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FIXED POINT PROPERTIES FOR SEMIGROUP OF NONEXPANSIVE MAPPINGS ON BI-TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, we shall outline our work on fixed point properties on bi-topological vector spaces for left reversible semitopological semigroups generalizing some classical results.

1. Introduction

A ***semitopological semigroup*** is a set S together with an associative operation and a Hausdorff topology such that for each $a \in S$, the two mappings from S into S defined by $a \mapsto as$ and $s \mapsto sa$ for all $s \in S$, are continuous. S is said to be ***left (resp. right) reversible*** if any two (and hence any finite number) non-empty closed right (resp. left) ideals of S has non-void intersection (see [3], p.34).

T. Mitchell [8] shows that if S is a ***discrete*** left reversible semigroup, then S has the following fixed point property (see also [4, 10]):

(F) Whenever $\mathcal{S} = \{T_s \ : \ s \in S\}$ is a representation of S as non-expansive mappings from a non-empty compact convex subset K of a Banach space into K, then K contains a common fixed point for $\mathcal{S}$.

In [6, Theorem 5.3], we show that if S is discrete and left reversible, then S also satisfies:

(F_*) Wheneve $\mathcal{S} = \{T_s \ : \ s \in S\}$ is a representation of S as weak*-weak* continuous and norm non-expansive mappings of a weak*-compact convex subset K of a norm separable dual Banach space, then K contains a common fixed point for $\mathcal{S}$.

It was shown by T. C. Lim [7], Theroem 4 that if S is left reversible and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as non-expansive seft-maps of a weak*-compact convex subset K of ℓ^1 (which is separable), then K contains a common fixed point for $\mathcal{S}$ without the assumption that each T_s, $s \in S$, is weak*-weak* continuous. However, this weak*-continutiy assumption connot be removed in general. Indeed, it follows from Alspach's example [1] that there exists a representation of the commutative semigroup $S = (\mathrm{N}, +), \mathrm{N} = \{1, 2, 3, \dots\}$, as non-expansive mappings of a weakly compact convex subset K of the separable Banach space $L_1[0,1]$ without a common fixed point. Then K regarded as a subset of $L_1[0,1]^{**}$ is norm separable, weak*-compact and convex.

In this paper, we shall outline our work on fixed point properties on bi-topological vector spaces for left reversible semitopological semigroups which, includes fixed point properties (F) and (F_*).

Defails of proof will appear elsewhere. The first author would like to thank Professor Wataru Takahashi for kindly in inviting him to speak at the Symposium on Nonlinear and Convex Analysis at Kyoto University in August, 2005 and his warm hospitality at the Tokyo Institute of Technology where discussions on main results of this work were carried out.

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2. Preliminaries and notations

If A is a subset of a topological space X, then $\overline{A}$ will denote the closure of A in X.

Throughout this paper, E will denote a separated locally convex (linear topological) space, Q a (fixed) family of seminorms which generates the topology of E, and S a Hausdorff semitopological semigroup.

We denote by $\ell^\infty(S)$ the Banach space of bounded real-valued functions on S with the supremum norm. Then a subspace of $\ell^*(S)$ is left (resp. right) translation invariant if $\ell_a(X) \subset X$ (resp. $r_a(X) \subset X$) for all $a \in S$, where $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(as)$ for all $s \in S$. Let $CB(S)$ denote the closed subalgebra of $\ell^\infty(S)$ consisting of all continuous functions. Let $LUC(S)$ be the subalgebra in $CB(S)$ of all left uniformly continuous functions on S, i.e., all $f \in CB(S)$ such that the map $a \mapsto \ell_a f$ from S in $(CB(S)), \|\cdot\|)$ is continuous. Then $LUC(S)$ is translation invariant and contains the constant functions (see [2]). We say that S is *left amenable* if $LUC(S)$ has a left invariant mean (LIM), i.e., $m \in LUC(S)^*$ such that $\|m\| = m(1) = 1$ and $m(\ell_a f) = m(f)$ for all $a \in S$ and $f \in LUC(S)$. If S is discrete, and left amenable, then S is left reversible. In general this is not ture.

A locally convex space which is metrizable and complete is called a *Fréchet space.*

Let (E, Q) be a locally convex topological vector space determined by a family of seminorms Q. We say that a locally convex topology τ on E is Q-admissible if

(i) each $p \in Q$ is τ-lower semicontinuous.

(ii) τ is weaker than τ_Q = topology generaled by Q.

The triple (E, τ, τ_Q) is called a *bi-topological vector space.* If E is a Banach space, then the weak topology is $\|\cdot\|$-admissible. Also if E^* is the dual of a Banach space E and τ = weak*-topology on E^*, then τ is also $\|\cdot\|$-admissible. In particular, $(E, \|\cdot\|, \text{weak})$ and $(E, \|\cdot\|, \text{weak}^*)$ are examples of bi-topological vecter spaces.

3. The main results

In this section, we shall state the main results of the paper.

Theorem 3.1. *Let S be a semitopolocigal semigroup. If S is left reversible, then S has the following fixed point property:*

Let (E, Q) be a separable Fréchet space determined by a sequence of seminorms Q and τ be a Hausdorff Q-admissible locally convex topology on E. If K is a τ-compact convex subset of E, and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as Q-non-expansive mappings on K such that the mapping $S \times K \to K$, $(s, x) \to T_s(x)$, is separately continous where K has the τ-topology, then K has a common fixed point for $\mathcal{S}$.

