_0}\sigma B\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= -\frac{1}{8}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\langle\sigma_2\sigma_1\rangle(k_1+k_2)\cdot k_2\frac{1}{E_2}\frac{1}{E_2}(e_1\cdot e_2 + e_2\cdot e_1)$$
$$= -\frac{1}{8}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2^2}4k_2\cdot(k_1+k_2)|k_1||k_2|(\hat{k}_1,\hat{k}_2)$$

28.

$$(\Phi_6^\mu, H_3\Phi_2^\mu) = \frac{1}{16}(P_{f\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}\sigma B\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= \frac{1}{16}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2}\frac{1}{E_2}(k_1+k_2)\cdot k_2(\langle\sigma_2\sigma_1\sigma_1\sigma_2\rangle\frac{1}{E_2} + \langle\sigma_2\sigma_1\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= \frac{1}{16}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}^2}\frac{1}{E_2^2}(k_1+k_2)\cdot k_2(\frac{4|k_1|^2|k_2|^2}{E_2} + \frac{-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1})$$

29.

$$(\Phi_7^\mu, H_1\Phi_1^\mu) = [3,(3.31)] = -\frac{1}{2}(\frac{1}{H_0}A^+A^+\varphi_{(0)}, A_\mu^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= -\frac{1}{2}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}(e_1\cdot e_2)((e_1\cdot e_2) + (e_2\cdot e_1))$$
$$= -\frac{1}{2}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}2(1+(\hat{k}_1,\hat{k}_2)^2)$$

30.

$$(\Phi_9^\mu, H_1\Phi_1^\mu) = \frac{1}{4}(\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, A_\mu^+\frac{1}{H_0}A_\mu^+\varphi_{(0)})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}(e_1\cdot e_2)(\langle\sigma_1\sigma_2\rangle\frac{1}{E_2} + \langle\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= \frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}2|k_1||k_2|(\hat{k}_1,\hat{k}_2)(\frac{1}{E_1} + \frac{1}{E_2})$$

31.

$$\begin{aligned}
&(\Phi_{11}^{\mu}, H_1\Phi_1^{\mu}) = [3, \text{RHS } of(3.30)] \\
&= -\frac{1}{2}(\frac{1}{H_0}A^+A^+\varphi_{(0)}, (P_{\rm f}\cdot A^+)\frac{1}{H_0}P_{{\rm f}\mu}\frac{1}{H_0}A_{\mu}^+\varphi_{(0)}) \\
&= -\frac{1}{2}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}(e_2\cdot k_2)(e_1\cdot k_2)(e_1\cdot e_2 + e_2\cdot e_1) = 0
\end{aligned}$$

32.

$$\begin{aligned}
&(\Phi_{13}^{\mu}, H_1\Phi_1^{\mu}) = \frac{1}{4}(P_{{\rm f}\mu}\frac{1}{H_0}(P_{\rm f}\cdot A^+)\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}A_{\mu}^+\varphi_{(0)}) \\
&\frac{1}{4}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}(e_2\cdot k_2)(k_2\cdot e_1)(\langle\sigma_1\sigma_2\rangle\frac{1}{E_2} + \langle\sigma_2\sigma_1\rangle\frac{1}{E_1}) = 0
\end{aligned}$$

33.

$$\begin{aligned}
&(\Phi_{15}^{\mu}, H_1\Phi_1^{\mu}) = \frac{1}{4}(P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\frac{1}{H_0}(P_{\rm f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}A_{\mu}^+\varphi_{(0)}) \\
&= \int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}(e_2\cdot k_2)((k_2\cdot e_1)\langle\sigma_1\sigma_2\rangle\frac{1}{E_2} + (k_1\cdot e_2)\langle\sigma_1\sigma_1\rangle\frac{1}{E_1}) = 0
\end{aligned}$$

34.

$$\begin{aligned}
&(\Phi_8^{\mu}, H_1\Phi_2^{\mu}) = -\frac{1}{4}(\frac{1}{H_0}(P_{\rm f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, A_{\mu}^+\frac{1}{H_0}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}) \\
&= -\frac{1}{4}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}(k_2\cdot e_1)((k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2} + (k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1}) \\
&= -\frac{1}{4}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^2}(\frac{2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2} + \frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1})
\end{aligned}$$

35.

$$\begin{aligned}
&(\Phi_{10}^{\mu}, H_1\Phi_2^{\mu}) = -\frac{1}{8}(P_{{\rm f}\mu}\frac{1}{H_0}2A^+A^-\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \frac{1}{H_0}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}) \\
&= -\frac{1}{4}(A_{\mu}^-\frac{1}{H_0}\sigma B^+\varphi_{(0)}, A_{\mu}^-\frac{1}{H_0}P_{{\rm f}\nu}\frac{1}{H_0}P_{{\rm f}\nu}\frac{1}{H_0}\sigma B^+\varphi_{(0)}) \\
&= -\frac{1}{4}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_1}|k_2|^2\langle\sigma_2\sigma_1\rangle(e_2\cdot e_1) \\
&= -\frac{1}{4}\int_D \frac{{\rm d}^3k_1{\rm d}^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_2^3}\frac{1}{E_1}2|k_1||k_2|^3(\hat{k}_1,\hat{k}_2)
\end{aligned}$$