Theorem 3.2. *Let S be a semitopological semigroup. If $LUC(S)$ has a LIM, then S has the following fixed point property:*

Let (E, Q) be a separable Fréchet space determind by a sequence of seminorms Q and τ be a Hausdorff Q-sdmissible locally convex topology on E. If K is a τ-compact convex subset of E and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as Q-non-expansive mappings on K such that the mapping $S \times K \to K$, $(s, x) \to T_s(x)$ is jointly continuous where K has the τ-topology, then K has a common fixed pont for $\mathcal{S}$.

Remark 3.3. *Note that the continuity condition in Theorem 3.2 may be weakened if spaces larger than $LUC(S)$ has a LIM.*

For example:

(i) If $C(S)$ has a LIM m, then it suffices to assume that for each $s \in S$, T_s is τ-τ continous and there exists $x \in K$ such that the map $s \mapsto T_s x$ from S to (K, τ) is continuous. In this case, if $f \in C(K,\tau)$, $(Q_x f)(s) = f(T_s x)$, is in $C(S)$. So $Q_x^* m$ defines a positive functional of norm one on $C(K,\tau)$. Then the same argument as given in Lemma 5.1 of [6] shows that if F is a mininal τ-closed $\mathcal{S}$-invariant subset of K, then $T_s(F) = F$ for each $s \in S$.

(ii) Let $WLUC(S)$ denote the set of all functions $f \in C(S)$ such that the map $S \mapsto (C(S), \text{weak})$, $s \mapsto \ell_s f$ is continuous. Then $WLUC(S)$ is a closed translation invariant subspace of $C(S)$ containing $LUC(S)$ (see [8]). If S is second countable, and $WLUC(S)$ has a LIM, then the "joint continuity condition" may be replaced by "separate continuity" in Theorem 3.2. Indeed, in this case, if $x \in K$, $f \in C(S)$, then $Q_x f \in WLUC(S)$: If $\{s_n\}$ is a sequence in S, $s_n \to s$, then $\ell_{s_n}(Q_x f) = Q_x(s_n f) \to Q_x(sf) = \ell_s(Q_x f)$ pointwise on S, where $sf(x) = f(T_s x)$. If $\phi \in C(S)^*$, $\phi \geq 0$, $\|\phi\| = 1$, then $Q_x^*\phi \in C(K,\tau)^*$, $Q_x^*\phi \geq 0$, and $\|Q_x\phi\| = 1$. So by Riesz representation theorem, there exists a probability measure μ on (K,τ) such that $\langle Q_x^*\phi, h\rangle = \int_K h(x)d\mu(x)$ for all $h \in C(K,\tau)$. Since $\|s_n f\| \leq \|f\|$ for all n, by the dominated convergence theorem, $\int_K s_n f(x)d\mu(x) \to \int_K f(x)d\mu(x)$. Consequently $\ell_{s_n}(Q_x f) \to \ell_s(Q_x f)$ weakly in $C(S)$.

Note that in general $WLUC(S) \neq LUC(S)$. For example, when S is the one point-compactification of $(\mathbf{R},+)$ or $(\mathbf{Z},+)$, then $WLUC(S) = C(S)$, but $LUC(S)$ consists of only constant functions (see [2, p.174]). But if S is a locally compact or complete metrizable group, then $WLUC(S) = LUC(S)$ (see Mithell [8]).

Question: Can "second countability" be dropped ?

Let (E,Q) be a locally convex space and τ be a Hausdorff Q-admissible locally convex topology of E. Let X be a τ-compact subset of E, and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as Q-non-expansive τ-τ continuous mappings from X into X. Let $\sum$ be the closure of $\mathcal{S}$ in the product space $(X,\tau)^X$. Then $\sum$ is a semigroup and a compact Hausdorff space such that

(i) $\forall \tau \in \sum$, the map $T' \mapsto T' \cdot T$ is continuous from $\sum \to \sum$
(ii) $\forall s \in S$ the map $T' \mapsto T_s \cdot T'$ is continuous from $\sum \to \sum$

Consequently $\sum$ is a *compact right topological semigroup*. $\sum$ contains minimal left ideals which are closed, pairwise algebraically isomorphic and topologically homeomorphic.

(iii) For each $T \in \sum$, T is Q-non-expansive

Theorem 3.4. *Let (E,Q) be a locally convex space, τ be a Q-admissible locally convex topology on E, and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S as Q-non-expansive and τ-τ continuous mappings from a τ-compact convex set X into X. Let $\sum$ denote the closure of $\mathcal{S}$ in $(X,\tau)^X$. Then $\sum$ is a compact right topological semigroup consisting of Q-non-expansive mappings from X into X. Furthermore, if X has τ-normal structure, L is a minimal left ideal of $\sum$ and Y is a $\mathcal{S}$-invariant τ-closed convex subset of X, then there exists a non-empty τ-closed $\mathcal{S}$-invariant subset C of Y such that L is constant on Y. Also, there exists $T_o \in \sum$ and $x \in X$ such that $T_oTx = T_ox$ for all $T \in \sum$. In particuler, T_ox is a common fixed point for the algebraic center of $\sum$.*

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