36.

$$\begin{aligned}
&(\Phi_{12}^{\mu}, H_1\Phi_2^{\mu}) \\
&= -\frac{1}{4}(\frac{1}{H_0}(P_{\rm f}\cdot A^+)\frac{1}{H_0}\sigma B^+\varphi_{(0)}, (P_{\rm f}\cdot A^+)\frac{1}{H_0}P_{{\rm f}\mu}\frac{1}{H_0}P_{{\rm f}\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})
\end{aligned}$$

$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}(k_2\cdot e_1)|k_2|^2((k_2\cdot e_1)\langle\sigma_2\sigma_2\rangle\frac{1}{E_2} + (k_1\cdot e_2)\langle\sigma_2\sigma_1\rangle\frac{1}{E_1})$$
$$= -\frac{1}{4}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^3}|k_2|^2(\frac{2|k_2|^4(1-(\hat{k}_1,\hat{k}_2)^2)}{E_2} + \frac{|k_1|^2|k_2|^2((\hat{k}_1,\hat{k}_2)^2-1)}{E_1})$$

37.

$$(\Phi_{14}^\mu, H_1\Phi_2^\mu) = \frac{1}{8}(\frac{1}{H_0}A^+A^+\varphi_{(0)}, \sigma B^+\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$\frac{1}{8}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}|k_2|^2\langle\sigma_2\sigma_1\rangle(e_1\cdot e_2 + e_2\cdot e_1)$$
$$= \frac{1}{8}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{4|k_1||k_2|^3(\hat{k}_1,\hat{k}_2)}{E_2^3}$$

38.

$$(\Phi_{16}^\mu, H_1\Phi_2^\mu) = \mathcal{E}_4$$
$$= -\frac{1}{16}(\frac{1}{H_0}\sigma B^+\frac{1}{H_0}\sigma B^+\varphi_{(0)}, \sigma B^+\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}P_{f\mu}\frac{1}{H_0}\sigma B^+\varphi_{(0)})$$
$$= -\frac{1}{16}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2}\frac{1}{E_2}\frac{1}{E_2}|k_2|^2(\langle\sigma_2\sigma_1\sigma_2\sigma_1\rangle\frac{1}{E_1} + \langle\sigma_2\sigma_1\sigma_1\sigma_2\rangle\frac{1}{E_2})$$
$$= -\frac{1}{16}\int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2}\frac{1}{E_{12}}\frac{1}{E_2^3}|k_2|^2(\frac{-2|k_1|^2|k_2|^2(1-(\hat{k}_1,\hat{k}_2)^2)}{E_1} + \frac{4|k_1|^2|k_2|^2}{E_2})$$

As is seen above, integrands in each term are functions of $|k_1|, |k_2|, (\hat{k}_1, \hat{k}_2)$. Changing variables $|k_1|, |k_2|, (\hat{k}_1, \hat{k}_2)$ as $r_1, r_2, X = \cos\theta$, $0 \le \theta < \pi$, each term has the form

$$F(\Lambda/m) = \int_D \frac{d^3k_1 d^3k_2}{4(2\pi)^6\omega_1\omega_2} f(|k_1|, |k_2|, (\hat{k}_1, \hat{k}_2)) = \frac{2\pi^2}{(2\pi)^6}\int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 f(r_1, r_2, X).$$

We see that

$$\frac{d}{d(\Lambda/m)}F(\Lambda/m) = \frac{2\pi^2}{(2\pi)^6}\int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr r(\Lambda/m)[f(\Lambda/m, r, X) + f(r, \Lambda/m, X)]. \quad (18)$$

To see the asymptotic behavior of $F(\Lambda/m)$, we estimate the right hand side of (18). We can see that

$$\lim_{\Lambda\to\infty}\frac{|(\Phi_i^\mu, H_j\Phi_k^\mu)|}{\sqrt{\Lambda/m}} < \infty, \quad \mu = 1, 2, 3,$$

where $(i, j, k) \ne (1, 8, 1), (2, 8, 2), (16, 1, 2)$. In the case of $(i, j, k) = (1, 8, 1), (2, 8, 2)$, each term is divided into two integrals. Although each integral diverges as $(\Lambda/m)^2$ as $\Lambda \to \infty$, cancelation happens. Then the terms $(i, j, k) = (1, 8, 1), (2, 8, 2)$ diverge as $[\log(\Lambda/m)]^2$. Finally we can directly see that the term $(16, 1, 2)$ diverges as $(\Lambda/m)^2$. Then the proof is complete. □

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